ORDER α'^2 EQUIVALENCE OF THE STRING EQUATIONS OF MOTION AND THE σ-MODEL WEYL INVARIANCE CONDITIONS. DEPENDENCE ON THE DILATON FIELD

M.C. BENTO [a,1], O. BERTOLAMI [a,b,2], A.B. HENRIQUES [a] and J.C. ROMÃO [a]

[a] *Centro de Física de Matéria Condensada, INIC, Av. Gama Pinto 2, P-1600 Lisbon Codex, Portugal*

[b] *Institut für Theoretische Physik der Universität Heidelberg, Philosophenweg 16, D-6900 Heidelberg, Fed. Rep. Germany*

Received 3 January 1989

We determine the order α'^2 terms in the closed string effective actions, when the dilaton field is included, by comparison with the four-point string amplitudes, both in the S-matrix parametrization and in the σ-model parametrization. We find that for the bosonic and heterotic string theories there are terms involving derivatives of the dilaton field which cannot be removed by field redefinitions and whose coefficients are non-zero. Our result is not in agreement with existing σ-model computations.

1. Considerable effort has been devoted to finding the effective action which describes the low-energy dynamics of the massless string modes [1,2]. The effective action is, at tree level, a perturbative expansion in powers of the string tension α' and, for the closed string theories, it is a functional of the gravitational, antisymmetric tensor and dilaton fields.

The structure of these actions to order α' is well known at present [3]. At order α'^2, there are recent results concerning the curvature cubed terms [4]; it is found that while these terms are absent in the case of the heterotic and superstring theories, they are indeed present in the bosonic string theory.

In this letter, we derive the dependence of string effective actions on the dilaton field at order α'^2, using the S-matrix approach, i.e., we construct an effective action which reproduces the (four-point) string scattering amplitudes at this order. We find that, while for the superstring there are no order α'^2 corrections at all, in the case of the bosonic and heterotic string theories there are six terms which cannot be removed by field redefinitions [2] and whose coefficients are non-zero, four of which involve derivatives of the dilaton field.

Another method to derive the effective action, the σ-model approach, has received much attention recently. It is believed that the equations of motion derived from string theory effective actions are equivalent to the conditions for conformal invariance of two-dimensional non-linear σ-models. Different arguments have been presented to support the general validity of this conjecture [5] and several low order explicit verifications have been performed [3,6,7]. At order α'^2, the equivalence has been checked in what concerns the curvature cubed terms [7]. Regarding the dependence of the action on the dilaton at this order, the authors of ref. [8] derived, in the torsion-free case, the $O(\alpha'^2)$ dilaton β-function from the $O(\alpha'^2)$ metric β-function using the Curci and Paffuti identity to conclude that the effective action can be expressed in such a way that there are no terms involving derivatives of the dilaton, a result which is in disagreement with our calculation.

2. The low-energy expansion of the gravitational sector of closed string effective actions has the following structure, up to order α'^2, when the dilaton is included [1–4]:

[1] Address after 1st October: Institut für Theoretische Physik der Universität Heidelberg, 16 Philosophenweg, D-6900 Heidelberg, Fed. Rep. Germany.

[2] Wishes to thank DFG for support.

$$S=-\frac{2}{\kappa^2}\int d^Dx\sqrt{g}\{R+\gamma(\partial\varphi)^2+\alpha'\exp(\gamma\varphi)\lambda_0\{G_2+\gamma^2[(D-4)/(D-2)](\partial\varphi)^4\}$$
$$+\alpha'^2\exp(2\gamma\varphi)[\lambda_1 I_1+\lambda_2 G_3+\lambda_3(D_\mu\partial_\nu\varphi)^2(\partial\varphi)^2+\lambda_4 D_\mu\partial_\nu\varphi D^\rho\partial^\mu\varphi D_\rho\partial^\nu\varphi$$
$$+\lambda_5 R_{\alpha\beta\mu\nu}D^\alpha\partial^\mu\varphi\partial^\beta\varphi\partial^{\nu b}\varphi+\lambda_6 R^2_{\mu\nu\alpha\beta}(\partial\varphi)^2+\lambda_7 R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}\partial^\mu\varphi\partial^\nu\varphi]+O(\alpha'^3)\}\,,$$
$$\gamma=-4/(D-2)\,, \tag{1}$$

where $\kappa^2\equiv 32\pi G$, G_2 is the curvature-squared Gauss–Bonnet invariant $(G_2=\Omega_2)$ and G_3 is related to the curvature-cubed Gauss–Bonnet invariant Ω_3:

$$G_2=R^2_{\mu\nu\alpha\beta}-4R^2_{\mu\nu}+R^2\,,\quad \Omega_3=G_3+\text{Ricci terms}\,,\quad G_3=I_1-2I_2\,,$$
$$I_1=R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\rho}R^{\lambda\rho}{}_{\mu\nu}\,,\quad I_2=R^\mu{}_{\nu\alpha\beta}R^{\nu\lambda\beta\gamma}R_{\lambda\mu\gamma}{}^\alpha\,. \tag{2}$$

The order α' part of the action (1) has been found in ref. [3], with $\lambda_0^{(B)}=\frac{1}{4}$, $\lambda_0^{(H)}=\frac{1}{8}$, $\lambda_0^{(S)}=0$ for the bosonic, heterotic and type II superstring theories, respectively. The order α'^2 part of the action has been put in its simplest form, with the help of local field redefinitions [2,4]. In (1) and throughout this letter, we use euclidean signature for the metric and the conventions

$$R^\lambda{}_{\mu\nu\rho}=\partial_\nu\Gamma^\lambda_{\mu\rho}-\ldots\,,\quad R_{\mu\nu}=R^\lambda{}_{\mu\lambda\nu}\,,\quad D_\mu A^\lambda=\partial_\mu A^\lambda+\Gamma^\lambda_{\mu\nu}A^\nu\,. \tag{3}$$

The coefficients $\lambda_1,\ldots,\lambda_7$ can be found by comparing the α'^2 four-point string theory and field theory amplitudes. The four-point amplitudes for the scattering of massless bosonic string states are given by [9]

$$\mathscr{T}_{\rm LM}=-\tfrac{1}{256}\kappa^2\alpha'^3\Gamma(s,t,u)e_1{}^{\mu\nu}e_2{}^{\alpha\beta}e_3{}^{\sigma\lambda}e_4{}^{\delta\gamma}K^{(\rm L)}_{\mu\alpha\sigma\delta}K^{(\rm M)}_{\nu\beta\lambda\gamma}\,, \tag{4}$$

with

$$\Gamma(s,t,u)=\frac{\Gamma(-\frac{1}{4}\alpha's)\Gamma(-\frac{1}{4}\alpha't)\Gamma(-\frac{1}{4}\alpha'u)}{\Gamma(1+\frac{1}{4}\alpha's)\Gamma(1+\frac{1}{4}\alpha't)\Gamma(1+\frac{1}{4}\alpha'u)}=-\frac{64}{\alpha'^3stu}-2\zeta(3)+\ldots\,,$$
$$s=-2k_1\cdot k_2\,,\quad t=-2k_1\cdot k_3\,,\quad u=-2k_1\cdot k_4\,,\quad s+t+u=0\,, \tag{5}$$

where (L, M) = (B, B), (S, B), (S, S) and $K^{(\rm B)}$ ($K^{(\rm S)}$) is the kinematic factor for the corresponding bosonic (supersymmetric) open string four-point amplitude:

$$K^{(\rm B)}_{\mu\alpha\sigma\delta}e_1{}^\mu e_2{}^\alpha e_3{}^\sigma e_4{}^\delta=\Big(\frac{st}{1+\frac{1}{4}\alpha'u}(\xi_{13}-\tfrac{1}{2}\alpha'k_{13}k_{31})(\xi_{24}-\tfrac{1}{2}\alpha'k_{24}k_{42})$$
$$-2s(k_{14}k_{32}\xi_{24}+k_{23}k_{41}\xi_{13}+k_{13}k_{42}\xi_{23}+k_{24}k_{31}\xi_{14})$$
$$+\alpha's[k_{13}k_{23}(k_{31}k_{41}+k_{32}k_{42})+\tfrac{1}{3}(k_{12}k_{23}k_{31}-k_{13}k_{21}k_{32})(k_{41}-k_{42})]\Big)+\text{two permutations}\,,$$

$$K^{(\rm S)}_{\mu\alpha\sigma\delta}e_1{}^\mu e_2{}^\alpha e_3{}^\sigma e_4{}^\delta=[st\xi_{13}\xi_{24}-2s(k_{14}k_{32}\xi_{24}+k_{23}k_{41}\xi_{13}+k_{13}k_{42}\xi_{23}+k_{24}k_{31}\xi_{14})]+\text{two permutations}\,, \tag{6}$$

where $\xi_{mn}=\xi_n{}^\mu\xi_{m\mu}$, $k_{nm}=\xi_n{}^\mu k_{m\mu}$ and $e_{\mu\nu}$ is the transverse polarization tensor, $e_n{}^{\mu\nu}=\xi_n{}^\mu\bar{\xi}_n{}^\nu$; the latter can be decomposed into the graviton and dilaton parts [10]:

$$e_{\mu\nu}(k)=h_{\mu\nu}(k)+[\delta^\perp_{\mu\nu}/(D-2)^{1/2}]\phi(k)\,,\quad \delta^\perp_{\mu\nu}=\delta_{\mu\nu}-\bar{k}_\mu k_\nu-k_\mu\bar{k}_\nu\,, \tag{7}$$

where $\bar{k}\cdot k=1$, $\bar{k}^2=0$ and $\delta^\perp_{\mu\nu}\delta^{\mu\nu}=D-2$.

Coefficients λ_1 and λ_2 have already been found in ref. [4] by matching the four-graviton amplitudes at order α'^2, with the result

$$\lambda_1^{(\rm S)}=\lambda_1^{(\rm H)}=0\,,\quad \lambda_1^{(\rm B)}=\tfrac{1}{48}\,,\quad \lambda_2^{(\rm S)}=\lambda_2^{(\rm H)}=0\,,\quad \lambda_2^{(\rm B)}=\tfrac{1}{24}\,. \tag{8}$$

As for the remaining coefficients $\lambda_3, ..., \lambda_7$, we can extract from eqs. (4)–(7) the order α'^2 four-point amplitudes which are relevant for their calculation. In the case of the superstring, it is trivial to see that there is no order α' and α'^2 contributions to these amplitudes, implying $\lambda_i^{(S)}=0$ ($i=1, ..., 7$). Regarding the bosonic and heterotic strings, we have, starting with the two-graviton–two-dilaton amplitudes,

$$\begin{aligned}\mathcal{T}^{(B)}_{\text{string}}[h_1 h_2 \phi_3 \phi_4] = {} & \frac{\alpha'^2}{(D-2)}\{\tfrac{1}{64}(3D-10)(h_1 h_2)stu + \tfrac{1}{8}(D-4)(k_2 h_1 h_2 k_1)tu \\ & + \tfrac{1}{16}(D-2)(k_2 h_1 h_2 k_3)su + \tfrac{1}{16}(D-2)(k_3 h_1 h_2 k_1)st - \tfrac{1}{16}(D-2)(k_3 h_1 h_2 k_3)s^2 \\ & + (k_2 h_1 k_2)[\tfrac{1}{16}(D-2)(k_1 h_2 k_1)ut/s + \tfrac{1}{8}(D-6)(k_1 h_2 k_3)u + \tfrac{1}{4}(k_3 h_2 k_3)su/t] \\ & + (k_2 h_1 k_3)[\tfrac{1}{8}(D-6)(k_1 h_2 k_1)t - \tfrac{1}{8}(D-10)(k_1 h_2 k_3)s - \tfrac{1}{2}(k_3 h_2 k_3)s^2/t] \\ & + (k_3 h_1 k_3)[\tfrac{1}{4}(k_1 h_2 k_1)st/u - \tfrac{1}{2}(k_1 h_2 k_3)s^2/u - \tfrac{1}{4}(k_2 h_2 k_3)(s^2/u + s^2/t)]\}\phi_3\phi_4 ,\end{aligned} \tag{9a}$$

$$\begin{aligned}\mathcal{T}^{(H)}_{\text{string}}[h_1 h_2 \phi_3 \phi_4] = {} & \frac{\alpha'^2}{(D-2)}\{\tfrac{1}{64}(D-7)(h_1 h_2)stu + \tfrac{1}{32}(D-8)(k_2 h_1 h_2 k_1)tu \\ & + \tfrac{1}{32}(D-6)(k_2 h_1 h_2 k_3)su + \tfrac{1}{32}(D-6)(k_3 h_1 h_2 k_1)st - \tfrac{1}{32}(D-6)(k_3 h_1 h_2 k_3)s^2 \\ & + (k_2 h_1 k_2)[\tfrac{1}{16}(D-8)(k_1 h_2 k_3)u + \tfrac{1}{16}(k_3 h_2 k_3)su/t] \\ & + (k_2 h_1 k_3)[\tfrac{1}{16}(D-8)(k_1 h_2 k_1)t - \tfrac{1}{16}(D-10)(k_1 h_2 k_3)s - \tfrac{1}{8}(k_3 h_2 k_3)s^2/t] \\ & + (k_3 h_1 k_3)[\tfrac{1}{16}(k_1 h_2 k_1)st/u - \tfrac{1}{8}(k_1 h_2 k_3)s^2/u - \tfrac{1}{16}(k_3 h_2 k_3)(s^2/u + s^2/t)]\}\phi_3\phi_4 .\end{aligned} \tag{9b}$$

The one-graviton–three-dilaton amplitudes are given by

$$\mathcal{T}^{(B)}_{\text{string}}[h_1 \phi_2 \phi_3 \phi_4] = -\frac{3D-10}{16(D-2)^{3/2}}\alpha'^2[(k_2 h_1 k_2)u^2 - 2(k_2 h_1 k_3)us + (k_3 h_1 k_3)s^2]\phi_2\phi_3\phi_4 , \tag{10a}$$

$$\mathcal{T}^{(H)}_{\text{string}}[h_1 \phi_2 \phi_3 \phi_4] = -\frac{3}{32}\frac{D-4}{(D-2)^{3/2}}\alpha'^2[(k_2 h_1 k_2)u^2 + (k_3 h_1 k_3)s^2 - 2us(k_2 h_1 k_3)]\phi_2\phi_3\phi_4 , \tag{10b}$$

and the four-dilaton amplitudes read:

$$\mathcal{T}^{(B)}_{\text{string}}[\phi_1 \phi_2 \phi_3 \phi_4] = \frac{(3D-10)^2}{64(D-2)^2}\alpha'^2 stu\phi_1\phi_2\phi_3\phi_4 , \tag{11a}$$

$$\mathcal{T}^{(H)}_{\text{string}}[\phi_1 \phi_2 \phi_3 \phi_4] = \frac{3}{64}\frac{(D-3)(D-6)}{(D-2)^2}\alpha'^2 stu\phi_1\phi_2\phi_3\phi_4 , \tag{11b}$$

where we have used momentum conservation, $k_1+k_2+k_3+k_4=0$, to eliminate k_4 from eqs. (9)–(11).

The evaluation of these string amplitudes is far from trivial, especially in the case of the bosonic string where the kinematical factor $K^{(B)}_{\mu\alpha\sigma\delta}$ has many terms. Also the substitution of the polarization vector $e_{\mu\nu}(k)$ by its dilaton part $\delta^{\perp}_{\mu\nu}$ is very complicated. In doing these calculations we have used the algebraic program REDUCE [11]. A check on our results is provided by the fact that the auxiliary vectors $\bar{k}_\mu$ drop out of the expressions for the amplitudes as they should on account of Lorentz invariance [10].

3. To evaluate the field theory contributions to the four-point amplitudes, we rescale the dilaton field $\varphi \to \frac{1}{4}\sqrt{D-2}\phi$ in order to have the standard normalization of the propagator, thus obtaining for the effective action in the s-parameterization

$$S=-\frac{2}{\kappa^2}\int d^Dx\sqrt{g}\{R-\tfrac{1}{4}(\partial\phi)^2+\alpha'\lambda_0\exp(m\phi)\{G_2+\tfrac{1}{16}[(D-4)/(D-2)](\partial\phi)^4\}$$
$$+\alpha'^2\exp(2m\phi)[\lambda_1 I_1+\lambda_2 G_3+\ldots+\bar{\lambda}_7 R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}\partial^\mu\phi\partial^\nu\phi]+O(\alpha'^3)\}\,, \tag{12}$$

where $m=-1/\sqrt{D-2}$ and $\bar{\lambda}_i$ $(i=3, ..., 7)$ include the effect of the ϕ redefinition.

There are essentially two types of contributions to the relevant field theory amplitudes, generated by this action: the contact and exchange contributions, depicted in the Feynman diagrams of figs. 1 and 2, respectively.

To find the vertices we first expand the metric in the form $g_{\mu\nu}=\delta_{\mu\nu}+h_{\mu\nu}+O(h^2)$, and then impose the dual gauge condition $k^{\mu i}h^i_{\mu\nu}=0$ (where i is a particle label), and the on-shell conditions, $k^2=0$ and $h^\mu{}_\mu=0$ for the external particles. Some of the vertices, especially those involving more than two gravitons, like e.g. the $O(h^3)$ contribution from $\sqrt{g}R$, have very complicated expressions which were computed with the assistance of REDUCE [11]. In the cases where the vertices were given in the literature we have checked that our results agree.

The evaluation of $\bar{\lambda}_3$ is straightforward since there is only the contact contribution of fig. 1a. We obtain

$$\mathcal{T}_{\text{cont.}}[\bar{\lambda}_3]=6\alpha'^2\bar{\lambda}_3 stu\phi_1\phi_2\phi_3\phi_4\,. \tag{13}$$

Comparing with the four-dilaton string amplitude results of eq. (11), we get

$$\bar{\lambda}_3^{(\mathrm{B})}=\frac{(3D-10)^2}{384(D-2)^2}\,,\quad \bar{\lambda}_3^{(\mathrm{H})}=\frac{1}{128}\frac{(D-3)(D-6)}{(D-2)^2}\,. \tag{14}$$

Similarly to find $\bar{\lambda}_4$ and $\bar{\lambda}_5$, we calculate the contact contributions of figs. 1b and 1c, respectively. We obtain

$$\mathcal{T}_{\text{cont.}}[\bar{\lambda}_4]=\tfrac{3}{2}\alpha'^2\bar{\lambda}_4(s^2+t^2+u^2)[(k_2h_1k_2)+(k_2h_1k_3)+(k_3h_1k_3)]\phi_2\phi_3\phi_4$$

and

$$\mathcal{T}_{\text{cont.}}[\bar{\lambda}_5]=-\tfrac{3}{2}\alpha'^2\bar{\lambda}_5[u^2(k_2h_1k_2)+s^2(k_3h_1k_3)-2us(k_2h_1k_3)]\phi_2\phi_3\phi_4\,. \tag{15}$$

Comparing with the corresponding one-graviton–three-dilaton string amplitudes, eq. (10), we get

$$\bar{\lambda}_4^{(\mathrm{B})}=0\,,\quad \bar{\lambda}_4^{(\mathrm{H})}=0\,,\quad \bar{\lambda}_5^{(\mathrm{B})}=\frac{3D-10}{24(D-2)^{3/2}}\,,\quad \bar{\lambda}_5^{(\mathrm{H})}=\frac{1}{16}\frac{(D-4)}{(D-2)^{3/2}}\,. \tag{16}$$

Finally, we calculate $\bar{\lambda}_6$ and $\bar{\lambda}_7$. The calculation of these coefficients is more involved since there are many contributions: the contact contributions of fig. 1d and the graviton-exchange contributions of fig. 2. The contact terms give

$$\mathcal{T}_{\text{cont.}}[\bar{\lambda}_6+\bar{\lambda}_7]=\{(\bar{\lambda}_6+\tfrac{1}{4}\bar{\lambda}_7)\alpha'^2[(h_1h_2)s^3+4(k_2h_1h_2k_1)s^2+4(k_2h_1k_2)(k_1h_2k_1)s]$$
$$-\tfrac{1}{4}\bar{\lambda}_7\alpha'^2[2(h_1h_2)stu+4(k_2h_1h_2k_1)ut+4(k_2h_1h_2k_3)su+4(k_3h_1h_2k_1)st-4(k_3h_1h_2k_3)s^2$$
$$+8u(k_2h_1k_2)(k_1h_2k_3)u+8(k_2h_1k_3)(k_1h_2k_1)t-8(k_2h_1k_3)(k_1h_2k_3)s]\}\phi_3\phi_4\,. \tag{17}$$

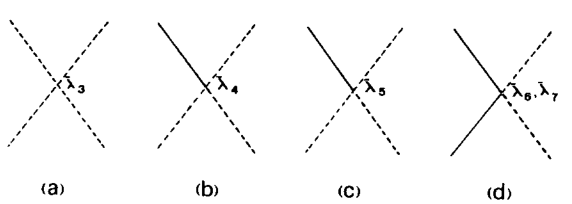

Fig. 1. Contact diagrams contributing to $\bar{\lambda}_3, ..., \bar{\lambda}_7$.

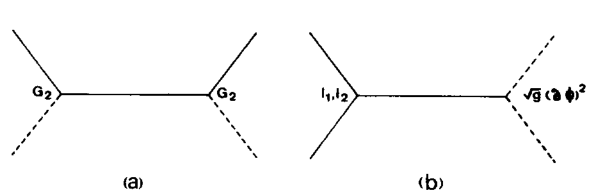

Fig. 2. Exchange diagrams contributing to the two-graviton–two-dilaton amplitude.

The contribution of the graviton exchange between the $\exp(m\phi)G_2$ vertices (fig. 2a) with the graviton propagator taken in the standard harmonic gauge,

$$D_{\mu\nu\alpha\beta}=\frac{1}{k^2}\left[\tfrac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha})-(D-2)^{-1}\delta_{\mu\nu}\delta_{\alpha\beta}\right],$$

is given by

$$\begin{aligned}\mathscr{T}_{\text{exch.}}[\exp(m\phi)G_2-\exp(m\phi)G_2]=&\lambda_0^2\alpha'^2m^2\{(h_1h_2)(s^3-3stu)+4(k_2h_1h_2k_1)s^2\\&-8(k_2h_1h_2k_1)ut-4(k_2h_1h_2k_3)su-4(k_3h_1h_2k_1)st+4(k_3h_1h_2k_3)s^2\\&+(k_2h_1k_2)[4(k_1h_2k_1)s-16(k_1h_2k_3)u+4(k_3h_2k_3)su/t]\\&+(k_2h_1k_3)[-16(k_1h_2k_1)t+24(k_1h_2k_3)s-8(k_3h_2k_3)s^2/t]\\&+(k_3h_1k_3)[4(k_1h_2k_1)st/u-8(k_1h_2k_3)s^2/u-4(k_3h_2k_3)(s^2/u+s^2/t)]\}\phi_3\phi_4\,. \end{aligned}\qquad(18)$$

Next, we compute the graviton exchange graphs between the vertices coming from the terms I_1, I_2 and $\sqrt{g}(\partial\phi)^2=\sqrt{g}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$. We find

$$\begin{aligned}\mathscr{T}_{\text{exch}}[I_1-\sqrt{g}(\partial\phi)^2]=&-\tfrac{3}{2}\alpha'^2(\lambda_1+\lambda_2)\{-(k_2h_1h_2k_1)ut+(k_2h_1h_2k_3)su+(k_3h_1h_2k_1)st-(k_3h_1h_2k_3)s^2\\&+(k_2h_1k_2)[-2(k_1h_2k_1)ut/s+2(k_1h_2k_3)u]+(k_2h_1k_3)[2(k_1h_2k_1)t-2(k_1h_2k_3)s]\}\phi_3\phi_4\,,\\ \mathscr{T}_{\text{exch}}[I_2-\sqrt{g}(\partial\phi)^2]=&-\tfrac{3}{16}\alpha'^2\lambda_2\{(h_1h_2)(4stu-s^3)+(k_2h_1h_2k_1)(16tu-4s^2)\\&+(k_2h_1k_2)(k_1h_2k_1)(16tu/s-4s)\}\phi_3\phi_4\,.\end{aligned}\qquad(19)$$

Summing up all the contributions, eqs. (17)-(19), and comparing with the two-graviton-two-dilaton string amplitudes, eq. (9), we find the following relations:

$$\begin{aligned}&\bar\lambda_6+\tfrac{1}{4}\bar\lambda_7+\lambda_0^2m^2+\tfrac{3}{16}\lambda_2=0 && (0)\,,\\ &-\tfrac{1}{2}\bar\lambda_7-3\lambda_0^2m^2-\tfrac{3}{2}\lambda_2=\frac{3D-10}{64(D-2)} && \left(\frac{D-7}{64(D-2)}\right),\\ &\bar\lambda_7-8\lambda_0^2m^2+\tfrac{3}{2}\lambda_1-\tfrac{3}{2}\lambda_2=\frac{D-4}{8(D-2)} && \left(\frac{D-8}{32(D-2)}\right),\\ &\bar\lambda_7-4\lambda_0^2m^2-\tfrac{3}{2}\lambda_1-\tfrac{3}{2}\lambda_2=\tfrac{1}{16} && \left(\frac{D-6}{32(D-2)}\right),\\ &3\lambda_1=\tfrac{1}{16} && (0)\,,\\ &-2\bar\lambda_7-16\lambda_0^2m^2-3\lambda_1-3\lambda_2=\frac{D-6}{8(D-2)} && \left(\frac{D-8}{16(D-2)}\right),\\ &4\lambda_0^2m^2=\frac{1}{4(D-2)} && \left(\frac{1}{16(D-2)}\right),\\ &2\bar\lambda_7+24\lambda_0^2m^2+3\lambda_1+3\lambda_2=-\frac{D-10}{8(D-2)} && \left(-\frac{D-10}{16(D-2)}\right),\end{aligned}\qquad(20)$$

where in the case of the heterotic string the right-hand side of eqs. (20) should be replaced by the quantities in parenthesis. These equations are satisfied for the previously quoted values of λ_0, λ_1 and λ_2 and for

$$\bar{\lambda}_6^{(B)}=\frac{1}{32}\,,\quad \bar{\lambda}_6^{(H)}=\frac{1}{128}\frac{(D-6)}{(D-2)}\,,\quad \bar{\lambda}_7^{(B)}=-\frac{1}{32}\frac{(5D-2)}{(D-2)}\,,\quad \bar{\lambda}_7^{(H)}=-\frac{1}{32}\frac{(D-4)}{(D-2)}\,. \tag{21}$$

The fact that we have six equations to fix the two coefficients $\bar{\lambda}_6$ and $\bar{\lambda}_7$ both for the bosonic and heterotic string and that they are consistent is a very good verification of our results.

In terms of the coupling constants of the action (1), our results are

$$\lambda_3^{(B)}=\frac{2}{3}\frac{(3D-10)^2}{(D-2)^4}\,,\quad \lambda_3^{(H)}=2\,\frac{(D-3)(D-6)}{(D-2)^4}\,,\quad \lambda_4^{(B)}=0\,,\quad \lambda_4^{(H)}=0\,,$$

$$\lambda_5^{(B)}=\frac{8}{3}\frac{(3D-10)}{(D-2)^3}\,,\quad \lambda_5^{(H)}=4\frac{(D-4)}{(D-2)^3}\,,\quad \lambda_6^{(B)}=\frac{1}{2(D-2)}\,,\quad \lambda_6^{(H)}=\frac{1}{8}\frac{(D-6)}{(D-2)^2}\,,$$

$$\lambda_7^{(B)}=-\frac{1}{2}\frac{(5D-2)}{(D-2)^2}\,,\quad \lambda_7^{(H)}=-\frac{1}{2}\frac{(D-4)}{(D-2)^2}\,. \tag{22}$$

Notice that we could have chosen different terms to parametrize the effective action at order α'^2, as pointed out in ref. [2], e.g. $RR_{\mu\nu}\partial^\mu\phi\partial^\nu\phi$, $R(\mathrm{D}_\mu\partial_\nu\phi)^2$, etc. However, none of the other possible terms contributes to the four-point amplitudes at order α'^2, either because they vanish at this order on account of the on shell and dual gauge conditions as, e.g. with $RR_{\mu\nu}\partial^\mu\phi\partial^\nu\phi$, or because their contact and exchange contributions cancel each other as, e.g. with $R(\mathrm{D}_\mu\partial_\nu\phi)^2$. This observation determined our choice of terms in the action, eq. (1).

Before closing this section we should mention that we have also computed the full three-graviton–one-dilaton amplitudes both from the string, eq. (4), and from the effective action, eqs. (1), (12). As these amplitudes only depend on the known coefficients λ_0, λ_1, λ_2 and they are very long we will not reproduce them here. We found a complete agreement for the values of these coefficients given in the literature [3,4]. We stress that this is a highly non-trivial consistency check as those coefficients have been determined by matching only a few structures in the four-graviton amplitudes [3,4] whereas the $hhh\phi$ full amplitude has 54 independent structures and they all match. This is another check on our computer programs and expressions for the vertices.

4. Finally, we calculate the $\mathrm{O}(\alpha'^2)$ coefficients in the σ-model parametrization, where [2]

$$\begin{aligned} S=-\frac{2}{\kappa^2}\int \mathrm{d}^Dx\sqrt{g}\exp(-2\varphi)\,[R+4(\partial\varphi)^2+\lambda_0\alpha' G_2 \\ +\alpha'^2(\rho_1 I_1+\rho_2 G_3+\rho_3(\mathrm{D}_\mu\partial_\nu\varphi)^2(\partial\varphi)^2+\rho_4\mathrm{D}_\mu\partial_\nu\varphi\mathrm{D}^\rho\partial^\mu\varphi\mathrm{D}_\rho\partial^\nu\varphi+\rho_5 R(\partial\varphi)^4 \\ +\rho_6 R^2_{\mu\nu\alpha\beta}(\partial\varphi)^2+\rho_7 R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}\partial^\mu\varphi\partial^\nu\varphi)+\mathrm{O}(\alpha'^3)]\,. \end{aligned} \tag{23}$$

We use the relations between coefficients $\rho_1, ..., \rho_7$ and $\lambda_1, ..., \lambda_7$ derived in ref. [2] (eqs. (18)) together with eqs. (22), to obtain

$$\rho_1^{(B)}=\tfrac{1}{48}\,,\quad \rho_1^{(H)}=0\,,\quad \rho_2^{(B)}=\tfrac{1}{24}\,,\quad \rho_2^{(H)}=0\,,\quad \rho_3^{(B)}=-8\frac{(D-3)(D-6)}{(D-2)^4}\,,\quad \rho_3^{(H)}=-2\frac{(D-3)(D-6)}{(D-2)^4}\,,$$

$$\rho_4^{(B)}=0\,,\quad \rho_4^{(H)}=0\,,\quad \rho_5^{(B)}=\frac{2}{3}\frac{(5D^3-24D^2+76D-128)}{(D-2)^6}\,,\quad \rho_5^{(H)}=\frac{(D^3-9D^2+40D-68)}{(D-2)^6}\,,$$

$$\rho_6^{(B)}=\frac{1}{2}\frac{(D-6)}{(D-2)^2}\,,\quad \rho_6^{(H)}=\frac{1}{8}\frac{(D-6)}{(D-2)^2}\,,\quad \rho_7^{(B)}=-\frac{2(D-4)}{(D-2)^2}\,,\quad \rho_7^{(H)}=-\frac{1}{2}\frac{(D-4)}{(D-2)^2}\,. \tag{24}$$

We conclude that the coefficients of the $\mathrm{O}(\alpha'^2)$ terms in the gravitational sector of the closed string effective actions are given by eqs. (8), (22), in the S-matrix parameterization and by eqs. (24) in the σ-model parameterization. We note that some features of string effective actions up to $\mathrm{O}(\alpha')$ are lost at $\mathrm{O}(\alpha'^2)$, namely:

(i) The curvature cubed terms do not appear in the generalized Gauss–Bonnet combination (Ω_3) [4], as conjectured in ref. [12].

(ii) The effective actions for the bosonic and heterotic string are no longer proportional.

(iii) The inclusion of the dilaton, in the σ-model parameterization, does not simply amount, as we have shown, to the exponential factor, $\exp(-2\phi)$, in front of the terms involving just curvatures.

According to the equivalence conjecture [5,6] it should be possible to reproduce the above results through a computation of the $O(\alpha'^2)$ σ-model β-functions. The authors of ref. [8] found the dilaton dependence of the effective action for the bosonic string theory at order α'^2 using a σ-model analysis; they computed the $O(\alpha'^2)$ (four-loop) dilaton function from the $O(\alpha'^2)$ (three-loop) metric β-function using the Curci and Paffuti identity and concluded that the $\exp(-2\phi)$ factor in eq. (23) is the only modification necessary to incorporate the effect of a non-zero dilaton background into the result of ref. [4] for the purely gravitational background. This result is not in agreement with our calculation since we find that, in order to match the four-point string amplitudes one needs in addition, four terms involving derivatives of the dilaton field, the ones with coefficients ρ_3, ρ_5, ρ_6 and ρ_7 in eq. (23).

References

[1] J. Scherk and J.H. Schwarz, Nucl. Phys. B 81 (1974) 118; Phys. Lett. B 52 (1974) 347;
T. Yoneya, Nuovo Cimento Lett. 8 (1973) 951; Prog. Theor. Phys. 51 (1974) 1907;
M.B. Green and J.H. Schwarz, Phys. Lett. B 149 (1984) 117;
D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. B 256 (1985) 253; B 267 (1986) 75;
A. Sen, Phys. Rev. D 32 (1985) 250; Phys. Rev. Lett. 55 (1985) 1846;
P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B 258 (1986) 46;
B. Zwiebach, Phys. Lett. B 156 (1985) 315;
R. Nepomechie, Phys. Rev. D 32 (1985) 3201;
D.J. Gross and E. Witten, Nucl. Phys. B 277 (1986) 1;
Y. Kikuchi, C. Marzban and Y.J. Ng, Phys. Lett. B 176 (1986) 57;
S. Deser and A.N. Redlich, Phys. Lett. B 176 (1986) 350.

[2] M.C. Bento, O. Bertolami, A.B. Henriques and J.C. Romão, Phys. Lett. B 218 (1989) 162.

[3] R.R. Metsaev and A.A. Tseytlin, Nucl. Phys. B 293 (1987) 385.

[4] R.R. Metsaev and A.A. Tseytlin, Phys. Lett. B 185 (1987) 52.

[5] C. Lovelace, Nucl. Phys. B 273 (1986) 413;
B.E. Fridling and A. Jevicki, Phys. Lett. B 174 (1986) 75;
A.B. Zamolodchikov, JETP Lett. 43 (1986) 730;
A.M. Polyakov, preprint SLAC Trans. - 0222 (1986);
A.A. Tseytlin, Phys. Lett. B 208 (1988) 221;
N. Mavromatos and L. Miramontes, Phys. Lett. B 212 (1988) 33;
N. Mavromatos, Oxford preprint 74/88.

[6] C.G. Callan, D. Friedan, E. Martinec and M.J. Perry, Nucl. Phys. B 262 (1985) 593;
T. Banks, D. Nemeschansky and A. Sen, Nucl. Phys. B 277 (1986) 67;
C.G. Callan, I.R. Klebanov and M.J. Perry, Nucl. Phys. B 278 (1986) 78;
A. Sen, Phys. Rev. Lett. 55 (1985) 1846; Phys. Rev. D 32 (1985) 2102;
M.T. Grisaru, A.E.M. van de Ven and D. Zanon, Phys. Lett. B 173 (1986) 423;
M.D. Freeman and C.N. Pope, Phys. Lett. B 174 (1986) 48;
M.T. Grisaru and D. Zanon, Phys. Lett. B 177 (1986) 347;
M.D. Freeman, C.N. Pope, M.F. Sohnius and K.S. Stelle, Phys. Lett. B 178 (1986) 199;
D. Zanon, Phys. Lett. B 186 (1987) 309;
I. Jack and D.R.T. Jones, Phys. Lett. B 200 (1988) 453;
U. Ellwanger, J. Fuchs and M.G. Schmidt, Phys. Lett. B 203 (1988) 244; Heidelberg preprint HD-THEP-88-9.

[7] S.J. Graham, Phys. Lett. B 197 (1987) 543;
A.P. Foakes and N. Mohamedhi, Phys. Lett. B 198 (1987) 359; Nucl. Phys. B 306 (1988) 343.

[8] I. Jack, D.R.T. Jones and D.A. Ross, Nucl. Phys. B 307 (1988) 130, 531.

[9] M.J. Green and J.H. Schwarz, Nucl. Phys. B 198 (1982) 252;
J.H. Schwarz, Phys. Rep. 89 (1982) 223;
H. Kawai, D.C. Lewellen and S.H.H. Tye, Nucl. Phys. B 269 (1986) 1;
Y. Cai and C.A. Núnez, Nucl. Phys. B 287 (1987) 279.
[10] J. Scherk and J.H. Schwarz, Nucl. Phys. B 81 (1974) 118.
[11] A.C. Hearn, REDUCE (1985).
[12] B. Zumino, Phys. Rep. 137 (1986) 109;
C. Aragone, Phys. Lett. B 186 (1987) 151.