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# The effect of nonlocal confining kernels on magnetic chiral condensates

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**Abstract**

The physics of spontaneous chiral symmetry breaking in the case of the simultaneous presence of a magnetic field and a fermionic quartic interaction is discussed for both local and nonlocal kernels in $2+1$ and $3+1$ dimensions. The approach is based on the use of Valatin–Bogoliubov canonical transformations, which allow, in the absence of fermionic quartic terms, to completely diagonalize the Hamiltonian and construct the vacuum state.


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## 1. Introduction

During the past few years, the behavior of QCD and, more specifically, the problem of dynamical chiral symmetry breaking in the presence of a strong magnetic field have attracted some attention [1–3]. It has been shown that a constant magnetic field acts as a strong catalyst of dynamical chiral symmetry breaking, thus leading to the generation of a fermion mass [1,2]. The physics behind this effect is easy to understand: the motion of a charged particle is restricted to directions perpendicular to the magnetic field and this leads to a dimensional reduction ($d \to d-2$) in the dynamics of the fermion pairing. Different methods can be used to study this problem.

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Among them, the Schwinger proper time method [4] allows to obtain the exact expression for the fermion propagator $S(x, y)$ in a constant magnetic field and evaluate the condensate in a limiting procedure, $\langle 0|\bar{\psi}\psi|0\rangle = -\lim_{x\to y} \mathrm{Tr}\, S(x, y)$. In an alternative approach, one can develop the propagator in terms of Landau poles [5] and obtain the condensate as the contribution coming from the lowest Landau level (LLL). In this sense, the LLL plays a role similar to the Fermi surface in the Bardeen–Cooper–Schrieffer (BCS) superconductivity theory. The drawback of the above methods is the fact that they do not reveal explicitly the vacuum structure in terms of fermion (antifermion) creation and annihilation operators. On the other hand, not only calculations are simpler in the operator formalism but also the interpretation of the pairing structure as well as its nonperturbative nature are more transparent [6]. In this paper we adhere to the spirit of Ref. [6] and develop a general operator formalism necessary to study the vacuum structure and fermion condensation in systems unyielding to the usual mean-field approach. In particular, we study the mechanism of fermion condensation in models where, besides an external magnetic field, a strong local or nonlocal confining interaction is also present.

The paper is organized as follows: in Section 2 we introduce the formalism, based on the use of successive Valatin–Bogoliubov canonical transformations. In Section 3 we exemplify its use by working out the complete diagonalization of the Hamiltonian of a free fermion in the presence of a constant magnetic field for both the $(2 + 1)$- and $(3 + 1)$-dimensional cases. Section 4 is devoted to test our formalism against the well-known results of the local Nambu–Jona-Lasinio (NJL) model [7], setting the stage for the new results concerning nonlocal kernels in the presence of magnetic fields. In particular, we show how to translate the formulae and results obtained within the present formalism to the ones usually obtained through the use of mean-field techniques [1,2]. This translation is a validation of our formalism. The new and far more difficult case of a nonlocal fermion kernel with a constant magnetic field is dealt in Section 5. For simplicity, we have chosen a harmonic confinement. Within the present formalism other fermion kernels, notably the linear confinement kernel, could be considered as well. However, the linear kernel, which involves integro-differential mass-gap equations versus the simpler differential equations proper of the harmonic confinement, puts, for the purpose of this paper, quite unessential calculation complexities. The end of this section is devoted to show that this harmonic confining, extended NJL model, already behaves, in the presence of a magnetic field, quite differently from the local-kernel NJL model. Our conclusions are presented in Section 6.

## 2. The approach

The Lagrangian of a relativistic fermion in an external field $A_\mu$ has the standard form

$$\mathcal{L} = \bar{\psi}(x)\big[i\gamma^\mu D_\mu - m\big]\psi(x), \tag{1}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative.[1] Different gauges can be used to solve the Dirac equation in a constant external magnetic field. In what follows we choose the Landau gauge $A_\mu = -By\, \delta_{\mu 1}$, where $B$ is the magnetic field strength. In $2 + 1$ dimensions, the presence of a homogeneous magnetic field breaks polar symmetry explicitly, while in $3 + 1$ dimensions, the spherical symmetry is broken. As is well known, this fact leads to the use of Landau levels as the appropriate basis for the remaining symmetry. The energy spectrum is given by $E^2 =$

[1] We recall that in $2 + 1$ dimensions there are two inequivalent representations of the Dirac algebra, described by the matrices $\tilde{\gamma}^\mu$ and $-\tilde{\gamma}^\mu$, with $\tilde{\gamma}^0 = \sigma_3$, $\tilde{\gamma}^1 = i\sigma_1$, $\tilde{\gamma}^2 = i\sigma_2$; $\sigma_i$ are the Pauli matrices. The corresponding chiral version of the problem can then be formulated using the 4-dimensional spinor representation $\gamma^\mu = \mathrm{diag}(\tilde{\gamma}^\mu, -\tilde{\gamma}^\mu)$.

$m^2 + p_z^2 + (2n + 1 - s)|eB|$, where $s = \pm 1$ defines the spin orientation and $p_z$ is the momentum along the $z$-direction.[2] The discrete number $n = 0, 1, 2, \ldots$ describes the Landau levels.

The Hamiltonian of the problem is given by

$$H = \int d^D x\, \psi^\dagger(\boldsymbol{x})(\beta m + \boldsymbol{\alpha} \cdot \boldsymbol{p} - e\boldsymbol{\alpha} \cdot \boldsymbol{A})\psi(\boldsymbol{x}), \tag{2}$$

where $D$ is the space dimension and we use the standard notation $\beta = \gamma^0$ and $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$. In order to construct the Dirac field in a magnetic field we start from the plane wave decomposition of the free field,

$$\psi(\boldsymbol{x}) = \sum_{\boldsymbol{p},s} \frac{1}{\sqrt{V}} \left\{ u_s(\boldsymbol{p})\, a_{s\boldsymbol{p}} + v_s(\boldsymbol{p}) b^\dagger_{s-\boldsymbol{p}} \right\} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}, \tag{3}$$

where $V$ is the volume. Clearly, in $2 + 1$ dimensions the spin quantum number $s$ has no meaning and it should be omitted in all the corresponding formulae. The operators $a_{s\boldsymbol{p}}$ and $b_{s\boldsymbol{p}}$ satisfy the usual anticommutation relations. The $u$ and $v$ spinors are the solutions of the Dirac equation for positive and negative energies, respectively.

Since the quantum field of a particle in a magnetic field is usually developed in a basis indexed by the Landau levels $n$, a change of basis is needed in order to relate it with the plane wave solution. To achieve this, we first perform the canonical transformation which relates the spinors $u_{s'}(\boldsymbol{p}), v_{s'}(\boldsymbol{p})$ with the momentum-independent ($\boldsymbol{p} = 0$) spinors $\tilde{u}_s, \tilde{v}_s$,

$$\begin{pmatrix} \tilde{a}_{s\boldsymbol{p}} \\ \tilde{b}^\dagger_{s-\boldsymbol{p}} \end{pmatrix} = \sum_{s'} O_{ss'}(\boldsymbol{p}) \begin{pmatrix} a_{s'\boldsymbol{p}} \\ b^\dagger_{s'-\boldsymbol{p}} \end{pmatrix}, \qquad \begin{pmatrix} \tilde{u}_s \\ \tilde{v}_s \end{pmatrix} = \sum_{s'} O^*_{ss'}(\boldsymbol{p}) \begin{pmatrix} u_{s'}(\boldsymbol{p}) \\ v_{s'}(\boldsymbol{p}) \end{pmatrix}, \tag{4}$$

where $O_{ss'}(\boldsymbol{p})$ is the rotation matrix. The resulting field maintains the canonical structure (3). Next, in order to index the spinors and operators by the Landau levels, we expand the field in a complete set of Hermite polynomials $H_n$ through the identity

$$e^{ip_y y} = e^{-i\ell^2 p_x p_y} \sqrt{2\pi} \sum_{n=0}^{\infty} i^n \omega_n(\xi)\, \omega_n(\ell p_y), \tag{5}$$

where

$$\omega_n(x) = \left(2^n n! \sqrt{\pi}\right)^{-1/2} e^{-x^2/2} H_n(x) \tag{6}$$

and

$$\xi = y/\ell + \ell p_x, \tag{7}$$

$\ell = |eB|^{-1/2}$ is the magnetic length. The corresponding change of basis is given by

$$\begin{pmatrix} \hat{a}_{sn\bar{p}} \\ \hat{b}^\dagger_{sn-\bar{p}} \end{pmatrix} = \sum_{s',\, p_y} \tilde{T}_{nss'}(\ell p_y) \begin{pmatrix} \tilde{a}_{s'\boldsymbol{p}} \\ \tilde{b}^\dagger_{s'-\boldsymbol{p}} \end{pmatrix}, \qquad \begin{pmatrix} \hat{u}_{sn}(\xi) \\ \hat{v}_{sn}(\xi) \end{pmatrix} = \sum_{s'} T_{nss'}(\xi) \begin{pmatrix} \tilde{u}_{s'} \\ \tilde{v}_{s'} \end{pmatrix}, \tag{8}$$

where $\bar{p} = (p_x, p_z)$ in $3 + 1$ dimensions and $\bar{p} = p_x$ in $2 + 1$ dimensions, and the operators satisfy the usual anticommutation relations $\{\hat{a}^\dagger_{sn\bar{p}}, \hat{a}_{s'n'\bar{p}'}\} = \{\hat{b}^\dagger_{sn\bar{p}}, \hat{b}_{s'n'\bar{p}'}\} = \delta_{ss'}\delta_{nn'}\delta_{\bar{p}\bar{p}'}$. The matrices $\tilde{T}_n(\ell p_y)$ and $T_n(\xi)$ can be obtained after a simple algebra.

[2] In $2 + 1$ dimensions, $p_z = 0$ and $s = +1$ ($s = -1$) for $E > 0$ ($E < 0$).

The last step consists in obtaining the mass gap equations which lead to the diagonalization of the Hamiltonian. There are several approaches one can adopt to obtain the mass gap equation. It can be derived as the condition for the vacuum energy to be a minimum [8,9], or in the form of a Dyson equation for the fermion propagator [9], or as a Ward identity [10]. Here we shall derived it by imposing the vanishing of the anomalous (nondiagonal) terms in the Hamiltonian, corresponding to the direct creation (or annihilation) of a fermion–antifermion pair [11,12]. Aiming at this, we perform a second canonical transformation defined as

$$\begin{pmatrix} a_{sn\bar{p}} \\ b^{\dagger}_{sn-\bar{p}} \end{pmatrix} = \sum_{s'} R_{nss'}(p_z) \begin{pmatrix} \hat{a}_{s'n\bar{p}} \\ \hat{b}^{\dagger}_{s'n-\bar{p}} \end{pmatrix}, \qquad \begin{pmatrix} u_{sn}(\xi) \\ v_{sn}(\xi) \end{pmatrix} = \sum_{s'} R^{*}_{nss'}(p_z) \begin{pmatrix} \hat{u}_{s'n}(\xi) \\ \hat{v}_{s'n}(\xi) \end{pmatrix}, \tag{9}$$

where the rotation matrix $R_{nss'}(p_z)$ is obtained by requiring the anomalous terms in the Hamiltonian to vanish.

After this diagonalization procedure, the vacuum state in the magnetic field can be easily constructed. Indeed, each Valatin–Bogoliubov transformation on the operators and spinors corresponds to a transformation of the vacuum such that the latter is annihilated by the new operators. In the next sections we give explicit realizations of this construction.

The approach presented above can also be extended to the study of systems having, besides an external magnetic field, a quartic fermion interaction of the form

$$H_{\text{int}} = \int d^D x \int d^D y\, \bar{\psi}(\boldsymbol{x}) \Gamma \psi(\boldsymbol{x}) V\big(|\boldsymbol{x}-\boldsymbol{y}|\big) \bar{\psi}(\boldsymbol{y}) \Gamma \psi(\boldsymbol{y}), \tag{10}$$

where $\Gamma$ denotes the vertex (Dirac, color) structure, and $V$ is an attractive potential. This type of kernels has been extensively used in hadronic phenomenology [12,13]. In this case, we write the total Hamiltonian in normal-ordered form with respect to the new vacuum. Using the Wick contraction technique, the Hamiltonian can be decomposed in three terms,

$$\begin{aligned}
&H = {:}H_0{:} + {:}H_2{:} + {:}H_4{:}, \\
&{:}H_0{:} \propto \psi^{\dagger}_{\alpha} K \psi_{\alpha} + \psi^{\dagger}_{\alpha} \Gamma \psi_{\beta} V \psi^{\dagger}_{\beta} \Gamma \psi_{\alpha} + \psi^{\dagger}_{\alpha} \Gamma \psi_{\alpha} V \psi^{\dagger}_{\beta} \Gamma \psi_{\beta}, \\
&{:}H_2{:} \propto \psi^{\dagger}_{\alpha} K \psi_{\beta} + \psi^{\dagger}_{\alpha} \Gamma \psi_{\beta} V \psi^{\dagger}_{\gamma} \Gamma \psi_{\alpha} + \psi^{\dagger}_{\alpha} \Gamma \psi_{\alpha} V \psi^{\dagger}_{\beta} \Gamma \psi_{\gamma} \\
&\qquad + \psi^{\dagger}_{\alpha} \Gamma \psi_{\beta} V \psi^{\dagger}_{\beta} \Gamma \psi_{\gamma} + \psi^{\dagger}_{\alpha} \Gamma \psi_{\beta} V \psi^{\dagger}_{\gamma} \Gamma \psi_{\gamma}, \\
&{:}H_4{:} \propto \psi^{\dagger}_{\alpha} \Gamma \psi_{\beta} V \psi^{\dagger}_{\gamma} \Gamma \psi_{\delta},
\end{aligned} \tag{11}$$

containing 0, 2 and 4 creation and/or annihilation operators, respectively. In the presence of a magnetic field, $K = \beta m + \boldsymbol{\alpha} \cdot \boldsymbol{p} - e\boldsymbol{\alpha} \cdot \boldsymbol{A}$. The Greek indices refer to the two different Dirac fields, $\psi_1$ and $\psi_2$, corresponding to two inequivalent representations of the Dirac algebra in the $(2+1)$-dimensional case. In $3+1$ dimensions, such indexing is not necessary.

The contribution to ${:}H_2{:}$ coming from $H_{\text{int}}$ can be diagrammatically expressed in $2+1$ dimensions as

$$\begin{aligned}
&\psi^{\dagger}_1 \Gamma \psi_1 V \psi^{\dagger}_1 \Gamma \psi_1 + \psi^{\dagger}_2 \Gamma \psi_2 V \psi^{\dagger}_2 \Gamma \psi_2 \longrightarrow \psi^{\dagger}_1 [A-B] \psi_1 + \psi^{\dagger}_2 [C-D] \psi_2, \\
&\psi^{\dagger}_1 \Gamma \psi_1 V \psi^{\dagger}_2 \Gamma \psi_2 + \psi^{\dagger}_2 \Gamma \psi_2 V \psi^{\dagger}_1 \Gamma \psi_1 \longrightarrow \psi^{\dagger}_1 [D] \psi_1 + \psi^{\dagger}_2 [B] \psi_2, \\
&\psi^{\dagger}_1 \Gamma \psi_2 V \psi^{\dagger}_2 \Gamma \psi_1 + \psi^{\dagger}_2 \Gamma \psi_1 V \psi^{\dagger}_1 \Gamma \psi_2 \longrightarrow \psi^{\dagger}_1 [C] \psi_1 + \psi^{\dagger}_2 [A] \psi_2,
\end{aligned} \tag{12}$$

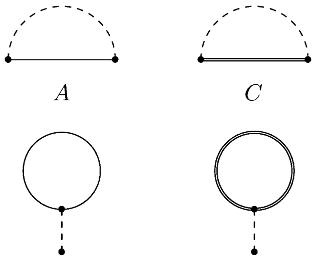


Fig. 1. Diagrams contributing to $:H_2:$. The dashed lines represent the interaction potential $V$ and the single (double) lines correspond to the fermion propagator of the field $\psi_1\,(\psi_2)$ in the case of $2+1$ dimensions.

where $A$, $B$, $C$ and $D$ correspond to the diagrams presented in Fig. 1. On the other hand, in $3+1$ dimensions the contribution is simply

$$\psi^\dagger \Gamma \psi V \psi^\dagger \Gamma \psi \longrightarrow \psi^\dagger [A - B]\psi. \tag{13}$$

Once the terms contributing to the quadratic part $:H_2:$ have been identified, its diagonalization proceeds through a sequence of Valatin–Bogoliubov transformations, as given in Eqs. (4), (8) and (9).

We note that, in the calculation of the diagrams of Fig. 1, the fermion propagators associated with $\psi_i$ $(i = 1, 2)$ in $2+1$ dimensions take the explicit form

$$S^{(i)}(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{\ell L_x} \sum_{n, p_x} \left[u_n^{(i)}(\xi_y)\, u_n^{(i)\dagger}(\xi_{y'}) - v_n^{(i)}(\xi_y)\, v_n^{(i)\dagger}(\xi_{y'})\right] e^{i(x-x')p_x}, \tag{14}$$

while in the $(3+1)$-dimensional case the propagator is given by

$$S(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{\ell L_x L_z} \sum_{n, \bar{p}, s} \left[u_{sn}(\xi_y) u_{sn}^\dagger(\xi_{y'}) - v_{sn}(\xi_y) v_{sn}^\dagger(\xi_{y'})\right] e^{i((x-x')p_x + (z-z')p_z)}. \tag{15}$$

## 3. Free fermions in a magnetic field

As a first example of the applicability of our approach, we consider the problem of a fermion in $2+1$ dimensions in the presence of a constant magnetic field. In this case, the rotation matrix $O(\boldsymbol{p})$ corresponding to the first canonical transformation (4) is simply given by

$$O(\boldsymbol{p}) = \begin{pmatrix} \cos\phi & -\sin\phi\,(\hat{p}_y + i\,\hat{p}_x) \\ \sin\phi\,(\hat{p}_y - i\,\hat{p}_x) & \cos\phi \end{pmatrix}, \qquad \hat{\boldsymbol{p}} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|}, \qquad \tan 2\phi = \frac{|\boldsymbol{p}|}{m}. \tag{16}$$

The vacuum associated to the operators $\tilde{a}_{\boldsymbol{p}}$ and $\tilde{b}_{\boldsymbol{p}}$ has the form

$$|\tilde{0}\rangle = \prod_{\boldsymbol{p}} \left(\cos\phi + \sin\phi\, a_{\boldsymbol{p}}^\dagger b_{-\boldsymbol{p}}^\dagger\right)|0\rangle, \tag{17}$$

so that $\tilde{a}_{\boldsymbol{p}}|\tilde{0}\rangle = 0$, $\tilde{b}_{\boldsymbol{p}}|\tilde{0}\rangle = 0$.

The second canonical transformation defined in Eq. (9) is performed with the rotation

$$R_n = \begin{pmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{pmatrix}, \tag{18}$$

where the angles $\theta_n$ are determined by imposing the vanishing of the anomalous terms in the Hamiltonian. A simple algebraic computation yields the following mass gap equations,

$$\begin{cases} m\sin 2\theta_0 = 0, & n = 0, \\ m\sin 2\theta_n - \ell^{-1}\sqrt{2n}\cos 2\theta_n = 0, & n > 0, \end{cases} \tag{19}$$

which for any value of $n$ have the solution

$$\tan 2\theta_n = \frac{\sqrt{2n}}{m\ell}. \tag{20}$$

It remains to construct the vacuum state in the magnetic field $|0\rangle_B$, which is annihilated by the operators $a_{np_x}$ and $b_{np_x}$. We find

$$|0\rangle_B = \prod_{n,p_x}\big(\cos\theta_n + \sin\theta_n \hat{a}^\dagger_{np_x}\hat{b}^\dagger_{n-p_x}\big)|\tilde{0}\rangle. \tag{21}$$

At this point, it is worth to comment on the question of dynamical symmetry breaking in the presence of the magnetic field [2,6]. This problem is connected with the appearance of a fermion condensate ${}_B\langle 0|\bar{\psi}\psi|0\rangle_B \neq 0$ in the limit where the mass parameter $m$ goes to zero. Writing the field $\psi$ in terms of the new canonical operators one can easily obtain

$$\begin{aligned} \lim_{m\to 0} {}_B\langle 0|\bar{\psi}\psi|0\rangle_B &= -\lim_{m\to 0}\frac{1}{2\pi\ell^2}\left(\cos 2\theta_0 + 2\sum_{n=1}^{\infty}\cos 2\theta_n\right) \\ &= -\lim_{m\to 0}\frac{1}{2\pi\ell^2}\left(1 + 2\sum_{n=1}^{\infty}\frac{m}{\sqrt{m^2+2n/\ell^2}}\right) = -\frac{|eB|}{2\pi}, \end{aligned} \tag{22}$$

i.e. the spontaneous breaking of the original $U(2)$ symmetry of the Lagrangian (1) occurs even in the absence of any additional interaction between fermions [2]. This result is specific to the $(2+1)$-dimensional Dirac theory in an external magnetic field.

Let us now consider the problem in $3+1$ dimensions. The rotation matrix $O_{ss'}(\boldsymbol{p})$ in Eq. (4) is given in this case by

$$O_{ss'}(\boldsymbol{p}) = \begin{pmatrix} \cos\phi\,\delta_{ss'} & -\sin\phi\, M_{ss'} \\ \sin\phi\, M^\dagger_{ss'} & \cos\phi\,\delta_{ss'} \end{pmatrix}, \qquad \tan 2\phi = \frac{|\boldsymbol{p}|}{m}, \qquad M = -(\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma})(i\sigma_2). \tag{23}$$

The new vacuum $|\tilde{0}\rangle$ can be generated from the $|0\rangle$ vacuum following the standard procedure and we find

$$\begin{aligned} |\tilde{0}\rangle &= \prod_{\boldsymbol{p}}\left(\frac{1+\cos 2\phi}{2} + \frac{\sin 2\phi}{2}C^\dagger_p + \frac{1-\cos 2\phi}{4}C^{\dagger 2}_p\right)|0\rangle, \\ C^\dagger_p &= \begin{pmatrix} a^\dagger_{\uparrow p} & a^\dagger_{\downarrow p} \end{pmatrix} M \begin{pmatrix} b^\dagger_{\uparrow -p} \\ b^\dagger_{\downarrow -p} \end{pmatrix}, \end{aligned} \tag{24}$$

which obviously satisfies $\tilde{a}_{s\boldsymbol{p}}|\tilde{0}\rangle = 0$, $\tilde{b}_{s\boldsymbol{p}}|\tilde{0}\rangle = 0$.

Next, we define the rotation matrix $R_{nss'}(p_z)$ of Eq. (9) as

$$R_{nss'}(p_z) = \begin{pmatrix} \cos\theta_n\,\delta_{ss'} & -\sin\theta_n M_{n\,ss'} \\ \sin\theta_n M^\dagger_{n\,ss'} & \cos\theta_n\,\delta_{ss'} \end{pmatrix}, \qquad M_n = \begin{pmatrix} -\sin\varphi_n & \cos\varphi_n \\ \cos\varphi_n & \sin\varphi_n \end{pmatrix}, \tag{25}$$

with $M_n$ chosen so that $M_n^\dagger M_n = \mathbb{1}$ and $\varphi_0 = 0$. Once more, the angles $\theta_n$ and $\varphi_n$ are determined by requiring the anomalous terms in the Hamiltonian to vanish. This leads to the following mass gap equations,

$$\begin{cases} m\sin 2\theta_0 - p_z\cos 2\theta_0 = 0, & n=0, \\ m\sin 2\theta_n M_n - \sqrt{p_z^2 + 2n/\ell^2}\,[\cos^2\theta_n \tilde{M}_n - \sin^2\theta_n M_n\tilde{M}_n M_n] = 0, & n>0, \end{cases} \tag{26}$$

where

$$\tilde{M}_n = \frac{1}{\sqrt{p_z^2 + 2n/\ell^2}} \begin{pmatrix} -\sqrt{2n}/\ell & p_z \\ p_z & \sqrt{2n}/\ell \end{pmatrix}. \tag{27}$$

The solution of the above equations is given for any value of $n$ by

$$\tan\varphi_n = \frac{\sqrt{2n}}{p_z\ell}, \qquad \tan 2\theta_n = \frac{\sqrt{p_z^2 + 2n/\ell^2}}{m}. \tag{28}$$

The vacuum state in the magnetic field reads in this case

$$|0\rangle_B = \prod_{n,\bar{p}} \left( \frac{1+\cos 2\theta_n}{2} + \frac{\sin 2\theta_n}{2} C_n^\dagger + \frac{1-\cos 2\theta_n}{4} C_n^{\dagger 2} \right) |\tilde{0}\rangle,$$

$$C_n^\dagger = \begin{pmatrix} \hat{a}^\dagger_{\uparrow n\bar{p}} & \hat{a}^\dagger_{\downarrow n\bar{p}} \end{pmatrix} M_n \begin{pmatrix} \hat{b}^\dagger_{\uparrow n-\bar{p}} \\ \hat{b}^\dagger_{\downarrow n-\bar{p}} \end{pmatrix}. \tag{29}$$

Let us now evaluate the fermion condensate ${}_B\langle 0|\bar{\psi}\psi|0\rangle_B$. In the limit when $m$ goes to zero we obtain

$$\begin{aligned} \lim_{m\to 0} {}_B\langle 0|\bar{\psi}\psi|0\rangle_B &= -\lim_{m\to 0} \frac{1}{2\pi\ell^2} \int \frac{dp_z}{2\pi} \left( \cos 2\theta_0 + 2\sum_{n=1}^{\infty} \cos 2\theta_n \right) \\ &= -\lim_{m\to 0} \frac{1}{2\pi^2\ell^2} \int_0^{\Lambda} dp_z \left( \frac{m}{\sqrt{m^2 + p_z^2}} + 2\sum_{n=1}^{\infty} \frac{m}{\sqrt{m^2 + p_z^2 + 2n/\ell^2}} \right) \\ &= -\lim_{m\to 0} \frac{|eB|}{4\pi^2} m \left[ \ln\frac{\Lambda^2}{m^2} + \mathcal{O}(m^0) \right] \to 0, \end{aligned} \tag{30}$$

where an ultraviolet cutoff $\Lambda$ has been introduced in order to regularize the integral. In contrast to the $(2+1)$-dimensional case, where a nonvanishing condensate appears in the limit when $m \to 0$ (see Eq. (22)), it is seen from Eq. (30) that this quantity vanishes in the $(3+1)$-dimensional case. Nevertheless, Eq. (30) indicates that a condensate in $3+1$ dimensions could in principle be generated in the presence of a small dynamical mass.

## 4. A test example: the NJL local kernel

At this stage it is convenient to test our formalism against the well-known results of the NJL model. This test, which is nontrivial, will then act as a validation criteria for the present formalism.

The Lagrangian density for the original NJL model in an external field reads

$$\mathcal{L} = \bar{\psi}(\boldsymbol{x})\left[i\gamma^{\mu} D_{\mu} - m\right]\psi(\boldsymbol{x}) + G\left[\left(\bar{\psi}(\boldsymbol{x})\psi(\boldsymbol{x})\right)^2 + \left(\bar{\psi}(\boldsymbol{x})i\gamma_5\psi(\boldsymbol{x})\right)^2\right]. \tag{31}$$

Let us start by discussing the $(2+1)$-dimensional NJL model. We recall that in $2+1$ dimensions the $\gamma_5$ matrix in the chiral representation is defined as

$$\gamma^5 = i\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{32}$$

where $\mathbb{1}$ is the $2\times 2$ unit matrix. The contribution of $H_{\mathrm{int}}$ to $:H_2:$ is easily obtained from Eq. (12),

$$\psi_1^{\dagger}\left[S^{(1)} + S^{(2)} - \mathrm{Tr}\left(S^{(1)} - S^{(2)}\right)\right]\psi_1 + (1 \leftrightarrow 2), \tag{33}$$

where the fermion propagators are given by (cf. Eq. (14))

$$S^{(i)} = \frac{1}{\ell L_x}\sum_{n,\, p_x}\left[u_n^{(i)}(\xi)u_n^{(i)\dagger}(\xi) - v_n^{(i)}(\xi)v_n^{(i)\dagger}(\xi)\right]. \tag{34}$$

Following the procedure of Section 2, the diagonalization of the Hamiltonian $:H_2:$ through Valatin–Bogoliubov transformations yields the following mass gap equations:

$$\begin{cases} (m+\rho)\,\sin 2\theta_0 = 0, & n = 0, \\ (m+\rho)\sin 2\theta_n - \ell^{-1}\sqrt{2n}\cos 2\theta_n = 0, & n > 0, \end{cases} \tag{35}$$

where

$$\rho = \frac{G}{\pi\ell^2}\left(\cos 2\theta_0 + 2\sum_{n=1}^{\infty}\cos 2\theta_n\right). \tag{36}$$

The solution of these equations is

$$\tan 2\theta_n = \frac{\sqrt{2n}}{(m+\rho)\ell}, \tag{37}$$

which when substituted back into Eq. (36) yields

$$\frac{\pi\ell^2}{G}\rho = 1 + 2\sum_{n=1}^{\infty}\frac{m+\rho}{\sqrt{2n/\ell^2 + (m+\rho)^2}}. \tag{38}$$

Using the identity [14]

$$2\sum_{n=0}^{\infty}\frac{1}{(2n+\beta)^{\mu}} = \beta^{-\mu} + \frac{1}{\Gamma(\mu)}\int_0^{\infty} ds\, s^{\mu-1}e^{-\beta s}\coth s, \tag{39}$$

we can recast Eq. (38) in the form

$$\frac{\pi^{3/2}\ell}{G}\rho = (m+\rho)\int_0^{\infty}\frac{ds}{s^{1/2}}e^{-(m+\rho)^2\ell^2 s}\coth s, \tag{40}$$

which reproduces the result obtained in leading order of the $1/N$ expansion [2].

In the weak coupling regime, i.e., when $G \ll \ell$, one expects the lowest Landau level to dominate. In this case, Eq. (38) implies

$$\rho = \frac{G}{\pi\ell^2} = G\frac{|eB|}{\pi}. \tag{41}$$

Recalling that $\langle 0|\bar{\psi}\psi|0\rangle = -\rho/(2G)$, we find $\langle 0|\bar{\psi}\psi|0\rangle = -|eB|/(2\pi)$. We notice that this result coincides with the value of the condensate previously obtained in Eq. (22), in the absence of the NJL interaction.

Let us now consider the $(3+1)$-dimensional NJL model. In this case, the contribution to $:H_2:$ is calculated from Eq. (13) and one obtains

$$\psi^\dagger\big[\gamma_0 S\gamma_0 - \gamma_0 \mathrm{Tr}(S\gamma_0) - \gamma_0\gamma_5 S\gamma_0\gamma_5 + \gamma_0\gamma_5\,\mathrm{Tr}(S\gamma_0\gamma_5)\big]\psi, \tag{42}$$

where, according to Eq. (15), the fermion propagator is

$$S = \sum_{n,\bar{p}} \frac{1}{\ell L_x L_z} \sum_s \big[u_{sn}(\xi)u^\dagger_{sn}(\xi) - v_{sn}(\xi)v^\dagger_{sn}(\xi)\big]. \tag{43}$$

Once again, $:H_2:$ is diagonalized by successive Valatin–Bogoliubov transformations. For the sake of comparison with the results obtained using the $1/N$-expansion technique, we shall assume that pions do not condensate. In this case, we end up with the following mass gap equations:

$$\begin{cases} (m+\rho)\sin 2\theta_0 - (p_z+\kappa)\cos 2\theta_0 = 0, & n=0, \\ (m+\rho)\sin 2\theta_n M_n - \sqrt{p_z^2+2n/\ell^2}(\cos^2\theta_n \tilde{M}_n - \sin^2\theta_n M_n\tilde{M}_n M_n) & \\ \quad -\kappa[\cos^2\theta_n\sigma_1 + \sin^2\theta_n(\sin 2\varphi_n\sigma_3 - \cos 2\varphi_n\sigma_1)] = 0, & n>0, \end{cases} \tag{44}$$

where $M_n$ and $\tilde{M}_n$ are defined in Eqs. (25) and (27), respectively;

$$\rho = \frac{G}{\pi\ell^2}\int\frac{dp_z}{2\pi}\left(\cos 2\theta_0 + 2\sum_{n=1}^{\infty}\cos 2\theta_n\right),$$

$$\kappa = \frac{G}{\pi\ell^2}\int\frac{dp_z}{2\pi}\left(\sin 2\theta_0 + 2\sum_{n=1}^{\infty}\sin 2\theta_n\cos\varphi_n\right). \tag{45}$$

The solution of these equations is given by

$$\tan\varphi_n = \frac{\sqrt{2n}}{(p_z+\kappa)\ell}, \qquad \tan 2\theta_n = \frac{\sqrt{(p_z+\kappa)^2+2n/\ell^2}}{m+\rho}. \tag{46}$$

Thus, from Eq. (45) we find

$$\frac{2\pi^2\ell^2}{G}\rho = \int dp_z\left(\frac{m+\rho}{\sqrt{p_z^2+(m+\rho)^2}} + 2\sum_{n=1}^{\infty}\frac{m+\rho}{\sqrt{2n/\ell^2+p_z^2+(m+\rho)^2}}\right), \tag{47}$$

which can be recast in the integral form

$$\frac{2\pi^2\ell^2}{G}\rho = (m+\rho)\int_0^\infty \frac{ds}{s} e^{-(m+\rho)^2\ell^2 s}\coth s, \tag{48}$$

through the identity (39).

In the weak coupling regime, when the LLL dominance approximation remains valid, from Eq. (47) we obtain in the limit $m\to 0$,

$$\frac{2\pi^2\ell^2}{G}\rho = \rho\ln\frac{\Lambda^2}{\rho^2}, \tag{49}$$

where $\Lambda$ is an ultraviolet cutoff which regularizes the integral in the momentum $p_z$. This result is consistent with the one previously found in the absence of the NJL interaction term (cf. Eq. (30)) and agrees with the results obtained by the use of other approaches [2]. We also notice that Eq. (49) gives the correct behavior near the singularity at $G = 0$,

$$\rho^2 = \Lambda^2 \exp\left(-\frac{2\pi^2}{G|eB|}\right). \tag{50}$$

On the other hand, as $G$ increases the expression (50) gets less accurate. This is due to the fact that for larger values of $G$ all the Landau levels give important contributions to the dynamical mass and the LLL approximation is no longer valid.

It is worth emphasizing that in the local NJL model the nontrivial chiral condensate is generated in the presence of an arbitrary small external magnetic field for arbitrary small values of the coupling constant. It turns out, as we shall see shortly, that this picture ceases to be valid when, in the presence of a homogeneous magnetic field $B$, we go from a local to a nonlocal fermionic kernel. In this case we cease to have a weak coupling limit no matter how small we set the kernel strength.

## 5. Nonlocal kernels: the harmonic oscillator case

The $(2+1)$-dimensional case of a nonlocal quartic kernel (given in this example, for the sake of simplicity, by a harmonic-oscillator kernel) constitutes, in the presence of a magnetic field, the first nontrivial application of our formalism.

The harmonic-oscillator quartic interaction is given by

$$H_{\text{int}} = \int d^2x\, d^2y\, \bar{\psi}(\boldsymbol{x})\gamma_0 \frac{\lambda^a}{2} \psi(\boldsymbol{x}) \frac{3}{4} K|\boldsymbol{x}-\boldsymbol{y}|^2 \bar{\psi}(\boldsymbol{y})\gamma_0 \frac{\lambda^a}{2} \psi(\boldsymbol{y}), \tag{51}$$

where $K$ is the interaction constant and $\lambda^a$ are the Gell-Mann matrices, which account for the color structure of the interaction. This structure prevents the appearance of tadpole-like terms. Notice that $H_{\text{int}}$ has a central (polar) symmetry.

In the absence of a magnetic field, the harmonic-oscillator nonlocal NJL Hamiltonian can be readily diagonalized through a "2D polar-symmetric" Valatin–Bogoliubov canonical transformation. The other limit, i.e., having a magnetic field and no quartic fermion interaction, has been already studied in the present paper and leads to a condensate with nonpolar, axial-like symmetry. It is therefore clear that we cannot accommodate both symmetries simultaneously: if we had started from a sufficiently strong magnetic field, and progressively turned on a quartic nonlocal interaction, there will be a value of the quartic kernel strength where the fermion condensate cannot hold any longer and the system becomes disordered.

The contribution to the Hamiltonian term $:H_2:$ in Eq. (12) is represented by the diagrams $A$ and $C$ of Fig. 1,

$$\psi_1^\dagger(\boldsymbol{x}) S^{(1)}(\boldsymbol{x},\boldsymbol{y}) V\big(|\boldsymbol{x}-\boldsymbol{y}|\big)\psi_1(\boldsymbol{y}) + (1 \leftrightarrow 2). \tag{52}$$

Here, $S^{(1)}(\boldsymbol{x},\boldsymbol{y})$ is the propagator defined in Eq. (14). The tadpole diagrams $B$ and $D$ do not contribute since the $\lambda^a$ matrices are traceless. Moreover, there is neither a contribution from diagram $C$ to the $\psi_1$ term nor a contribution from diagram $A$ to the $\psi_2$ term, because the $\gamma_0$ vertex is block diagonal.

Applying successive Valatin–Bogoliubov transformations and making use of the simple relations

$$(\xi - \partial_\xi)\omega_n(\xi) = \sqrt{2(n+1)}\omega_{n+1}(\xi),$$
$$(\xi + \partial_\xi)\omega_n(\xi) = \sqrt{2n}\omega_{n-1}(\xi), \tag{53}$$

with $\omega_n(\xi)$ defined in Eq. (6), we arrive at the following recurrence system of mass gap equations:

$$\begin{cases} \left[\frac{m}{2} + K\ell^2(\cos 2\theta_0 + \cos 2\theta_1)\right]\sin 2\theta_0 = 0, & n = 0, \\ m \sin 2\theta_n - \sqrt{2n}/\ell \cos 2\theta_n & \\ \quad + K\ell^2[-((2n-1)\cos 2\theta_{n-1} + (2n+1)\cos 2\theta_{n+1})\sin 2\theta_n & \\ \quad + 2\sqrt{n}(\sqrt{n-1}\sin 2\theta_{n-1} + \sqrt{n+1}\sin 2\theta_{n+1})\cos 2\theta_n] = 0, & n > 0. \end{cases} \tag{54}$$

We impose the boundary condition $\theta_0 = 0$, so that the standard expression for the $n = 0$ spinor, $u_0(\xi) = (\omega_0(\xi), 0)^T$, is recovered. We also note that Eqs. (54) enforce $\theta_n \to \pi/4$ as $n \to \infty$.

Using the second equation in (54) we can calculate the chiral angle $\theta_{n+1}$ from $\theta_n$ and $\theta_{n-1}$. This leaves $\theta_1$ as a free parameter. By varying $\theta_1$, we can find the solution that minimizes the vacuum energy,

$$H_0 = -m\left(\cos 2\theta_0 + 2\sum_{n=1}^{\infty}\cos 2\theta_n\right) + K\ell^2\sin^2 2\theta_0 - \frac{2}{\ell}\sum_{n=1}^{\infty}\sqrt{2n}\sin 2\theta_n. \tag{55}$$

The nonlinear mass gap equations (54) have no obvious analytical solutions and we have to solve them numerically. As an example, we present in Fig. 2 their numerical solution for $m = 0$ and $K\ell^3 = 0.5$. Having obtained the chiral angles we can proceed to calculate physical quantities. The chiral condensate $\langle\bar{\psi}\psi\rangle$ was numerically shown to converge for each value of $K\ell^3$ and, therefore, no renormalization is needed. This is in contradistinction to the NJL local kernel, which needs to be regularized. The results are shown in Fig. 3. Notice that when $K\ell^3 \to 0$ we recover the value of the condensate previously obtained without the harmonic-oscillator interaction, i.e.,

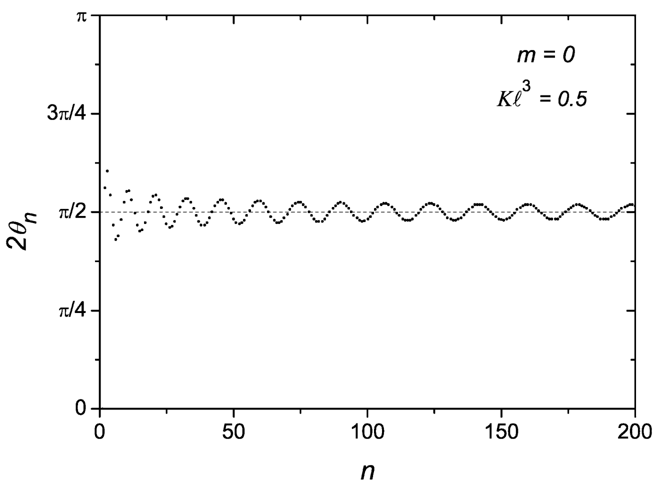


Fig. 2. Chiral angle as a function of the Landau level $n$ for $m = 0$ and $K\ell^3 = 0.5$ in the presence of a harmonic-oscillator potential and a constant magnetic field in $2 + 1$ dimensions.

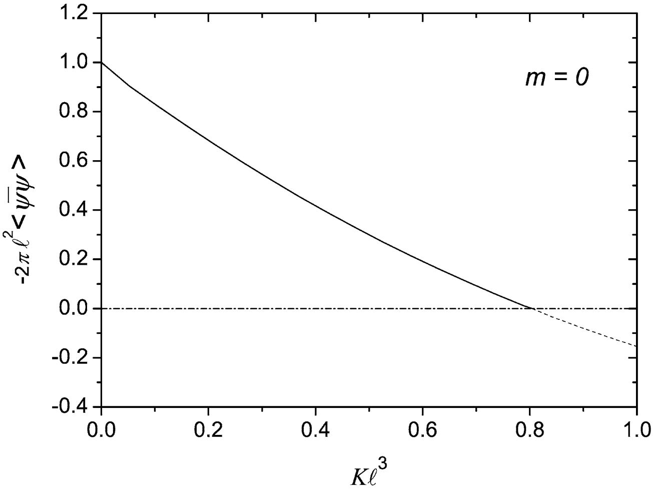


Fig. 3. Chiral condensate as a function of the dimensionless parameter $K\ell^3$ for $m = 0$, in the presence of a harmonic-oscillator potential and a constant magnetic field in $2 + 1$ dimensions.

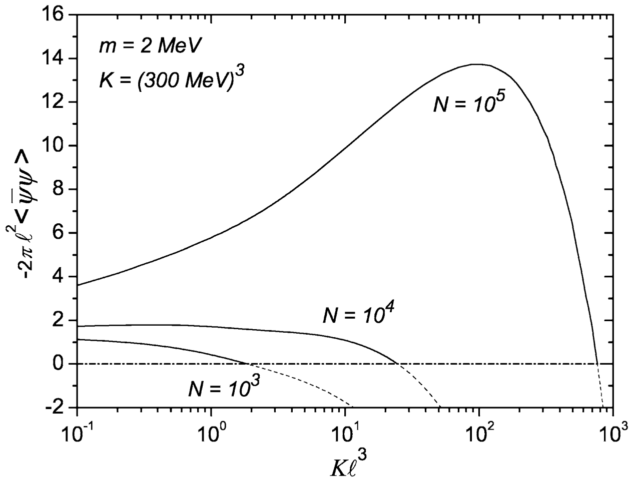


Fig. 4. Chiral condensate as a function of the dimensionless parameter $K\ell^3$ for a current mass $m = 2$ MeV, in the presence of a harmonic-oscillator potential and a constant magnetic field in $2 + 1$ dimensions.

$-2\pi\ell^2\langle 0|\bar{\psi}\psi|0\rangle = 1$. When $K\ell^3 \simeq 0.8$ the chiral condensate vanishes.[3] This signals the onset of a phase transition due to the presence of both the magnetic field and the fermionic quartic kernel (in this case, a confining harmonic-oscillator kernel).

In the presence of a nonvanishing current fermion mass, $m \neq 0$, the picture is quite different. As an example, in Fig. 4 we present a plot of the chiral condensate for $m = 2$ MeV. Notice that

[3] Taking, for instance, $K = (300\text{ MeV})^3$, a value of $K\ell^3 = 0.8$ would correspond to a magnetic field strength of $B = 1.5 \times 10^{18}$ G.

the chiral condensate diverges with the number $N$ of Landau levels and, consequently, a renormalization scheme is needed for this case.

## 6. Conclusion

In this paper we have studied the problem of spontaneous chiral symmetry breaking in the case of the simultaneous presence of a homogeneous magnetic field and a fermionic quartic interaction, for both local and nonlocal kernels in $2+1$ and $3+1$ dimensions. The operator formalism presented here is based on the use of a sequence of Valatin–Bogoliubov canonical transformations, which allowed us to construct explicitly the vacuum of the system and also to calculate the chiral condensate. As illustrative examples, we have considered the original NJL model and the harmonic-oscillator potential. In the case of the NJL model we were able to recover the well-known mean-field results. For the $(2+1)$ harmonic oscillator, we have shown that in the presence of the magnetic field the system (with nonlocal interaction) exhibits new properties, namely, the existence of a critical magnetic field $B_c$ such that for $B > B_c$ a chiral condensate is induced. It would interesting to extend this analysis to the $(3+1)$-dimensional case. However, in the latter case the mass gap equations turn out to be not only recurrent but also differential and, therefore, their solution is more involved.

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