$e^+e^- \to \nu\bar{\nu}\gamma$: THE IMPORTANCE OF AN EXACT CALCULATION

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We make a detailed study of the reaction $e^+e^- \to \nu\bar{\nu}\gamma$. We derive the helicity amplitudes using the spinor product formalism and discuss the effects of beam polarization. Our exact and analytical results are compared with the ones derived using the local approximation for the W-exchange diagram.

The importance of the reaction $e^+e^- \to \nu\bar{\nu}\gamma$ to count the number of neutrino species, N_ν, was pointed out [1] several years ago. However, with the forthcoming operation of the SLAC and CERN colliders one sees a renewed interest in this reaction. Some of these latter works [2] study the general case of photon plus missing energy–momentum reactions in e^+e^- collisions while others address the question of a possible existence of extra Z-bosons [3], examine the various background reactions [4] or deal with radiative corrections [5]. Unfortunately, in all these papers the standard model calculation is done using the local approximation for the W-exchange diagram.

In a previous publication [6], we have done, for the first time, an exact calculation of the $e^+e^- \to \nu\bar{\nu}\gamma$ differential cross section and recently a similar calculation was performed by Chiappetta et al. [7]. Comparing their results with ours leads us to re-examine and extend our work. In doing so we have three aims. Firstly, we want to show that, in some cases, the error associated with the local approximation is larger than some predicted deviations characteristic of non-standard models. Hence, it is inconsistent to use the local approximation and try to learn about these extensions of the standard model. Our second aim is to point out the advantages of using our analytic expressions [6] for the differential cross section after the integration over the neutrino phase space. Finally, the third purpose of this letter is to consider the effects of beam polarization. Besides Chiappetta et al. [7], other authors [8] have also examined this problem. However, there is no agreement between them. In this letter, we derive the helicity amplitudes using the spinor product formalism [9,10]. Using them we have an alternative evaluation of the cross section which uses a four-dimensional numerical integration. As we shall see, both calculations are in excellent agreement.

Let us denote by p_1 (p_2) the momentum of the incoming electron (positron), by q_1 (q_2) the momentum of the neutrino (antineutrino) and by k the photon momentum. After integration over the neutrino phase space we obtain the differential cross section $d\sigma/dx\,dy$, where $x=2\omega/\sqrt{s}$ is the photon energy (ω) in units of the beam energy and $y=\cos\theta$ with θ the angle between $\boldsymbol{k}$ and $\boldsymbol{p}_1$. In our previous paper [6] we gave a complete expression for $d\sigma/dx\,dy$ in terms of the integrals

$$\hat{I}_{ij,lm,\dots}[f(q_1,q_2)] = \int \frac{d^3q_1}{2E_1}\frac{d^3q_2}{2E_2}\frac{\delta^4(q_1+q_2-\Delta)}{N_{ij}N_{lm}\cdots} f(q_1,q_2)\,, \tag{1}$$

with

$$\Delta = p_1+p_2-k \tag{2}$$

and

$$N_{ij}=M_W^2-(p_i-q_j)^2\ , \quad i,j=1,2\ . \tag{3}$$

Since these results are rather lengthy and are correctly printed in ref. [6] there is no need to repeat them here. However, we have found that the value of one of the integrals $\hat{I}$ was missing. Hence, for completeness we list in the appendix the value of this integral.

In our equations for $d\sigma/dx\,dy$ it is very easy to include the effect of longitudinal polarized beams. In fact, all that one has to do is an appropriate redefinition of the vector (g_V) and axial-vector (g_A) coupling constants. However, for the general case it is simpler to calculate the helicity amplitudes $M(\sigma_-, \sigma_+, \lambda)$ where $\sigma_-(=\pm)$, σ_+ and λ [#1] denote the electron, positron and photon helicity respectively. At high energy it is a very good approximation to neglect the electron mass and so the amplitudes $M(\sigma_-, \sigma_+, \lambda)$ are easily derived using the spinor product formalism [9,10].
If $u_\pm(p)$ are chiral spinors satisfying the equation

$$u_\pm(p)\bar{u}_\pm(p)=\gamma_\pm \not{p}\ , \tag{4}$$

with $\gamma_\pm=(1\pm\gamma_5)/2$, the non-zero spinor products are

$$s(p_1,p_2)=\bar{u}_+(p_1)u_-(p_2)=-s(p_2,p_1) \tag{5a}$$

and

$$t(p_1,p_2)=\bar{u}_-(p_1)u_+(p_2)=s^*(p_2,p_1)\ . \tag{5b}$$

For the polarization vector of the photon we take the form originally presented by the CALKUL Collaboration [11] modified to be expressed in terms of spinor products [10], i.e.,

$$\not{\epsilon}_\pm(k,p_1,p_2)=N_\pm(\gamma_\pm \not{k}\not{p}_1\not{p}_2-\gamma_\mp \not{p}_1\not{p}_2\not{k}\pm p_1\cdot p_2\not{k}\gamma_5)\ ,$$

with

$$N_+=\sqrt{2}/s(p_1,k)s(p_2,k)s^*(p_1,p_2)\ , \quad N_-=(N_+)^*\ ,$$

where p_1 and p_2 are arbitrary four-momenta not proportional to k or to each other. To simplify we take them to be the electron and positron momenta. We should point out that contrary to the situation in ref. [10], the last term in eq. (6) gives a non-zero contribution to our process.

From the Z and W exchange Feynman diagrams it is fairly easy to obtain the helicity amplitudes. Defining

$$C_Z=4eG_F\frac{M_Z^2}{\Delta^2-M_Z^2+iM_Z\Gamma_Z}\ , \quad C_{W_1}=8eG_F\frac{M_W^2}{N_{22}}\ , \quad C_{W_2}=8eG_F\frac{M_W^2}{N_{11}}\ , \quad C_{W_3}=8eG_F\frac{M_W^2}{N_{11}N_{22}}\ , \tag{7}$$

where M_Z and Γ_Z denote the mass and width of the Z-boson, G_F is the Fermi coupling constant, $g_A=-1/2$ and $g_V=-1/2+2\sin^2\theta_W$, our results are

$$M_Z(+,-,+)=C_Z(g_V-g_A)\frac{s(p_2,q_2)}{s(p_1,k)s(p_2,k)}[s^*(k,q_1)s(p_2,k)-s^*(p_1,q_1)s(p_2,p_1)]\ , \tag{8a}$$

$$M_Z(-,+,+)=C_Z(g_V+g_A)\frac{s(q_2,p_1)}{s(p_1,k)s(p_2,k)}[s(p_1,p_2)s^*(q_1,p_2)-s^*(q_1,k)s(p_1,k)] \tag{8b}$$

and

[#1] + means right-handed.

$$M_W(-,+,+)=\{s(p_1,q_2)[C_{W_1}s(p_1,p_2)s^*(q_1,p_2)-C_{W_2}s^*(q_1,k)s(p_1,k)]$$
$$+C_{W_3}[s^*(k,q_1)s(p_1,k)s^*(p_2,k)[s(p_1,k)s(p_2,q_2)+s(p_2,p_1)s(k,q_2)]$$
$$+s^*(q_1,p_2)s(p_1,q_2)[s(p_1,p_2)\,|s(p_1,k)|^2-s(p_1,p_2)\,|s(k,q_1)|^2$$
$$+s^*(k,q_1)s(q_1,p_1)s(p_2,k)]]\}/s(p_1,k)s(p_2,k)\;. \tag{9}$$

$M_Z(+,-,-)$, $M_Z(-,+,-)$ and $M_W(-,+,-)$ can be obtained from $M_Z(+,-,+)$, $M_Z(-,+,+)$ and $M_W(-,+,+)$, respectively, with the replacements $C_Z\to -C_Z$, $C_{W_i}\to -C_{W_i}$, $1\leftrightarrow 2$ and taking the complex conjugate, i.e., $s\leftrightarrow s^*$. The total amplitude for the reaction with electron-neutrinos is

$$M(\sigma_-,\sigma_+,\lambda)=M_Z(\sigma_-,\sigma_+,\lambda)+M_W(\sigma_-,\sigma_+,\lambda)\;.$$

It is interesting to notice that these relations among the helicity amplitudes can be derived on general grounds. In fact, the CP invariance of the theory implies

$$CP|\boldsymbol{p}_1,\lambda_1,\boldsymbol{p}_2\lambda_2\rangle=|-\boldsymbol{p}_2-\lambda_2,\,-\boldsymbol{p}_1-\lambda_1\rangle\;, \tag{10}$$

where, in each ket, the first pair of variables denotes the momentum and helicity of the e^-. Hence, recalling that the photon has $C=-1$, we have

$$\langle\boldsymbol{q}_1-,\boldsymbol{q}_2+,\boldsymbol{k}\lambda|T|\boldsymbol{p}_1\lambda_1,\boldsymbol{p}_2\lambda_2\rangle=-\langle-\boldsymbol{q}_2-,\,-\boldsymbol{q}_1+,\,-\boldsymbol{k}-\lambda|T|-\boldsymbol{p}_2-\lambda_2,\,-\boldsymbol{p}_1-\lambda_1\rangle\;. \tag{11}$$

On the other hand, combining the T-invariance with the hermiticity of the Born amplitude T_B one has

$$\langle\boldsymbol{p}_f\lambda_f|T_B|\boldsymbol{p}_i\lambda_i\rangle^*=\langle-\boldsymbol{p}_f\lambda_f|T_B|-\boldsymbol{p}_i\lambda_i\rangle\;. \tag{12}$$

Finally, using this result in eq. (11) we obtain

$$\langle\boldsymbol{q}_1-,\boldsymbol{q}_2+,\boldsymbol{k}\lambda|T_B|\boldsymbol{p}_1\lambda_1,\boldsymbol{p}_2\lambda_2\rangle=-\langle\boldsymbol{q}_2-,\boldsymbol{q}_1+,\boldsymbol{k}-\lambda|T_B|\boldsymbol{p}_2-\lambda_2,\boldsymbol{p}_1-\lambda_1\rangle^*\;. \tag{13}$$

The restriction to the Born amplitude explains why, in the Z amplitude, one does not take the complex conjugate of C_Z.

Expressions for the cross section, for different beam polarizations, in terms of the helicity amplitudes can be found in ref. [10]. However, as an example, let us write the result for unpolarized beams. The spinor product $s(p,q)$ as a function of the components of the four-vectors p and q is [9]

$$s(p,q)=(p^2+\mathrm{i}p^3)[(q^0-q^1)/(p^0-p^1)]^{1/2}-(q^2+\mathrm{i}q^3)[(p^0-p^1)/(q^0-q^1)]^{1/2} \tag{14}$$

and the unpolarized differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\Omega^*}=\frac{x}{128(2\pi)^4}\,\Sigma_{\mathrm{unp}}(x,y,\theta^*,\varphi^*)\;, \tag{15}$$

where

$$\Sigma_{\mathrm{unp}}=\tfrac{1}{4}\{(N_\nu-1)[\,|M_Z(+,-,+)|^2+|M_Z(-,+,+)|^2+|M_Z(+,-,-)|^2+|M_Z(-,+,-)|^2]$$
$$+|M(+,-,+)|^2+|M(-,+,+)|^2+|M(+,-,-)|^2+|M(-,+,-)|^2\} \tag{16}$$

and Ω^* is the solid angle in the center of mass of the neutrinos. To obtain σ the four-dimensional phase space integration was done by Monte Carlo and Gauss. In table 1 the values obtained with both methods are compared with the ones derived from the analytic expression for $\mathrm{d}\sigma/\mathrm{d}x\,\mathrm{d}y$. We used 16 points Gauss integration for $\mathrm{d}y$, and for $\mathrm{d}x$ we divided the region of integration in three intervals in such a way that the central interval contained the peak of the cross section. Again, in each region 16 points were used. With this approach the results were stable up to five significant digits when we changed from 16 to 32 points. The agreement between

Table 1
Comparison between the total cross section obtained by doing analytically the integration over the neutrino phase space and by using the helicity amplitudes and numerical integration.

s (GeV)	σ (pb) analytical integration	σ (pb) helicity amplitudes + Gauss integration	σ (pb) helicity amplitudes + Monte Carlo integration
50	5.1557×10^{-2}	5.1557×10^{-2}	$(5.158\pm0.004)\times10^{-2}$
100	4.3607	4.3607	4.363 ± 0.003
150	5.3487	5.3487	5.350 ± 0.005
200	2.3520	2.3520	2.352 ± 0.003
750	5.230×10^{-1}	5.23×10^{-1}	$(5.23\pm0.02)\times10^{-1}$

the calculation based on the helicity amplitudes and the previous one [6] is excellent. Since they are quite independent this is a powerful check on both. Comparing with the calculation of Chiappetta et al. [7] the agreement is good with differences of the order of a few percent. Perhaps, these differences can be attributed to round-off errors. Notice that Chiappetta et al. [7] used Monte Carlo for the integration but squared the amplitude and used numerical methods (Reduce) to calculate the traces. With the helicity formalism the amplitudes are summed before being squared. In general this gives a more stable algorithm. After establishing the accuracy of our results, we believe that the advantage of having an analytic expression for $d\sigma/dx\,dy$ will be useful in the forthcoming data analysis. Perhaps, one of these advantages can be felt comparing the amount of computer time needed to calculate each value in table 1. With the analytic expression of ref. [6] every entry in table 1 took roughly 30 s while the programme using the helicity amplitudes used nearly 3.5 h with 16 point Gauss integration and about 7 h with Monte Carlo integration. For the numerical results we took $M_Z=92.0$ GeV, $M_W=80.7$ GeV, $\sin^2\theta_W=1-M_W^2/M_Z^2=0.23$ and $\Gamma_Z=2.78$ GeV or 2.95 GeV for $N_\nu=3$ or 4 respectively. By expressing the final results in terms of G_F and α we took into account a large part of the radiative corrections [12].

Let us now turn to the main question of our study. How good is the local approximation? The answer is given in fig. 1 where we plot

$$\epsilon=(\sigma_{approx}-\sigma_{exact})/\sigma_{exact} \tag{17}$$

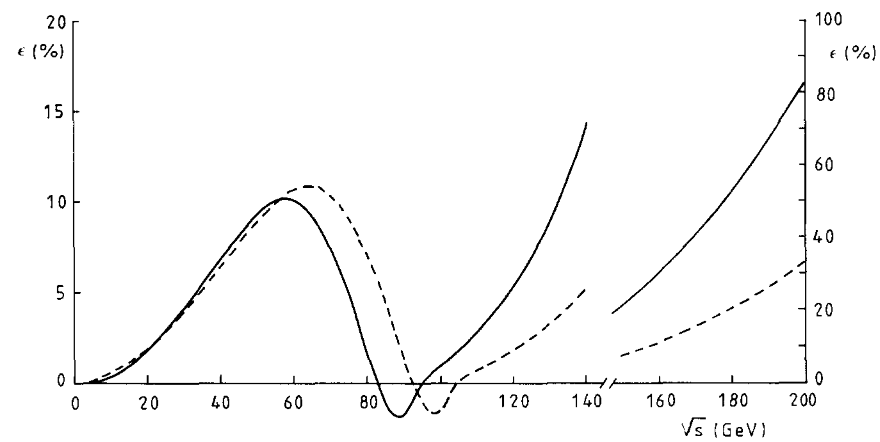

Fig. 1. Relative error of the local approximation. The dashed curve corresponds to $0.2\leqslant x\leqslant 1.0$ and $|y|\leqslant0.94$ while the solid line corresponds to $2/[\sqrt{s}(\text{GeV})\sin\theta]\leqslant x\leqslant 1$ and $5°\leqslant\theta\leqslant75°$.

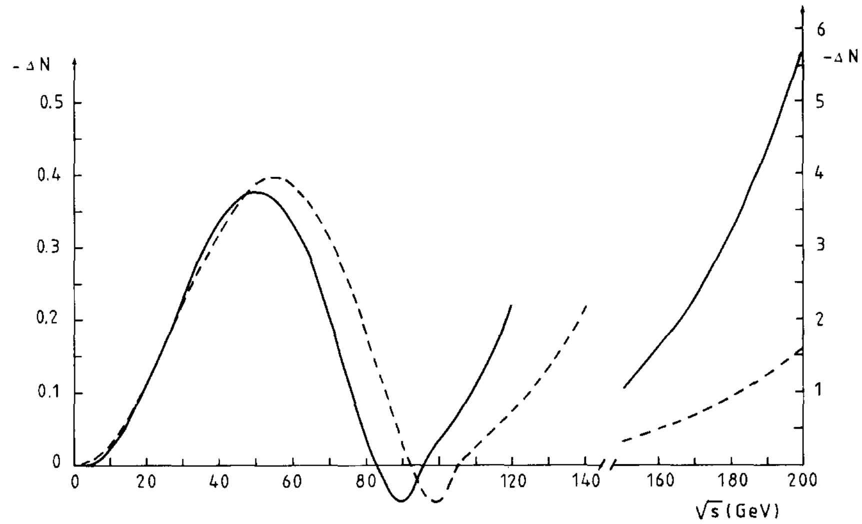

Fig. 2. The quantity ΔN defined in eq. (20) with the same cuts as in fig. 1.

as a function of $\sqrt{s}$, for two different kinematical cuts. The solid line corresponds to the cuts of ref. [7], i.e., $k_{\perp\gamma}>1$ GeV or $2/(\sqrt{s}\sin\theta)\leqslant x\leqslant 1.0$ and $5°\leqslant\theta\leqslant 175°$, while for the dashed line we used $0.2\leqslant x\leqslant 1.0$ and $|y|\leqslant 0.94$. As one would have expected ϵ goes through zero in the vicinity of the Z-boson mass. Obviously, in this small region the local approximation is very good. On the other hand, one should not forget that ϵ is cut dependent. For instance, for the cuts shown in fig. 1, at $\sqrt{s}=80$ GeV we have $\epsilon=2\%$ and 7% respectively, while at $\sqrt{s}=200$ GeV the corresponding figures are 84% and 33%. Clearly, at LEP-II energies no one would have used the local approximation. However, it is not so widely felt that, even below the Z peak ($\sqrt{s}=60$ GeV for instance), the local approximation could induce an error of the order of 10%. This is certainly larger than any radiative correction and, in some cases, even larger than the effect of right-handed neutrinos or extra gauge bosons. Even the more modest claim for the existence of a fourth left-handed neutrino could be misleading if based in the local approximation. To stress this point we plot in fig. 2 the quantity

$$\Delta N=[\sigma_{\text{exact}}(N_\nu=3)-\sigma_{\text{approx}}(N_\nu=3)]/[\sigma_{\text{exact}}(N_\nu=4)-\sigma_{\text{exact}}(N_\nu=3)] \tag{18}$$

as a function of $\sqrt{s}$. The curves correspond to the cuts used in fig. 1. Let us remark that, around 60 GeV, the difference between the local approximation and the exact result can be almost one half of the increase in the cross section due to the existence of a fourth neutrino.

In fig. 3 we show the longitudinal polarization,

$$A_{\parallel}=(\sigma_{\mathrm{R}}-\sigma_{\mathrm{L}})/(\sigma_{\mathrm{R}}+\sigma_{\mathrm{L}})\ , \tag{19}$$

where $\sigma_{\mathrm{R(L)}}$ denotes the cross section for a right (left)-handed polarized electron beam. The full and the dashed curves are for the cuts used before while the dash-dotted curve corresponds to the cuts of ref. [8], i.e., $4/\sqrt{s}\leqslant x<1$ and $3°\leqslant\theta\leqslant 177°$. $A_{\parallel}$ changes sign around the Z peak but since the positive values are at most of the order of a few percent, this is of negligible interest. On the contrary, the large negative values obtained below and above the cross section peak can be exploited to distinguish this reaction from others where neutral supersymmetric particles are produced [6,7]. Within a few percent our results agree with those of ref. [7] but are in disagreement with ref. [8]. In the region 90 GeV$\leqslant\sqrt{s}\leqslant$100 GeV our results are also in agreement, within a few percent, with Caffo et al. [4] who used the infinite W mass approximation. This is to be expected because, as can be seen from fig. 1, in this region the error in the approximation is less than 2%.

Fig. 4 displays the behaviour of the transverse polarization asymmetry $A_{\perp}$, namely,

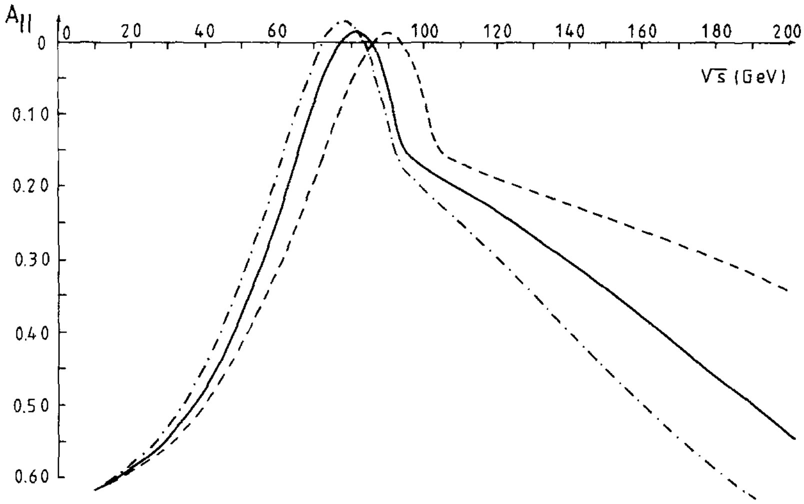

Fig. 3. The longitudinal asymmetry. The dashed and solid curves correspond to the cuts of fig. 1 while the dash-dotted curve corresponds to $4/\sqrt{s}\,(\mathrm{GeV}) \leqslant x \leqslant 1$. and $3° \leqslant \theta \leqslant 177°$.

$$A_\perp = -\frac{\sigma_\perp}{\sigma} = \frac{2\int d\varphi \cos 2\varphi \; d\sigma/d\varphi}{\int d\varphi \; d\sigma/d\varphi}, \tag{20}$$

where the angle φ is the azimuthal angle of the photon using the electron spin direction as the y-axis. A simple calculation gives

$$\sigma_\perp = \int dx \, dy \, d\Omega^* \frac{x}{128(2\pi)^4} \Sigma_\perp , \tag{21}$$

where

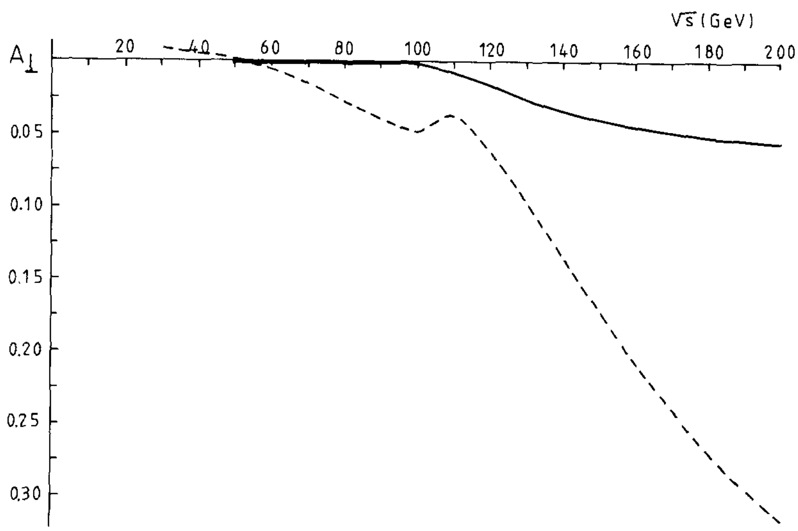

Fig. 4. The transverse asymmetry as defined in eq. (22). The solid line corresponds to $2/[\sqrt{s}\,(\mathrm{GeV}) \sin\theta] \leqslant x \leqslant 1$ and $5° \leqslant \theta \leqslant 175°$ and the dashed line to $0.3 \leqslant x \leqslant 1$ and $|y| \leqslant 0.4$.

$$\Sigma_\perp = \tfrac{1}{2}\,\mathrm{Re}[2M_Z(+,-,+)M_Z^*(-,+,+)+2M_Z(+,-,-)M_Z^*(-,+,-)+M(+,-,+)M^*(-,+,+)$$

$$+M(+,-,-)M^*(-,+,-)]\,. \tag{22}$$

In fig. 4 the solid line corresponds to the cut defined before and the dashed line corresponds to $0.3 \leqslant x \leqslant 1.0$ and $|y| \leqslant 0.4$. In agreement with ref. [7], $A_\perp$ is extremely small. To obtain large asymmetries $A_\perp$ one needs cuts that exclude most of the forward angles but then the cross section is also small. This is so because $d\sigma_\perp$ is proportional to $\sin^2\theta$ and cancels the $\sin^{-2}\theta$ behaviour of the total cross section responsible for the large values of σ. For instance, at $\sqrt{s}=100$ GeV, for the first cut we have $\sigma = 107$ pb while for the other one the figure is 0.12 pb. With such a small σ the transverse asymmetry will be unmeasurable in the near future.

Appendix

$$\hat{I}_{ij}[u\cdot q\, v\cdot q] = \frac{3(u,p_i)(v,p_i) - B_i^2(u,v)}{2B_i^4}\,\hat{I}_{ij}[p_i\cdot q\, p_i\cdot q] + \frac{B_i^2(u,v)-(u,p_i)(v,p_i)}{2B_i^2}\, I_{ij}\,, \tag{A1}$$

with

$$B_i = (p_i, p_i)^{1/2}\,,\quad \delta = M_W^2 - m_e^2\,,\quad \epsilon_m = (-1)^{m+1}\,,\quad q = q_1 - q_2\,,\quad A_i = \delta + \Delta\cdot p_i\,, \tag{A2}$$

u and v are any four-vector but q_i. For convenience we define the "scalar product" $(a,b) = \Delta\cdot a\,\Delta\cdot b - \Delta^2 a\cdot b$.

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