GAUGE SYMMETRY BREAKING IN COMPACT MULTIPLY CONNECTED MANIFOLDS

F. FREIRE [a], J.C. ROMÃO [a] and A. BARROSO [b]

[a] *CFMC, Instituto Nacional de Investigação Cientifica, Av. Gama Pinto 2, 1699 Lisbon Codex, Portugal*

[b] *CFN, Universidade de Lisboa, Av. Gama Pinto 2, 1699 Lisbon Codex, Portugal*

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In a multiply connected manifold, $M_4 \otimes S_3/Z^2$, we compute at one-loop level the gauge symmetry breaking due to Wilson loops. For an SU(3) model without matter fields a non-trivial vacuum, which breaks the gauge symmetry has lower energy.

The study of spontaneous symmetry breaking of non-abelian gauge theories has been a topical subject since the early days of the discovery of the Higgs mechanism [1]. In a field theory, formulated in the usual four-dimensional spacetime, M_4, it is quite clear that only scalar fields can acquire a vacuum expectation value (VEV). However, in Kaluza–Klein theories, formulated in $M_4 \otimes J$, where J is a compact multiply connected manifold, there is an alternative gauge breaking mechanism [2] due to Wilson loops. To some extent this mimics the usual Higgs breaking but, the VEV is acquired by some components of the gauge field which correspond to space dimensions that are compactified.

Recently, this mechanism has been generally assumed to be at work in the framework of superstring theories (see ref. [3] for a review). Consistent and anomaly free theories of this kind require gauge groups with enormous dimensions (for instance $E_8 \otimes E_8$ has 496 dimensions) formulated in a ten-dimensional spacetime. Hence, to obtain any phenomenological interesting model one needs several symmetry breakings and, in the initial stage of this breaking chain only the Wilson-loop mechanism is available.

Despite the fact that these ideas have been widely advocated, it is fair to point out that there are not many cases where explicit examples have been studied. As far as we know, only one simple model was studied by Evans and Ovrut [4] but their results were contradicted by Shiraishi [5]. In this letter we re-examine the same example studied by these two groups and solve the problem of understanding their conflicting claims. As we will see, neither of them is entirely correct.

Let K be a simple connected manifold and G a discrete symmetry group which acts freely on K, i.e., $g\cdot y \doteq y$ implies $g=1$ for any $g \in G$ and y any point on K. Then K/G is a multiply connected manifold. On K we define a gauge field $A^a_\mu(y)$ where a is the gauge group index referring to the adjoint representation of the gauge group E and μ is the world index on K. If D_μ denotes the covariant derivative on K and C^{abc} are the structure constants of E, the field tensor $F^a_{\mu\nu}$ is

$$F^a_{\mu\nu}(y) = D_\mu A^a_\nu(y) - D_\nu A^a_\mu(y) + C^{abc} A^b_\mu(y) A^c_\nu(y) . \tag{1}$$

Let B^a_μ be a field configuration such that $F^a_{\mu\nu}|_B = 0$. This is called a vacuum gauge field. The Wilson loop U is defined by the path-ordered (P) integral

$$U = \mathrm{P}\exp\left(-\mathrm{i}\int_\gamma T^a B^a_\mu(y)\,\mathrm{d}y^\mu\right), \tag{2}$$

where γ is a closed line on K and T^a are the generators of E. Since U is invariant for continuous deformations of γ, we have $U = \mathbb{1}$. However, this is not so for the multiply connected manifold K/G. On K/G the gauge fields are such that

$$A^a_\mu(g\cdot y) = U^{ab}_g A^b_\mu(y) , \quad g \in \mathrm{G},\ y \in \mathrm{K/G} , \tag{3}$$

where U^{ab}_g is an element of the adjoint representation

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of E. The generators of the unbroken gauge group are the ones which are invariant under U_g for any $g \in \mathrm{G}$, i.e.,

$$U_g T U_g^{-1} = T . \tag{4}$$

This is a necessary condition for symmetry breaking by the Wilson-loop mechanism but, as we will show, it is not a sufficient condition to determine completely the symmetry breaking pattern.

After these preliminary definitions we can examine our toy model. The spacetime structure is $\mathrm{M}_4 \otimes \mathrm{S}_3/\mathrm{Z}^2$, where S_3 is the three-sphere and $\mathrm{Z}^2 = \{-1,1\}$. We parametrize S_3 in terms of the angles (θ, ψ, ϕ) and, embedding S_3 in $\mathbb{R}^4$, their definitions are

$$x + \mathrm{i}y = r \cos\theta \exp[\tfrac{1}{2}\mathrm{i}(\psi+\phi)] ,$$

$$z + \mathrm{i}t = r \sin\theta \exp[\tfrac{1}{2}\mathrm{i}(\psi-\phi)] , \tag{5}$$

with $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \psi \leqslant 4\pi$ and $0 \leqslant \phi \leqslant 2\pi$. The gauge group E is SU_3 and the quantum lagrangian of the system involves a gauge fixing term, $\mathscr{L}_{\mathrm{gf}}$, and a Faddeev–Popov term, $\mathscr{L}_{\mathrm{FP}}$, besides the classical Yang–Mills term $\mathscr{L}_{\mathrm{YM}}$. For convenience we split the world index μ into m and M, where $m = 0, 1, 2, 3$ refers to the Minkowski space.

Clearly, $B^a_\mu = 0$ is a trivial vacuum. Other non-trivial possibilities are of the form

$$B^a_\mu(y) = (B^a_m(y) = 0, B^a_M(y) \neq 0) . \tag{6}$$

In any case, because $F^a_{\mu\nu}|_B = 0$, at tree-level any vacuum configuration has zero energy. One-loop corrections break this degeneracy and the relevant question is whether or not a non-trivial vacuum, which breaks the gauge symmetry, has a lower energy than the trivial one. To compute the energy E^i corresponding to different background fields B^i ($i = 1, 2, \ldots$) we compute the partition function $Z[B^i]$, i.e.,

$$\exp(-E^i) = Z^*[B^i] , \tag{7}$$

where Z^* is the restriction of Z to the quadratic terms in the lagrangian. If we write

$$A^a_\mu(y) = B^a_\mu(y) + Q^a_\mu(y) \tag{8}$$

we obtain, for the quadratic terms in $\mathscr{L}$, the expression

$$\begin{aligned}\mathscr{L}^{(2)} = {} & \tfrac{1}{4} e^{-2} F^a_{\mu\nu}(x)|_A F^{\mu\nu a}(x)|_A \\ & - \tfrac{1}{2}\alpha^{-1} e^{-2}\{[\partial_m A^{ma}(x)]^2 + [D_M{}^{ab}|_B Q^{Mb}(x)]^2\} \\ & - \bar{C}^a(x)\, \partial_m \partial^m C^a(x) \\ & - \bar{C}^a(x)\, D_M{}^{ab}|_B D^{Mbc}|_B C^c(x) \\ & - \Phi^{i\dagger}(x)\, D_m{}^{ik} D^{mkj} \Phi^j(x) \\ & - \Phi^{i\dagger}(x)\, D_M{}^{ik}|_B D^{Mkj}|_B \Phi^j(x) \\ & + \mathrm{i}\bar{\Psi}^A(x)\, \gamma^m D_m{}^{AB} \Psi^B(x) \\ & + \mathrm{i}\bar{\Psi}^A(x)\, \gamma^M D_M{}^{AB}|_B \Psi^B(x) ,\end{aligned} \tag{9}$$

where

$$D_M{}^{ab}|_B = \mathrm{D}_M \delta^{ab} - C^{abc} B^c_M , \tag{10}$$

C^a are the Faddeev–Popov ghosts and, for completeness, we have included also scalar (Φ) and spinor (Ψ) matter fields.

If in eq. (2), instead of a closed path, we stop the integration at a point x, U will be a function of x, i.e.,

$$U_B = \mathrm{P} \exp\left(-\mathrm{i}\int^x T^c B^c_M(y)\,\mathrm{d}y^M\right) . \tag{11}$$

Taking a covariant derivative we obtain

$$\mathrm{D}_\mu U_B{}^{\alpha\beta}(x) = -\mathrm{i} U_B{}^{\alpha\gamma}(x) (T^a)^{\gamma\beta} B^a_\mu(x) , \tag{12}$$

where the greek indices in superscript denote a representation of the gauge group, either the adjoint representation or one of the representations of the matter fields. For instance, if $\Phi^\alpha(x)$ is one of these fields it can easily be shown that

$$D_M{}^{\alpha\beta}|_B \Phi^\beta(x) = U_B^{-1\alpha\gamma} \mathrm{D}_M [U_B{}^{\gamma\beta}(x) \Phi^\beta(x)] . \tag{13}$$

Then, defining

$$\Phi_B{}^\alpha(x) = U_B{}^{\alpha\beta}(x) \Phi^\beta(x) \tag{14}$$

and integrating by parts, $\mathscr{L}^{(2)}$ can be cast in the form

$$\begin{aligned}\mathscr{L}^{(2)} = {} & \tfrac{1}{2} e^{-2}\{Q_{B\mu}{}^a(x)[-\eta^{\mu\nu}\Box + R^{\mu\nu} \\ & + (1 - 1/\alpha)\mathrm{D}^\mu \mathrm{D}^\nu] Q_{B\nu}{}^a(x)\} \\ & + \bar{C}_B{}^a(x)(-\Box) C_B{}^a(x) + \Phi_B{}^{i\dagger}(x)(-\Box)\Phi_B{}^i(x) \\ & + \bar{\Psi}_B{}^A(x)(\mathrm{i}\gamma^\mu \partial_\mu + \mathrm{i}\gamma^M \mathrm{D}_M)\Psi_B{}^A(x) ,\end{aligned} \tag{15}$$

where $R^{\mu\nu}$ is the Ricci tensor on $\mathrm{M}_4 \otimes \mathrm{K}$. Hence,

$$Z^*[B] = Z^*_{\mathrm{YM}} Z^*_{\mathrm{G}} Z^*_{\mathrm{S}} Z^*_{\mathrm{F}} \tag{16}$$

with

$$Z^{*}_{\mathrm{YM}} \propto \prod_{\substack{\text{adjoint}\\ \text{rep}}} [\det(-g^{\mu\nu}\square_{\mathrm{V}} + R^{\mu\nu})]^{-1/2} , \tag{17a}$$

$$Z^{*}_{\mathrm{G}} \propto \prod_{\substack{\text{adjoint}\\ \text{rep}}} \det(-\square_{\mathrm{S}}) , \tag{17b}$$

$$Z^{*}_{\mathrm{S}} \propto \prod_{\substack{\text{scalar}\\ \text{rep}}} [\det(-\square_{\mathrm{S}})]^{-1} , \tag{17c}$$

$$Z^{*}_{\mathrm{F}} \propto \prod_{\substack{\text{fermion}\\ \text{rep}}} [\det(-\square_{\mathrm{F}} + R\tfrac{1}{4})]^{1/2} . \tag{17d}$$

Now, the problem is reduced to the search of the eigenvalues, of the operators that appear in eqs. (17). This is a well-known problem in differential geometry which we solved following a procedure used by Pilch and Schellekens [6]. For a unit n-sphere, S_n, the eigenvalues of the rank-m tensor harmonics, $E^{(l)}(n,m)$, are

$$E^{(l)}(n,m) = -[l(l+n-1)-m] , \tag{18}$$

and the degeneracies corresponding to the $m=0$ and $m=1$ cases are

$$D^{(l)}(n,0) = \frac{(l+n-2)!}{l!(n-1)!}(2l+n-1) , \tag{19a}$$

and

$$D^{(l)}(n,1) = \frac{(l+n-3)!}{(l+1)!(n-2)!} l(l+n-1)(2l+n-1), \tag{19b}$$

respectively. The details of this calculation can be found elsewhere [7].

Using the S_3 coordinate system introduced earlier, let us consider the following vacuum gauge field configuration:

$$B^{a(1)}_{M}(x) = (0,0,0) , \tag{20a}$$

$$B^{a(2)}_{M}(x) = (0,0,2\sqrt{3}\,\delta^{a8}) \tag{20b}$$

and

$$B^{a(3)}_{M}(x) = (0,0,2\delta^{a3}) . \tag{20c}$$

Calculating the Wilson loops in S_3/Z^2 we obtain

$$U^{(1)}_{B} = \mathbb{1} \tag{21}$$

for all SU_3 representations, and

$$U^{(2)}_{B} = \mathbb{1}_3 \oplus \mathbb{1}_1 \oplus (-\mathbb{1}_2) \oplus (-\mathbb{1}_2) \tag{22a}$$

or

$$U^{(2)}_{B} = (-\mathbb{1}_2) \oplus \mathbb{1}_1 \tag{22b}$$

for the adjoint and fundamental representations, respectively. For the third background we obtain

$$U^{(3)}_{B} = U^{(2)}_{B} . \tag{23}$$

In the first case, B^1, there is no symmetry breaking but in the other two cases the gauge symmetry is broken. However the breaking pattern is different. For B^2 the unbroken subgroup is $SU_2 \otimes U_1$ while for B^3 it is $U_1 \otimes U_1$. This conclusion can be easily obtained examining the generators of SU_3 which, in each case, commute with the exponent of the Wilson loop U. It is also interesting and straightforward to look at the quartic term in $\mathscr{L}_{\mathrm{YM}}$ and check that when the symmetry is broken the gauge bosons associated with the broken generators acquire a mass. For instance, for the vacuum B^2 we obtain

$$\mathscr{L}_{\mathrm{mass}} = \frac{9}{2e^2r^2}\sum_{a=4}^{7} [2Q_m{}^a(x)Q^{ma}(x) + Q_M{}^a(x)Q^{Ma}(x)] + \frac{9}{r^2}\sum_{a=4}^{7}\bar{C}^a(x)C^a(x) , \tag{24}$$

where e is the coupling constant and r the radius of S_3.

Let us proceed with the evaluation of the difference in energy, ΔE between the first and second backgrounds, i.e.,

$$\Delta E = E^1 - E^2 . \tag{25}$$

Recalling eqs. (18), (19a), (21) and (22a) and noting that on M^4 the operator $\square$ has a continuous spectrum, it is easy to obtain the ghost contribution to ΔE, namely

$$\Delta E_{\mathrm{G}} = 4\mathrm{i}\mu^{-(n-4)}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\sum_{l=0}^{\infty}(-1)^l(2l+N-1) \times \frac{\Gamma(l+N-1)}{\Gamma(N)\cdot l!}\ln[k^2 + l(l+N-1)/r^2] , \tag{26}$$

where μ is a mass parameter used in dimensional regularization. Notice that the sum over the adjoint representation in eq. (17b) gives, in view of eq. (21) and eq. (22a) four times the difference between the even and odd harmonics in S_3/Z^2. This justifies the factor $4(-1)^l$ in eq. (26). In this equation the dimensions of M^4 and S_3 are assumed to be n and N respectively and both are considered as continuous parameters living in the complex plane. The integration in k can easily be done and we get

$$\Delta E_{\mathrm{G}}=\frac{8\mu^{-(n-4)}\Gamma(-n/2)}{(4\pi)^{n/2}r^{n}}\sum_{l=0}^{\infty}(-1)^{l}\frac{\Gamma(l+N-1)}{\Gamma(N)\cdot l!}\times[l+\tfrac{1}{2}(N-1)][l(l+N-1)]^{n/2}. \tag{27}$$

In this case eq. (26) is convergent for $\mathrm{Re}(n+N)<0$. Then, making an analytic continuation to the physical values $n=4$ and $N=3$ the result can be expressed in terms of ζ-functions. This regularization method was previously used by Candelas and Weinberg [8] and further details can be found in ref. [7]. The result is

$$\Delta E_{\mathrm{G}}=\frac{4}{r^{4}}\left(-\frac{7\zeta(3)}{(2\pi)^{4}}+\frac{1209\zeta(5)}{(2\pi)^{6}}-\frac{11430\zeta(7)}{(2\pi)^{8}}\right)=4.0925\times10^{-2}/r^{4}, \tag{28}$$

which means that the ghost contribution favours the symmetry breaking. For the Yang–Mills fields we obtained after the k integration

$$\Delta E_{\mathrm{YM}}=-\frac{4\mu^{-(n-4)}\Gamma(-n/2)}{(4\pi)^{n/2}r^{n}}\times\sum_{l=1}^{\infty}(-1)^{l}\frac{\Gamma(l+N-2)}{\Gamma(N-1)(l+1)!}l(l+N-1)\times[l+\tfrac{1}{2}(N-1)][l(l+N-1)+1]^{n/2} \tag{29}$$

and, using the same method for the analytic continuation, the result is

$$\Delta E_{\mathrm{YM}}=-\frac{8}{r^{4}}\left(\frac{46.5\zeta(5)}{(2\pi)^{6}}+\frac{5715\zeta(7)}{(2\pi)^{8}}\right)=-2.5248\times10^{-2}/r^{4}, \tag{30}$$

which favours the trivial vacuum. However the total result $\Delta E_{\mathrm{YM}}+\Delta E_{\mathrm{G}}$ is positive which implies that, for a theory without matter fields, the vacuum with SU_3 spontaneously broken to $SU_2\otimes U_1$ is the true minimum. The scalar fields contribution to ΔE depend on the group representation but, in all cases except for the singlet, it favours the unbroken vacuum. This can be easily seen comparing eqs. (17b) and (17c). On the other hand, the spinor fields give a null contribution to ΔE because in S_3 fermions cannot be made invariant under the discrete group Z^2.

Now we can compare our results with the previous ones. The calculation of Shiraishi agrees [5] with our result for the gauge fields but he did not consider the Faddeev–Popov term. Hence it is obvious why he reached a different conclusion. However, we should point out that both calculations were done in the covariant 't Hooft–Feynman gauge. So, it is clear that the Faddeev–Popov ghost contribution should be added. This is also done in the work of Evans and Ovrut [4]. But, these authors simply considered the eigenvalues of the laplacian on S_3/Z^2 rather than on $M_4\otimes S_3/Z^2$. In this case, even the gauge fields give a positive ΔE. Following again a ζ-function regularization scheme, rather than the Pauli–Villars methods used by Evans and Ovrut [4], we obtain

$$\Delta E_{\mathrm{YM}}=32[\tfrac{1}{2}\ln 2\,\zeta(0)+\tfrac{7}{4}\zeta'(-2)-\tfrac{1}{4}\zeta'(0)]=0.1012192. \tag{31}$$

Thus, we confirm their results for this restrictive case.

In conclusion, we have shown by explicit computation that in a particular example the Wilson-loop mechanism for symmetry breaking works. It is easy to generalize this analysis for a general sphere S_n and for other gauge groups. However, it seems at this moment out of the question to carry the computation onto Calabi–Yau manifolds. It would be interesting to consider the case of orbifolds which are simpler than Calabi–Yau manifolds and where, perhaps, the evaluation of the harmonics could be done. If this turns out to be the case, then a more realistic model could be analyzed.

References

[1] P.W. Higgs, Phys. Rev. Lett. 12 (1964) 132.
[2] Y. Hosotani, Phys. Lett. B 129 (1983) 193.
[3] E. Witten, Phys. Lett. B 149 (1984) 351; J.D. Breit, B.A. Ovrut and G. Segrè, Phys. Lett. B 158 (1985) 33.
[4] M. Evans and B.A. Ovrut, Phys. Lett. B 174 (1986) 63.
[5] K. Shiraishi, Prog. Theor. Phys. 78 (1987) 535.
[6] K. Pilch and A.N. Schellekens, J. Math. Phys. 25 (1984) 3455.
[7] F. Freire, M.Sc. Thesis, Universidade de Lisboa (1988).
[8] P. Candelas and S. Weinberg, Nucl. Phys. B 237 (1984) 397.