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Pion rescattering operator in the *S*-matrix approach

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The pion rescattering operator for pion production, derived recently in time-ordered perturbation theory, is compared with the one following from the simple *S*-matrix construction. We show that this construction is equivalent to the on-shell approximation introduced in previous articles. For a realistic *NN* interaction, the *S*-matrix approach, and its simplified fixed threshold-kinematics version, work well near threshold.

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I. INTRODUCTION

The detailed analysis of the irreducible pion rescattering operator was recently performed [1] for the reaction $pp \to pp\pi^0$. The pion rescattering is certainly part of the pion production mechanism, but its importance relative to other contributions varies considerably dependent on the approximations made in evaluation of the effective operators (see Ref. [1] and references therein). The nature and extent of this uncertainty are reexamined in this article.

To this end we deal with retardation effects in the exchanged pion propagator, that is, its energy dependence, as well as with the energy dependence of the πN scattering amplitude in the vertex, from which the produced pion is emitted. Although the approximations employed in a previous article [1] yield rather different results, we show here that the deviations between them are significantly reduced when the approximations are applied consistently in the whole effective operator.

We also show that the *S*-matrix approach, which has been successfully used below the pion production threshold, also yields above-threshold results rather close to those obtained with the energy-dependent operator following from time-ordered perturbation theory.

The article is organized as follows: after this introduction, Sec. II describes the *S*-matrix technique for deriving effective nuclear quantum-mechanical operators, Sec. III describes the results, and Sec. IV presents a summary and conclusions.

II. PION RESCATTERING OPERATOR

To derive the effective pion production operators, and other effective nuclear operators in general, one starts from the relativistic (effective) Lagrangian written in terms of hadronic fields. The interactions mediated by meson exchanges before and after the production reaction takes place are included in the effective *NN* (and nucleon-meson) interaction, whereas from the irreducible parts connected to the reaction mechanism (e.g., pion production), one obtains effective operators whose expectation values are to be evaluated between the initial and final nucleonic wave functions. One aims to get such effective operators consistent with the realistic description of the *NN* interaction, which can then be used in studies of the corresponding reactions not only on the simplest (one- or two-nucleon) systems but preferably also on heavier nuclei.

The covariant techniques based on the Bethe-Salpeter equation or its quasipotential rearrangements are these days practically manageable only below the meson production threshold. However, above the threshold the dressings of the single hadron propagators and interaction vertices via the meson loops have to be included explicitly. For this reason the construction of the production operator is so far realized mostly in the Hamiltonian quantum-mechanical framework (usually nonrelativistic or with leading relativistic effects included perturbatively within the decomposition in powers of p/m, where p is the typical hadronic momentum and m is the nucleon mass).

The derivation of the nuclear effective operators below the meson production threshold within the Hamiltonian framework—leading to Hermitian and energy-independent *NN* and 3*N* potentials and conserved e.m. and partially conserved weak current operators—can be done in many different ways (see discussion in Ref. [2] and references therein). At the nonrelativistic order the results are determined uniquely. As for the leading-order relativistic contributions, they were shown to be unitarily equivalent. The unitary freedom allows us to choose the *NN* potentials to be static (in the c.m. frame of the two-nucleon system) to identify them with the successful static semiphenomenological potentials.

Also, above the threshold the static limit is commonly employed, because more elaborate descriptions that include the mesonic retardation and loop effects are technically considerably more complex [3,4], especially for systems of more than two nucleons. Both the static approaches and the ones including "retardation" typically consider contributions of several one-meson exchanges and the potentials are fitted to describe the data. It is therefore difficult to assess how well they approximate the covariant amplitudes (corresponding to the same values of physical masses and coupling constants) that are so far outside the scope of existing calculational schemes but that we believe provide in principle the consistent description of the considered reactions.

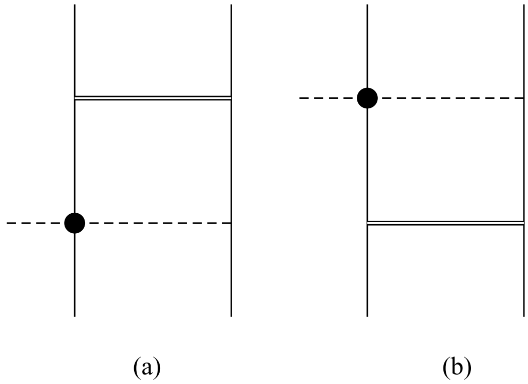

FIG. 1. Feynman diagrams for pion rescattering. The pion field is represented by a dashed line, the *NN* interaction by a solid double line, and the nucleons by solid lines.

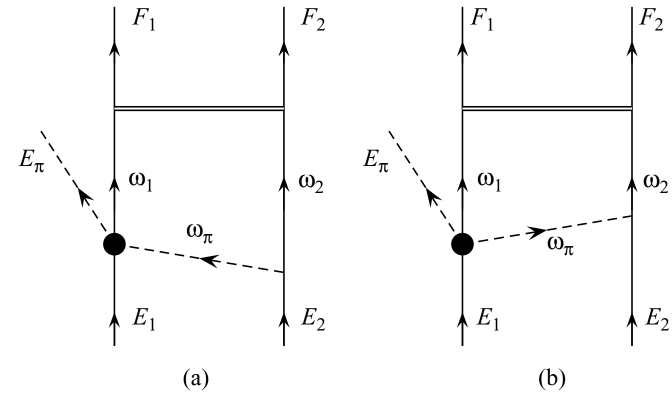

FIG. 2. The two time-ordered diagrams for FSI considered here; additional stretched box diagrams are neglected.

Thus, the ultimately exact approach to the description of pion production (and in particular of the pion rescattering contribution) would be the covariant Bethe-Salpeter or quasipotential frameworks (extended above the pion threshold) or the quantum mechanical coupled-channel technique, including retardation. In these approaches one has to treat the non-Hermitian energy-dependent *NN* interaction (fitted to the data also above pion production threshold) and consider the effects of renormalization of vertices, masses, and wave functions via meson loops.

In this article we instead (following Refs. [5]) numerically estimate the range of the predictions from several commonly used simplifying approximations [5,6] and compare them to the result obtained from the reduction of the corresponding covariant Feynman diagrams for the pion rescattering operator. This reduction coincides with the time-ordered perturbation theory [1].

A. Factorization of the effective rescattering operator

In a previous article [1] we made the connection to the usual representation of the pion rescattering operator for nonrelativistic calculations by following the approach of Ref. [5]. We started from the covariant two-nucleon Feynman amplitudes, including final and initial state interaction (FSI and ISI, respectively), shown in Figs. 1(a) and 1(b).

To obtain the effective rescattering operator the negative energy contributions in the nucleon propagators (to be included in the complete calculations) were neglected. By integrating subsequently over the energy of the exchanged pion the resulting Feynman amplitudes were transformed into those following from the time-ordered perturbation theory. We have shown in Ref. [1] that the irreducible "stretched box diagrams" (i.e., those with more than one meson in flight in the intermediate states) give a very small contribution and can therefore also be neglected. Thus, the full covariant amplitude is in the lowest order Born approximation well approximated by the product of the *NN* potential and the effective pion rescattering operator, which can be extracted from these time-ordered diagrams (Fig. 2).

The effective pion rescattering operator was in Ref. [1] factorized into an effective pion rescattering vertex $\tilde{f}$ and an effective pion propagator G_π. For the diagram with FSI (Fig. 2) this factorization reads as follows:

$$\mathcal{M}_{FSI} = \int \frac{d^3q'}{(2\pi)^3}\, V_\sigma\, \frac{1}{F_1 + F_2 - \omega_1 - \omega_2 + i\varepsilon}\, \hat{O}_{rs}, \tag{1}$$

$$\begin{aligned}\hat{O}_{rs} &= -\frac{1}{2\omega_\pi}\left[\frac{f(\omega_\pi)}{E_2 - \omega_2 - \omega_\pi} + \frac{f(-\omega_\pi)}{E_1 - \omega_1 - E_\pi - \omega_\pi}\right] \\ &= \frac{1}{2}\,\tilde{f}\,G_\pi,\end{aligned} \tag{2}$$

$$\begin{aligned}\tilde{f} &= \frac{1}{\omega_\pi}\,[(E_1 - \omega_1 - E_\pi - \omega_\pi) f(\omega_\pi) \\ &\quad + (E_2 - \omega_2 - \omega_\pi) f(-\omega_\pi)],\end{aligned} \tag{3}$$

$$G_\pi = -\frac{1}{(E_1 - \omega_1 - E_\pi - \omega_\pi)(E_2 - \omega_2 - \omega_\pi)}, \tag{4}$$

$$V_\sigma = \frac{1}{2\omega_\sigma}\left[\frac{1}{F_2 - \omega_2 - \omega_\sigma} + \frac{1}{F_1 - \omega_1 - \omega_\sigma}\right], \tag{5}$$

where, adopting the notation of Ref. [1], $\vec{q}\,'$ is the momentum of the exchanged pion, $\omega_\pi^2 = m_\pi^2 + \vec{q}\,'^2$ is its on-mass-shell energy, and $f(\omega_\pi)$ is the product of the πN amplitude with the πNN vertex (as in Ref. [1], the standard χPT rescattering vertex is employed here). We note that f in fact depends on the three-momenta and energies of both (the exchanged and produced) pions and also on the nucleon spin. However, for simplicity, we indicate explicitly only the dependence on the exchanged pion energy, because it is the important variable for the main considerations in this article. The variables E_i, ω_i, and F_i are the on-mass-shell energies of the ith nucleon in the initial, intermediate, and final state, respectively, E_π is the energy of the produced pion, $E_1 + E_2 = F_1 + F_2 + E_\pi$.

The inclusion of some pieces of the integrand of Eq. (1) into the propagator G_π and of others in the modified vertex $\tilde{f}$ is somewhat arbitrary. The appearance of the unusual effective propagator G_π and the effective vertex $\tilde{f}$ is the result of

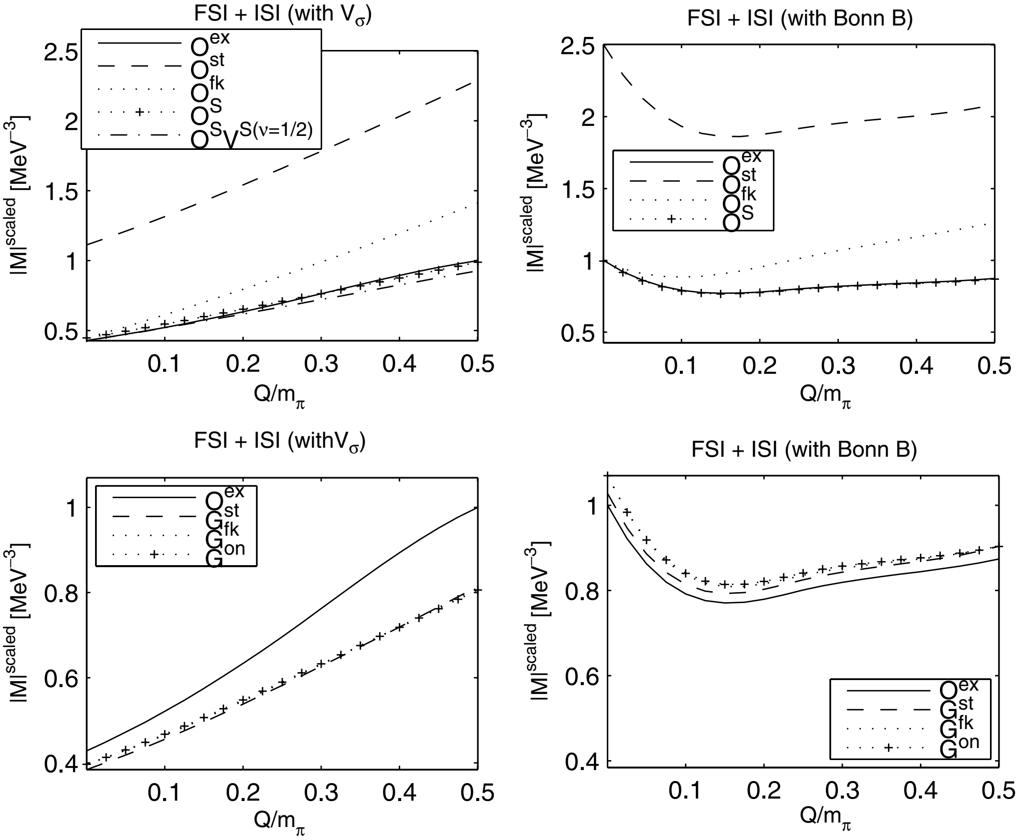

FIG. 3. Absolute values of the FSI + ISI amplitude as a function of the excess energy $Q = 2E - 2M - E_\pi$ (in units of m_π). The right (left) panels correspond to the amplitudes with σ exchange (Bonn B) for the *NN* interaction. The amplitudes are taken at the maximum pion momentum $q_\pi^{\max}$, determined by Q. The upper panels correspond to approximations for the whole operator $\hat{O}_{rs}$, the lower panels to approximations for the pion propagator G_π only. The solid line (O^{ex}) is the reference calculation. The dashed, dotted, and cross-dotted lines correspond to the static, fixed threshold-kinematics, and on-shell approximations, respectively. The corresponding operators are O^{st}, O^{fk}, O^S and G^{st}, G^{fk}, G^{on}. All amplitudes were normalized by a factor defined by the maximum value of the reference result.

combining *two* time-ordered diagrams with different energy dependence into a *single* effective operator.

The *NN* interaction is in Fig. 2 and in Eq. (1) simulated by a simple σ-exchange potential. Though not realistic, this interaction suffices for model studies of approximations employed in derivations of the effective pion rescattering operator, as done in Ref. [5]. Because some results do depend on the behavior of the *NN* scattering wave function, in particular in the region of higher relative momenta, we perform our calculations (as in Ref. [1]) also with V_σ replaced by a full *NN* T matrix, generated from the realistic Bonn B potential.

We note that the meson poles are not neglected in the integration over the energy Q_0' of the exchanged pion, which generates Eq. (1). A result similar to Eq. (1) can also be obtained for the amplitude with the ISI. The two amplitudes differ, however, in the contribution from the pion poles to the remaining integration over the three-momentum. For the amplitude with FSI there are no such poles. However, for the ISI there are values of the exchanged pion three-momentum for which the propagator G_π has poles. These poles have been considered in all our numerical calculations for the cross section. As shown, they are one of the main reasons for deviations between several approximations and the reference results calculated from Eqs. (1)–(5).

It is worth mentioning that although the FSI and ISI diagrams graphically separate the *NN* interaction and the pion rescattering part (when the stretched boxes are neglected), they do not define a *single* effective operator (as a function of nucleon three-momenta and the energy of emitted pion). Because in these time-ordered diagrams energy is not conserved at individual vertices, each of these diagrams defines a different off-energy shell extension of the pion rescattering amplitude. This is an unpleasant feature, because one would have to make an analogous construction for diagrams with both FSI and ISI. Moreover, one would have to repeat the whole analysis for systems of more than two nucleons. Only after the on-shell approximation is made (in the next subsection) do the pion rescattering parts of FSI and ISI diagrams coincide and one can identify them with a single effective rescattering operator.

B. Rescattering operator in the *S*-matrix technique

The *S*-matrix technique is a simple prescription to derive the effective nuclear operators from the corresponding covariant

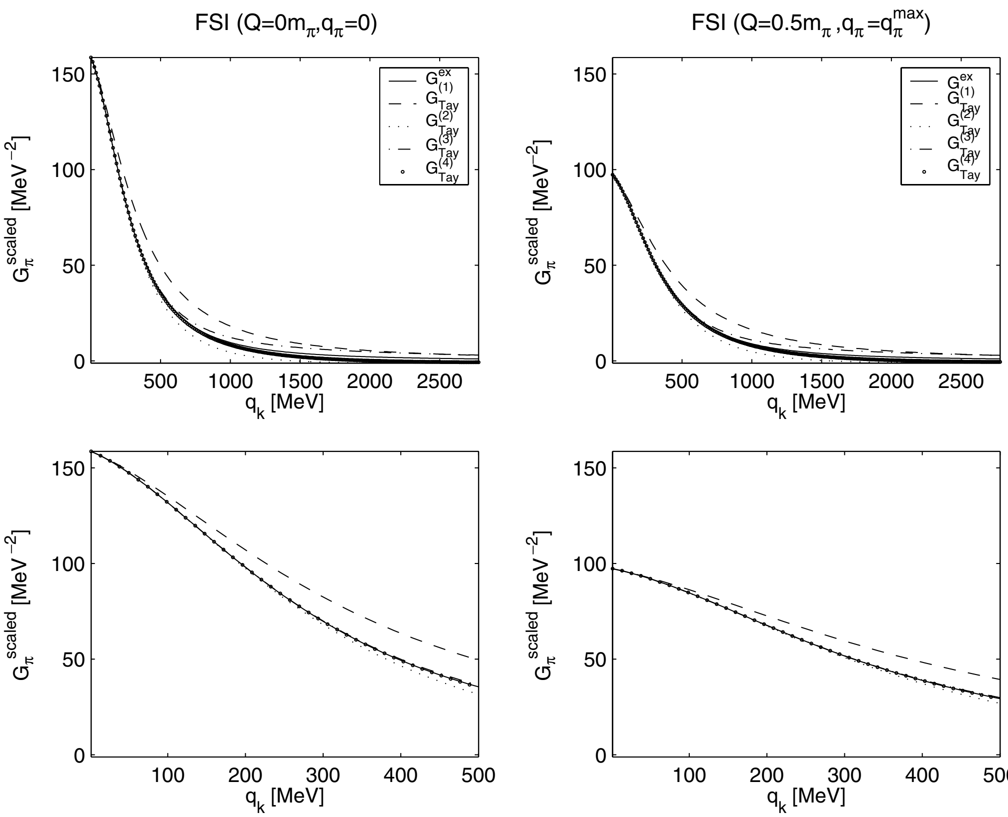

FIG. 4. Convergence of the Taylor expansion for the pion propagator G_π in the FSI amplitude as a function of the two nucleon relative momentum [Eq. (12)]. (Left panel) At threshold; (right panel) above threshold at maximum pion momentum for an excess energy Q of $0.5m_\pi$. Bottom panels zoom into the region of low relative momentum.

Feynman diagrams [2]. For electromagnetic operators and also for *NN* and 3*N* potentials the *S*-matrix approach reproduces the results of more laborious constructions, based on time-ordered or nonrelativistic diagram techniques.

The two-nucleon effective operators are by definition identified with the diagrams describing the irreducible mechanism of the corresponding reaction. The only exceptions are the nucleon Born diagrams from which the iteration of the one-nucleon operator has to be subtracted. The operators of the nuclear e.m. and weak currents, as well as the pion absorption operators and nuclear potentials, are obtained by a straightforward nonrelativistic reduction of the corresponding Feynman diagrams, in which the intermediate particles are off-mass-shell and energy is conserved at each vertex: therefore the derived effective operators are also defined on-energy-shell. The nuclear currents and other transition operators are defined to be consistent with a Hermitian energy independent *NN* potential, which has the usual one boson exchange form employed in realistic models of the *NN* interaction and can be used also in systems of more than two nucleons. This approach is well defined and understood below the meson production threshold, but as a simple tool it is employed also above the threshold, for instance in Refs. [7,8], to derive the Z-diagram operators.

For the σ exchange potential the *S*-matrix technique in the lowest order of nonrelativistic reduction yields the following:

$$V_\sigma \rightarrow V_\sigma^{on} = -\frac{1}{m_\sigma^2 + \vec{q}_\sigma^{\,2} - \Delta^2}, \tag{6}$$

with $\Delta = \Delta_1 = -\Delta_2$ and $\Delta_i = \epsilon_i' - \epsilon_i$; ϵ_i' and ϵ_i being the on-shell energies of the *i*th nucleon after and before the meson exchange, respectively. As pointed out above, this defines the potential only on-energy-shell. However, the Lippmann-Schwinger equation and even the first-order Born approximation to the wave function require the potential off-energy-shell. The extended *S*-matrix approach [2] defines the most general off-energy-shell continuation of V_σ as a class of unitarily equivalent potentials parameterized by the "retardation parameter" ν. The particular choice $\nu = 1/2$ leads to the static potential in the *NN* c.m. frame. This choice corresponds to the substitution $\Delta^2 = (\Delta_1 - \Delta_2)^2/4$. Most realistic *NN* potentials, namely those fitted to the data below the pion threshold, in particular the Bonn B potential used in this article, are energy independent and static in the nucleon c.m. frame and can be therefore considered to be consistent with this construction.

For the pion rescattering diagram, the *S*-matrix prescription leads to a single effective operator (for both FSI and ISI diagrams) of the following form:

$$\hat{O}_{rs}^S = \frac{f(\Omega)}{m_\pi^2 + \vec{q}^{\,\prime 2} - \Omega^2}, \tag{7}$$

where $\Omega = \epsilon_2' - \epsilon_2 = \epsilon_1 - \epsilon_1'$.

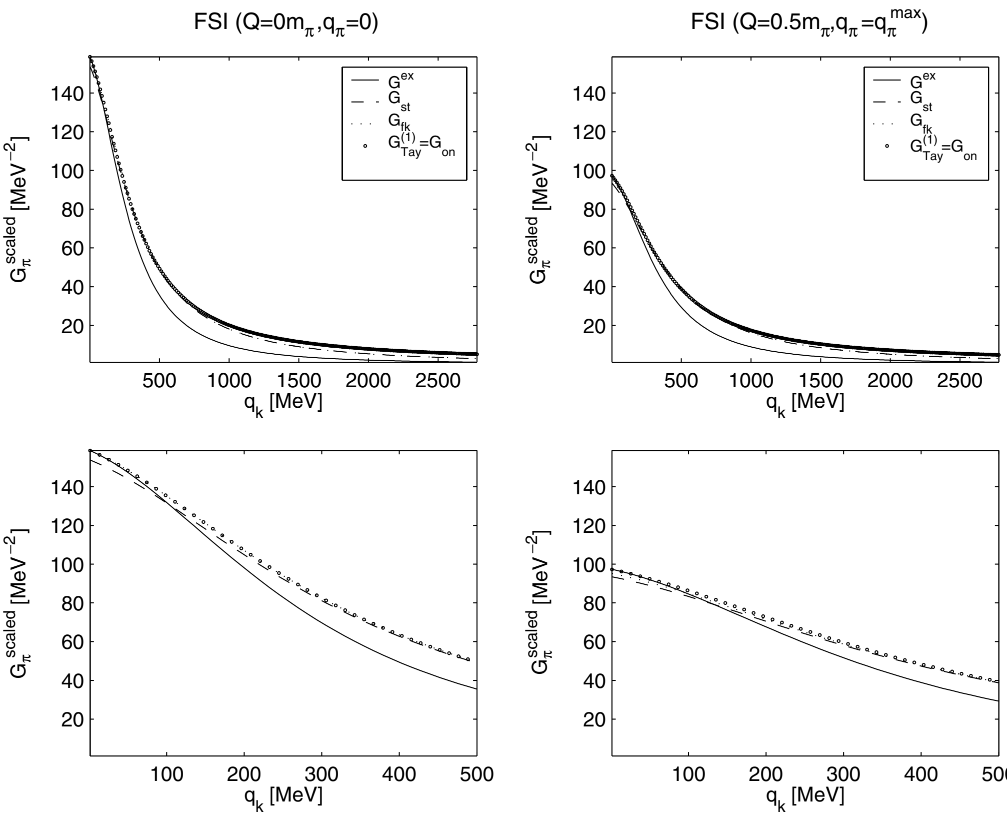

FIG. 5. Approximations for G_π in the FSI diagram and the first term of the Taylor series. Left and right panels are designated as in Fig. 4.

Let us finally introduce approximations to the time-ordered PT result given by Eqs. (1)–(5) and explain how the S-matrix technique fits in. The first of them is *the on-shell approximation*, in which the two nucleons in the intermediate state are put on-energy shell. That is, we put $\omega_1 + \omega_2 = F_1 + F_2$, which also implies $E_1 + E_2 = \omega_1 + \omega_2 + E_\pi$. For the scalar potential in the FSI diagram this leads to Eq. (6), where now $\Delta = F_1 - \omega_1 = \omega_2 - F_2$ is the energy transfer in the corresponding vertices.

For the rescattering operator (2) the on-shell replacement implies the following:

$$\hat{O}_{rs} \to \hat{O}_{rs}^{on} = -\frac{1}{2\omega_\pi}\left[\frac{f(\omega_\pi)}{E_2 - \omega_2 - \omega_\pi} + \frac{f(-\omega_\pi)}{\omega_2 - E_2 - \omega_\pi}\right]. \tag{8}$$

Clearly, the S-matrix pion rescattering operator $\hat{O}_{rs}^S$ follows from $\hat{O}_{rs}^{on}$ if one assumes the energy conservation at each vertex. The exchanged pion is then no longer on-mass-shell and we have to replace $f(\omega_\pi) \to f(E_2 - \omega_2)$ and $f(-\omega_\pi) \to f(E_2 - \omega_2)$: in the first time-ordered diagram the virtual pion is entering the rescattering vertex and in the second one it is emitted from this vertex (as defined on Fig. 2).

The on-shell approximation as introduced in Refs. [5,6] actually coincides with the S-matrix approximation defined above. The rescattering operator in Eq. (7) can be obtained directly from Eqs. (3) and (4) by the substitutions following from the on-energy-shell prescription and the energy conservation in individual vertices $\omega_\pi = E_2 - \omega_2 = -(E_1 - \omega_1 - E_\pi)$ as follows:

$$\begin{aligned}\hat{O}_{rs} &\to -\frac{1}{2(E_2 - \omega_2)} \\ &\quad\times \frac{-2(E_2 - \omega_2) \times f(E_2 - \omega_2) + 0 \times f(\omega_2 - E_2)}{(E_2 - \omega_2 - \omega_\pi)(\omega_2 - E_2 - \omega_\pi)} \\ &= \hat{O}_{rs}^S.\end{aligned} \tag{9}$$

In Ref. [1] the extra kinematical factors in Eq. (3) multiplying the function $f(\omega_\pi)$ were interpreted as form factors and kept unaltered, that is, the substitution above was made only in G_π and $f(\omega_\pi)$ of (2), not in the kinematical factors included in the function $\tilde{f}$.

In Eq. (9) the effective pion propagator G_π is seen to take its Klein-Gordon form as follows:

$$G_\pi \to G_\pi^{on} = 1\big/\big[(E_2 - \omega_2)^2 - \omega_\pi^2\big]. \tag{10}$$

The replacement, Eq. (10), does not significantly alter the results, as show below, but the corresponding substitution alone in the pion rescattering vertex $f(\pm\omega_\pi) \to f(\pm(E_2 - \omega_2))$ present in Eq. (3) leads to a large enhancement of the amplitude (1). According to Ref. [6] it increases the cross section by almost factor of 3. Because the splitting of $\hat{O}_{rs}$ is not defined unambiguously, in the work reported here the on-shell replacement is made in the whole rescattering operator. We demonstrate in this article that making the on-shell replacement in the whole operator leads to a significantly smaller deviation from the reference result compared to the replacement $f(\pm\omega_\pi) \to f(\pm(E_2 - \omega_2))$ involving f only.

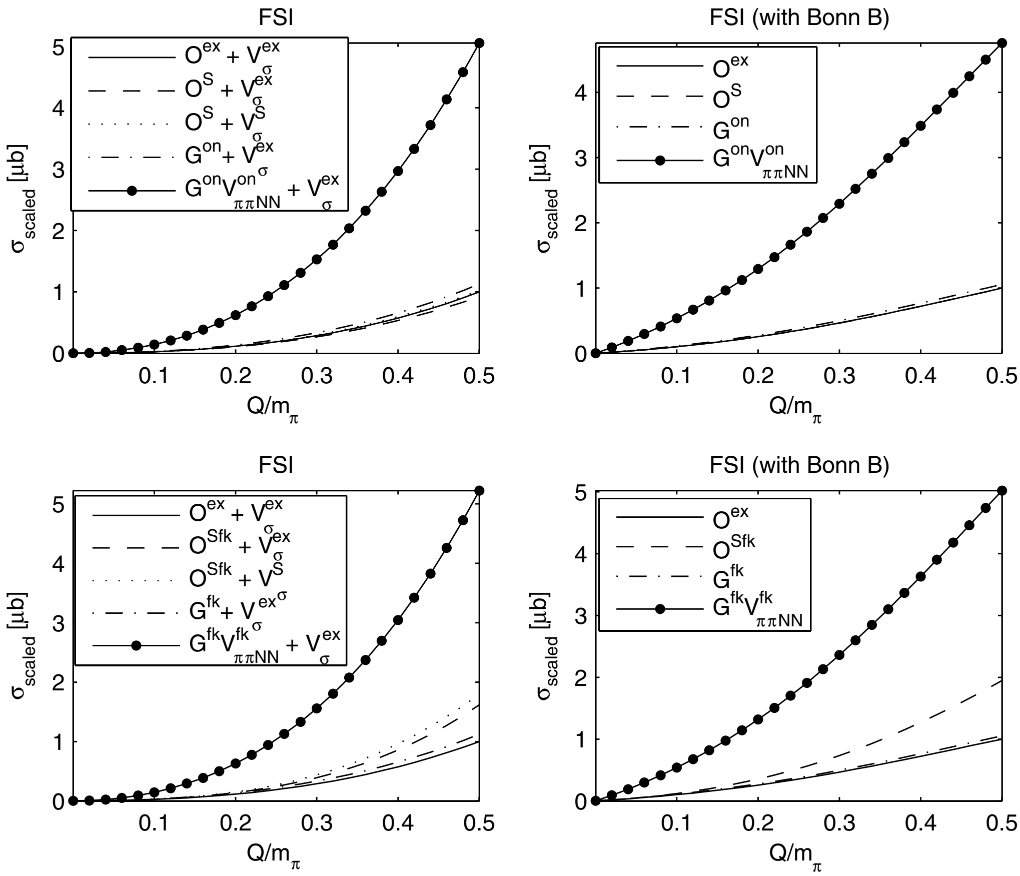

FIG. 6. Effects of the approximations for the rescattering operator $\hat{O}_{rs}$ and for the effective pion propagator G_π as a function of the excess energy Q. The cross-section curves shown correspond to the FSI amplitude alone. The upper panels correspond to the rescattering operator $\hat{O}$ given by Eq. (7) and the lower panels to fixed threshold-kinematics approximation. The solid line is the reference calculation (1). The dashed line is the S-matrix calculation for the rescattering operator $\hat{O}$ given by Eq. (7) (upper panels) and the fixed threshold-kinematics approximation for Eq. (7) (lower panels). The dotted line corresponds to take the S-matrix approximation not only for $\hat{O}$ but also for the σ-exchange interaction (6). The dashed-dotted line corresponds to the on-shell (upper panels) and fixed threshold-kinematics (lower panels) prescriptions only for G_π. The solid lines with bullets refer to the energy prescriptions taken for the propagator G_π and for $f(\omega_\pi)$ in Eq. (3), as in [1] but not for extra kinematical factors in $\tilde{f}$. All the cross sections were normalized with a factor defined by the maximum value of the reference result.

In our previous article [1] we considered also other approximations (in addition to the *on-shell* one): the so-called *static* and *fixed threshold-kinematics* approximations, defined by the replacement of the energy of the exchanged pion $E_2 - \omega_2$ by zero and $m_\pi/2$ (its threshold value), respectively. The static approximation in Ref. [1] was considered only for G_π, and the fixed threshold-kinematics one for G_π and also for $f(\pm\omega_\pi)$ (the additional kinematical factors in $\tilde{f}$ were again kept unchanged). In this article we again make these approximations in the full rescattering operator (7), replacing $\Omega \to 0$ and $\Omega \to m_\pi/2$, respectively. For V_σ the static approximation is defined by $\Delta \to 0$.

III. RESULTS

For numerical calculations we consider the $NN \to (NN)\pi$ transition in partial waves ${}^3P_0 \to ({}^1S_0)s_0$. Amplitudes and cross sections are evaluated both with the simple interaction V_σ and with the Bonn B potential. We test the S-matrix prescription for the rescattering operator [Eq. (7)] and also the fixed threshold-kinematics and the static approximations discussed in the previous section. Moreover, to compare to the previous articles we include also the results for the on-shell [Eq. (10)], fixed threshold-kinematics, and the static approximations for the effective pion propagator G_π.

In Fig. 3 we show that the amplitudes with the S-matrix operator O^S (dotted line with crosses on the upper panels) are the closest to the reference result (solid line). Using the same approach both for the operator and for V_σ increases slightly the gap from the reference result (dashed-dotted line versus solid line on the upper left panel). The fixed threshold-kinematics version of $\hat{O}_{rs}$, denoted as $\hat{O}^{fk}$, works well for small values of the excess energy $Q = 2E - 2M - E_\pi$ but starts to deviate rapidly with increasing Q (dotted line in the upper panels of Fig. 3). The static approximation for the rescattering operator ($\hat{O}^{st}$) overestimates significantly the amplitude [Eq. (1)] (dashed versus solid lines in the upper panels).

From the lower panels of Fig. 3 one sees that all considered approximations taken only for the effective pion propagator

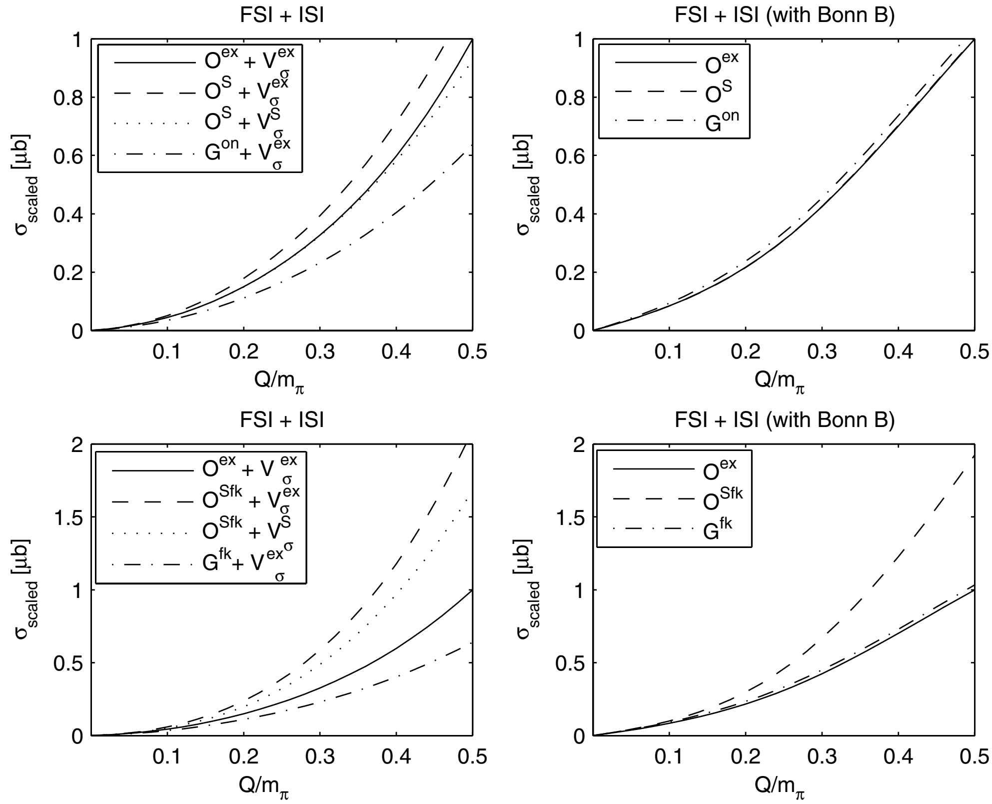

FIG. 7. The same as in Fig. 6, but for the total (FSI+ISI) cross section and considering only the approximations for $\hat{O}$ and G_π.

do not differ much from each other. This had already been found in Ref. [1]. It means that the choices for the energy of the exchanged pion in the effective propagator alone are not very decisive (solid line versus dotted, dashed, and cross-dotted lines). We notice, however, that there is a considerable deviation (dependent on the *NN* interaction employed) of all these approximations from the reference result.

To understand this we considered the expansion of the effective pion propagator G_π in Eq. (4) in terms of an "off-mass-shell" dimensionless parameter y:

$$y = -\frac{2E - E_\pi - \omega_1 - \omega_2}{\omega_1 - \omega_2 + E_\pi}, \tag{11}$$

which measures the deviation of the total energy from the energy of the intermediate state with all three particles on-mass-shell. This Taylor series expansion gives insight on the small effect of retardation effects in the propagator, and it reads as follows:

$$G_\pi = \underbrace{\frac{1}{\left(\frac{E_\pi+\omega_1-\omega_2}{2}\right)^2 - \omega_\pi^2}}_{G_{Tay}^{(1)}} \times \left[1 + \frac{(-2E + E_\pi + \omega_1 + \omega_2)}{\left(\frac{E_\pi+\omega_1-\omega_2}{2}\right)^2 - \omega_\pi^2} + \cdots\right], \tag{12}$$

where $G_{Tay}^{(1)}$ has the form of the usual Klein-Gordon propagator.

We notice here that in the case of the ISI amplitude, the representation of the pion propagator G_π by its Taylor series, the first term of which is $G_{Tay}^{(1)}$, fails because of the presence of a pole in the propagator.

Figure 4 compares the first four terms $G_{Tay}^{(i)} (i = 1, \ldots, 4)$ of this expansion with the full effective propagator in Eq. (4), as a function of the two-nucleon relative momentum q_k, for two different values of the excess energy $Q = 2E - 2M - m_\pi$. The convergence of the series demands at least four terms. Moreover, as expected, this convergence is momentum dependent. We have also compared the first term of this expansion with the already considered on-shell, fixed threshold-kinematics, and static approximations for the pion propagator. These results are shown on Fig. 5. We realize that all these approximations are very near to the first-order term of the Taylor series. The corrections arising from higher order terms in the expansion are negligible only for low-momentum transfer, more precisely in the range $q_k < 100$ MeV.

The deviations of G^{st}, G^{fk}, and G^{on} from the effective propagator G_π given by Eq. (4) cannot explain the relatively large deviations obtained on the bottom-left panel of Fig. 3 between considered approximations and the reference result. These deviations follow from the ISI contribution. The weight of the ISI term depends on the *NN* interaction employed. It is comparable to the FSI term for V_σ (for which the deviations are large, as seen on the bottom-left panel of Fig. 3), but it is much less important for the full Bonn B potential (and therefore the corresponding deviations on the bottom-right panel of Fig. 3 are indeed much smaller).

All the findings for the amplitudes manifest themselves also in the results for the cross section. We show in Fig. 6 the effects of the considered approximations on the cross section, first taking only the FSI contribution. In the left panel the amplitude includes V_σ for the *NN* interaction; in the right panel the Bonn B T matrix is used. The curves compare the reference result (solid line in all panels) with the *S*-matrix results (upper panels) and their fixed threshold-kinematics version (lower panels). The *S*-matrix approach (dashed line) is the closest to the reference result (upper panels of Fig. 6).

For the case of the *NN* interaction described by V_σ we also show the result following from the *S*-matrix prescription applied to the *NN* interaction (dotted line on left panels in Fig. 6). For the fixed threshold-kinematics versions (bottom-left panel), the deviations from the reference result increase more pronouncedly with the excess energy Q, as expected. The approximations for the energy of the exchanged pion taken in the pion propagator G_π and in the rescattering vertex $f(\omega_\pi)$, but not in the kinematical factors of (3), overestimate the cross section by a factor of 5 (solid line with bullets).

Finally, we present in Fig. 7 the comparison between the approximated total cross sections with both FSI and ISI included. The approximation dictated by the *S*-matrix approach (dashed and dotted lines on the upper panels of Fig. 7) is clearly seen as the best one. For the Bonn potential calculation, it practically coincides with the reference result. As shown in the previous section, this procedure amounts to extend the on-shell approximation, used in Ref. [1] for G_π and $f(\omega_\pi)$ alone, also to the multiplicative kinematical factors showing up in the operator $\tilde{f}$ [see Eqs. (2)–(4)].

To conclude we notice, moreover, that for the realistic *NN* interaction, the difference between the *S*-matrix approach (upper right panel of Fig. 7) and its fixed threshold-kinematics version (lower right panel of the same figure) is not very important near threshold, provided that the excess energy does not exceed ≈30 MeV ($Q/m_\pi \sim 0.2$).

IV. CONCLUSIONS

a. The usual approximations to the effective pion propagator [1,6,9], obtained from a quantum-mechanical reduction of the Feynman diagram describing the pion rescattering process, are rather close to the first-order term of a Taylor series in a parameter measuring off-mass-shell effects in the intermediate states. The series converges rapidly for the FSI amplitude near threshold. As a consequence, retardation effects are not decisive in the pion rescattering mechanism near the threshold energy for pion production.

b. As for the pion energy in the πN rescattering amplitude, the on-shell approach when used only in $f(\omega_\pi)$ overestimates significantly the reference result. Nevertheless, and this is the key point of this article, this deviation is dramatically reduced if the approximation coming from the *S*-matrix approach is used consistently in the whole effective operator. This procedure amounts to extending the on-shell approximation used in Ref. [1] for G_π and $f(\omega_\pi)$ to the full operator $\tilde{f}$, including kinematical factors that differently weight the two dominant time-ordered diagrams. The amplitudes and cross sections obtained with the *S*-matrix effective operator are very close to those obtained with the time-ordered one in the considered kinematical region.

The rescattering operator for the neutral pion production in the isoscalar πN channel indeed seems to be relatively unimportant: its enhancement reported in previous articles followed from inconsistent or too crude (static or fixed threshold-kinematics) treatment of the energy dependence of the effective operator. Our findings explain why the calculation of Ref. [6], where the on-shell approximation is used, artificially enhances the contribution of the isoscalar rescattering term. Conversely, importantly and in retrospect, our results support the choice done in Refs. [7–9] for the different production operators considered.

The rescattering mechanism is filtered differently by other spin/isospin channels in pion production reactions. For charged pion production reactions the general irreducible rescattering operator comprises also the dominant isovector Weinberg-Tomozawa term of the πN amplitude, and its importance is therefore enhanced. Investigation of these channels within the approach of this article is in progress.

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