

# Charged-fermion masses in $SO(10)$ : Analysis with scalars in $\mathbf{10} \oplus \mathbf{120}$

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## Abstract

We consider the scenario in which the mass matrices of the charged fermions in the  $SO(10)$  grand unified theory are generated exclusively by renormalizable Yukawa couplings to one  $\mathbf{10} \oplus \mathbf{120}$  representation of scalars. We analyze, partly analytically and partly numerically, this scenario in the three-generations case. We demonstrate that it leads to unification of the  $b$  and  $\tau$  masses at the GUT scale. Testing this scenario against the mass values at the GUT scale, obtained from the renormalization-group evolution in the minimal SUSY extension of the Standard Model, we find that it is not viable: either the down-quark mass or the top-quark mass must be unrealistically low. If we include the CKM mixing angles in the test, then, in order that the mixing angles are well reproduced, either the top-quark mass or the strange-quark mass together with the down-quark mass must be very low. We conclude that, assuming a SUSY  $SO(10)$  scenario, charged-fermion mass generation based exclusively on one  $\mathbf{10} \oplus \mathbf{120}$  representation of scalars is in contradiction with experiment.

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## 1. Introduction

All the fermionic multiplets of one Standard Model generation, plus one right-handed neutrino singlet, fit exactly into the 16-dimensional irreducible representation of the grand unification group  $SO(10)$ . This is the unique and distinguishing feature of the unified gauge theories (GUTs)

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based on this group [1]. As a bonus, the presence of three (for three generations, as we shall assume in this paper) right-handed neutrino singlets allows one to incorporate into this GUT the seesaw mechanism of type I [2].

However, when it comes to the scalar sector and to fermion mass generation the uniqueness of the  $SO(10)$  GUT is lost and numerous ramifications exist. One possible strategy to limit the freedom in the scalar sector is to confine oneself to renormalizable terms—for a review see, for instance, [3]. In that case, the scalar representations coupling to the fermions are determined by the relation [4,5]

$$\mathbf{16} \otimes \mathbf{16} = (\mathbf{10} \oplus \mathbf{126})_S \oplus \mathbf{120}_{AS}, \quad (1)$$

where the subscripts “S” and “AS” denote, respectively, the symmetric and antisymmetric parts of the tensor product. Thus, scalars with renormalizable Yukawa couplings to the fermions must transform under  $SO(10)$  either as  $\mathbf{10}$ ,  $\overline{\mathbf{126}}$ , or  $\mathbf{120}$  (the  $\mathbf{10}$  and  $\mathbf{120}$  are real representations; the  $\mathbf{126}$  is complex). A minimal supersymmetric (SUSY) scenario—which has built-in the gauge-coupling unification of the minimal SUSY extension of the Standard Model (MSSM)—making use of one  $\mathbf{10}$  and one  $\overline{\mathbf{126}}$  for the Yukawa couplings [6] has recently received a lot of attention. This attention was triggered by the observation [7] that maximal atmospheric neutrino mixing may in this theory be related to  $b$ – $\tau$  unification via the type II seesaw mechanism [8]. Detailed and elaborate studies of this minimal theory have been performed for its Yukawa couplings [9–12] and scalar potential [13,14]. This “minimal SUSY  $SO(10)$  GUT” works very well, since its Yukawa couplings are able to fit all fermion masses and mixings, allowing in particular for small quark mixings simultaneously with large leptonic mixings. However, in this context a minimal Higgs scalar sector is too constrained [15] and does not allow to produce large enough neutrino masses [16].

As a way out, the  $\mathbf{120}$  scalar representation—which had been somewhat arbitrarily left out—may be used for a rescue [17,18]. In [10,11] that representation was only taken as a perturbation of the minimal scenario, to cure minor deficiencies in the fermionic sector. However, in [19] it was pointed out that the antisymmetric coupling matrix of the  $\mathbf{120}$  could be responsible for the different features of quark and lepton mixing, since that matrix has different weights in all four Dirac-type mass matrices—i.e., in the Dirac mass matrices for the up-type quarks, down-type quarks, charged leptons, and neutrinos. Thinking along this line, the roles of the  $\mathbf{120}$  and  $\overline{\mathbf{126}}$  could be interchanged in the charged-fermion sector: the brunt could be borne by one  $\mathbf{10} \oplus \mathbf{120}$ , and the Yukawa couplings of the  $\overline{\mathbf{126}}$  would be just a perturbation. This thought is realized in the model of [17], where the scalar  $\overline{\mathbf{126}}$  is still a protagonist in the neutrino sector, through the type I seesaw mechanism.

In this paper we investigate the extreme form of this scenario of [17], namely we assume that the  $\overline{\mathbf{126}}$  plays no role whatsoever in the Yukawa couplings to the charged fermions,<sup>1</sup> and may be important only in the neutrino sector, where it would be responsible for large Majorana neutrino masses. Thus, we base our investigation on the following assumptions:

- (i) The charged-fermion mass matrices result solely from the Yukawa couplings of one  $\mathbf{10}$  and one  $\mathbf{120}$  scalar multiplets.
- (ii) The mechanism for the generation of the light-neutrino mass matrix is the type I seesaw mechanism, possibly with some admixture of type II.

<sup>1</sup> This idea was previously put forward in [20].

Due to the first assumption, the Yukawa-coupling matrix of the  $\overline{\mathbf{126}}$  can be used freely for the neutrino mass matrix and, therefore, one can accommodate any neutrino masses and lepton mixing that one wants, through either the type I or type II seesaw mechanisms. The tight connection between the charged-fermion and neutrino sectors is lost, and the predictive power of the model for the neutrino sector too. The subject of this paper is then only the discussion of the charged-fermion masses and of quark (CKM) mixing under the assumption (i), and the working out of where this assumption is successful and where it might fail.

The charged-fermion sector and the Yukawa couplings in [18] coincide with ours. Still, our results do *not*, in general, apply to that model. The reason is that its authors assume split supersymmetry, where the renormalization-group evolution of the fermion masses differs from the one of the MSSM. Indeed, in order to test any specific scenario one must use the charged-fermion masses and the quark mixings at the GUT scale. Having in mind a SUSY  $SO(10)$  GUT and the MSSM, we use in this paper the values computed in [21] with the renormalization-group evolution of the MSSM.

This paper is organized as follows. In Section 2 we discuss the mass matrices and count the number of parameters. Basis-invariant quantities are introduced in Section 3. The derivation of some inequalities, and  $b$ - $\tau$  unification, are discussed in Section 4. In Section 5 we show that a partly analytical treatment of our scenario is possible when the Yukawa-coupling matrices are assumed to be real. Section 6 explains our procedure for the numerical fit of the mass matrices to the fermion masses and to the CKM mixing angles at the GUT scale. We present our results in Section 7, which is followed by a brief summary in Section 8.

## 2. The charged-fermion mass matrices

The mass Lagrangian that we are concerned with is

$$\mathcal{L}_M = -\bar{d}_L M_d d_R - \bar{\ell}_L M_\ell \ell_R - \bar{u}_L M_u u_R + \text{H.c.} \quad (2)$$

The symmetric and antisymmetric Yukawa couplings of one  $\mathbf{10}$  and one  $\mathbf{120}$  scalar representations, respectively [4], generate the mass matrices, which at the GUT scale may be parametrized as

$$\begin{aligned} M_d &= S + e^{i\psi} A, \\ M_\ell &= S + r e^{i\theta} A, \\ M_u &= pS + q e^{i\xi} A, \end{aligned} \quad (3)$$

$S$  being symmetric while  $A$  is antisymmetric. The parameters  $p$ ,  $q$ , and  $r$  are real and positive. The matrix  $S$  is proportional to the Yukawa-coupling matrix of the  $\mathbf{10}$ , while  $A$  is proportional to the Yukawa couplings of the  $\mathbf{120}$ . The factors  $e^{i\psi}$ ,  $r e^{i\theta}$ ,  $p$ , and  $q e^{i\xi}$  depend on some ratios of vacuum expectation values.

We may perform changes of weak basis

$$\begin{aligned} S &\rightarrow U S U^T, \\ A &\rightarrow U A U^T, \end{aligned} \quad (4)$$

where  $U$  is unitary. In this way we may reach convenient weak bases. We may for instance use  $U$  to diagonalize  $S$ :

$$S = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}, \quad (5)$$

with real and nonnegative  $a$ ,  $b$ , and  $c$ . Alternatively, we may use  $U$  to force  $A$  to have only two nonzero matrix elements, and moreover two matrix elements of  $S$  to vanish:

$$S = \begin{pmatrix} a & f & 0 \\ f & b & d \\ 0 & d & c \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & -x & 0 \end{pmatrix}. \quad (6)$$

In the weak basis (6), again, we may choose  $a$ ,  $b$ , and  $c$  to be real and nonnegative.

As for the number of degrees of freedom in the mass matrices (3), we consider two cases:

- In the complex-Yukawa-couplings case, the matrices  $S$  and  $A$  are a priori complex and contain nine independent matrix elements (six in  $S$  and three in  $A$ ), hence nine moduli and nine phases. One of three phases  $\psi$ ,  $\theta$ , and  $\xi$  may be absorbed in the definition of  $A$ . Through a weak-basis transformation we may eliminate the three moduli and six phases which parameterize  $U$ . In that case the model has, therefore, nine real parameters and five phases.
- In the real-Yukawa-couplings case, in which CP violation is considered to be spontaneous, the matrices  $S$  and  $A$  are a priori real and contain nine independent moduli. If we want to preserve the reality of  $S$  and  $A$ , the matrix  $U$  of the weak-basis transformation (4) must be chosen real (orthogonal),<sup>2</sup> hence it contains three real parameters. One ends up with nine real parameters as before, but only three phases.

With these 14 (in the complex case) or 12 (in the real case) parameters we must try and fit 13 observables: nine charged-fermion masses and four parameters of the CKM matrix. Even if there is, in the complex case, an excessive number of parameters, the fitting may prove impossible, due to the fact that a large number of those parameters are phases.

### 3. Fermion masses and invariants

We first confine ourselves to the masses. For brevity of notation we introduce

$$\begin{aligned} \sigma_d &= m_d^2 + m_s^2 + m_b^2, & \rho_d &= m_d^2 m_s^2 + m_s^2 m_b^2 + m_b^2 m_d^2, & \pi_d &= m_d^2 m_s^2 m_b^2, \\ \sigma_\ell &= m_e^2 + m_\mu^2 + m_\tau^2, & \rho_\ell &= m_e^2 m_\mu^2 + m_\mu^2 m_\tau^2 + m_\tau^2 m_e^2, & \pi_\ell &= m_e^2 m_\mu^2 m_\tau^2, \\ \sigma_u &= m_u^2 + m_c^2 + m_t^2, & \rho_u &= m_u^2 m_c^2 + m_c^2 m_t^2 + m_t^2 m_u^2, & \pi_u &= m_u^2 m_c^2 m_t^2. \end{aligned} \quad (7)$$

We define the matrices  $H_a \equiv M_a M_a^\dagger$  ( $a = d, \ell, u$ ), which have eigenvalue equations

$$\det(m^2 \mathbb{1} - H_a) = m^6 - \sigma_a m^4 + \rho_a m^2 - \pi_a = 0. \quad (8)$$

With the mass matrices (3) we obtain the relations

$$\sigma_d = s_2 + 2a_2, \quad (9)$$

$$\sigma_\ell = s_2 + 2r^2 a_2, \quad (10)$$

$$\sigma_u = p^2 s_2 + 2q^2 a_2, \quad (11)$$

<sup>2</sup> If  $S$  and  $U$  are assumed to be real, then one may obtain a weak basis of the form (5), but  $a$ ,  $b$ , and  $c$  must be allowed to be negative. It is only when we allow  $U$  to include some  $i$  factors that we may obtain nonnegative  $a$ ,  $b$ , and  $c$ ; but then  $x$ ,  $y$ , and  $z$  will not necessarily be real.

where

$$s_2 = \text{tr}(SS^*), \quad (12)$$

$$a_2 = -\frac{1}{2} \text{tr}(AA^*); \quad (13)$$

also,

$$\rho_d = s_4 + a_2^2 + 2z_4 + 2 \text{Re}(e^{2i\psi} \bar{z}_4), \quad (14)$$

$$\rho_\ell = s_4 + r^4 a_2^2 + 2r^2 z_4 + 2r^2 \text{Re}(e^{2i\theta} \bar{z}_4), \quad (15)$$

$$\rho_u = p^4 s_4 + q^4 a_2^2 + 2p^2 q^2 z_4 + 2p^2 q^2 \text{Re}(e^{2i\xi} \bar{z}_4), \quad (16)$$

where

$$s_4 = \frac{1}{2} [s_2^2 - \text{tr}(SS^* SS^*)], \quad (17)$$

$$z_4 = s_2 a_2 + \text{tr}(SS^* AA^*), \quad (18)$$

$$\bar{z}_4 = -\frac{1}{2} \text{tr}(AS^* AS^*); \quad (19)$$

finally,

$$\pi_d = |s_3 + e^{2i\psi} z_3|^2, \quad (20)$$

$$\pi_\ell = |s_3 + r^2 e^{2i\theta} z_3|^2, \quad (21)$$

$$\pi_u = |p^3 s_3 + p q^2 e^{2i\xi} z_3|^2, \quad (22)$$

where

$$s_3 = \det S, \quad (23)$$

$$z_3 = \text{tr}(SA^2) - \frac{1}{2} \text{tr} S \text{tr}(A^2). \quad (24)$$

#### 4. $b$ - $\tau$ unification

In [18] the mass matrices for the charged-fermion sector are the same as in this paper, but the discussion is confined to the two-generations case. In that paper, approximate  $b$ - $\tau$  unification is traced back to some inequalities derived from the specific structure of the mass matrices. Here we show that analogous inequalities hold in the three-generations case.

It is convenient to use the weak basis of Eq. (5). We remind that, in that weak basis,  $a$ ,  $b$ , and  $c$  are real and nonnegative, while  $x$ ,  $y$ , and  $z$  are in general complex. One has

$$s_2 = a^2 + b^2 + c^2, \quad (25)$$

$$a_2 = |x|^2 + |y|^2 + |z|^2, \quad (26)$$

$$s_4 = a^2 b^2 + b^2 c^2 + c^2 a^2, \quad (27)$$

$$z_4 = a^2 |x|^2 + b^2 |y|^2 + c^2 |z|^2, \quad (28)$$

$$\bar{z}_4 = bcx^2 + cay^2 + abz^2, \quad (29)$$

$$s_3 = abc, \quad (30)$$

$$z_3 = ax^2 + by^2 + cz^2. \quad (31)$$

Note that  $z_3$  and  $\bar{z}_4$  are in general complex, while the other parameters are real.

From Eqs. (16), (29), and (28) we derive

$$\rho_u \geq p^4 s_4 + q^4 a_2^2 + 2p^2 q^2 z_4 - 2p^2 q^2 (bc|x|^2 + ca|y|^2 + ab|z|^2) \quad (32)$$

$$= p^4 s_4 + q^4 a_2^2 + 2p^2 q^2 [(a^2 - bc)|x|^2 + (b^2 - ca)|y|^2 + (c^2 - ab)|z|^2]. \quad (33)$$

Without loss of generality we assume that

$$b \geq a, \quad c \geq a. \quad (34)$$

Since  $a + b + c$  is nonnegative, the inequalities (34) are equivalent to

$$b^2 - ca \geq a^2 - bc, \quad c^2 - ab \geq a^2 - bc. \quad (35)$$

Applying the inequalities (35) to the inequality (33) and remembering Eq. (26), we obtain

$$\rho_u \geq p^4 s_4 + q^4 a_2^2 + 2p^2 q^2 (a^2 - bc)a_2. \quad (36)$$

We next rewrite Eqs. (11) and (25) as

$$a_2 = \frac{1}{2q^2} [\sigma_u - p^2(a^2 + b^2 + c^2)]. \quad (37)$$

We plug this equation into inequality (36) and find after some algebra that

$$\rho_u \geq \frac{1}{4} [\sigma_u - p^2(b + c)]^2 + F, \quad (38)$$

where

$$F = \frac{p^2 a^2}{2} \left[ \sigma_u + p^2(b + c)^2 - \frac{3p^2 a^2}{2} \right]. \quad (39)$$

The inequalities (34) give  $b + c \geq 2a$ , hence

$$F \geq \frac{p^2 a^2}{2} \left( \sigma_u + \frac{5p^2 a^2}{2} \right) \quad (40)$$

$$\geq \frac{p^2 a^2 \sigma_u}{2} \quad (41)$$

$$\geq 0. \quad (42)$$

From inequalities (38) and (42),

$$\rho_u \geq \frac{1}{4} [\sigma_u - p^2(b + c)]^2. \quad (43)$$

This inequality may equivalently be written

$$\sigma_u - 2\sqrt{\rho_u} \leq p^2(b + c)^2 \leq \sigma_u + 2\sqrt{\rho_u}. \quad (44)$$

It is obvious that, in an exactly analogous fashion, one may derive

$$\begin{aligned} \sigma_d - 2\sqrt{\rho_d} &\leq (b + c)^2 \leq \sigma_d + 2\sqrt{\rho_d}, \\ \sigma_\ell - 2\sqrt{\rho_\ell} &\leq (b + c)^2 \leq \sigma_\ell + 2\sqrt{\rho_\ell}. \end{aligned} \quad (45)$$

Inequalities (45) should be compared with those of [18]. One reaches the same conclusion as in [18]: the intervals  $[\sigma_d - 2\sqrt{\rho_d}, \sigma_d + 2\sqrt{\rho_d}]$  and  $[\sigma_\ell - 2\sqrt{\rho_\ell}, \sigma_\ell + 2\sqrt{\rho_\ell}]$  must overlap. This overlap—at the GUT scale—implies that, at that scale,  $m_b \simeq m_\tau$ . Notice that this conclusion was reached without making use of the quantities  $\pi_a$ .

Comparing inequalities (44) and (45), one also finds that the parameter  $p$  is approximately given by

$$p \simeq \frac{m_t}{m_b} \quad (46)$$

at the GUT scale.

Inequality (38) also delivers  $F \leq \rho_u$ . Taking into account inequality (41), one has

$$p^2 a^2 \leq \frac{2\rho_u}{\sigma_u}. \quad (47)$$

With Eq. (46) in mind, this gives, approximately,

$$a \lesssim \frac{\sqrt{2}m_c m_b}{m_t}. \quad (48)$$

Numerically, using the values of the quark masses in the MSSM at the GUT scale, as given in [21], one obtains for instance  $a \lesssim 3.8$  MeV for  $\tan\beta = 10$ .

## 5. Analytical treatment of the real case

In this section we analyze the case of real Yukawa-coupling matrices, i.e., the case of real  $S$  and  $A$ . In this case it is convenient to define

$$x_1 = \text{tr} S, \quad (49)$$

$$x_2 = \frac{1}{2}[x_1^2 - \text{tr}(S^2)]. \quad (50)$$

Then,

$$s_2 = \text{tr}(S^2) = x_1^2 - 2x_2, \quad (51)$$

$$s_4 = x_2^2 - 2x_1 s_3. \quad (52)$$

With  $S$  and  $A$  real,  $\bar{z}_4$  is real, and moreover it is not independent from  $z_4$ , rather

$$\bar{z}_4 - z_4 = x_2 a_2 - x_1 z_3. \quad (53)$$

This allows one to write Eqs. (14) and (15) as

$$\begin{aligned} \rho_d &= x_2^2 - 2x_1 s_3 + a_2^2 + 2z_4 + 2\cos(2\psi)\bar{z}_4 \\ &= x_2^2 - 2x_1 s_3 + a_2^2 - 2x_2 a_2 + 2x_1 z_3 + 2[1 + \cos(2\psi)]\bar{z}_4 \\ &= (x_2 - a_2)^2 + 2x_1(z_3 - s_3) + 2[1 + \cos(2\psi)]\bar{z}_4, \end{aligned} \quad (54)$$

$$\begin{aligned} \rho_\ell &= x_2^2 - 2x_1 s_3 + r^4 a_2^2 + 2r^2 z_4 + 2r^2 \cos(2\theta)\bar{z}_4 \\ &= x_2^2 - 2x_1 s_3 + r^4 a_2^2 - 2r^2 x_2 a_2 + 2r^2 x_1 z_3 + 2r^2[1 + \cos(2\theta)]\bar{z}_4 \\ &= (x_2 - r^2 a_2)^2 + 2x_1(r^2 z_3 - s_3) + 2r^2[1 + \cos(2\theta)]\bar{z}_4. \end{aligned} \quad (55)$$

Now, plugging Eq. (51) into Eqs. (9) and (10), one obtains

$$\sigma_d = x_1^2 - 2(x_2 - a_2), \quad (56)$$

$$\sigma_\ell = x_1^2 - 2(x_2 - r^2 a_2). \quad (57)$$

Hence, Eqs. (54) and (55) may be rewritten as

$$\rho_d - \frac{1}{4}(x_1^2 - \sigma_d)^2 = 2x_1(z_3 - s_3) + 2[1 + \cos(2\psi)]\bar{z}_4, \quad (58)$$

$$\rho_\ell - \frac{1}{4}(x_1^2 - \sigma_\ell)^2 = 2x_1(r^2 z_3 - s_3) + 2r^2[1 + \cos(2\theta)]\bar{z}_4. \quad (59)$$

In the trivial case  $\cos(2\psi) = \cos(2\theta) = -1$ , the mass matrices  $M_d$  and  $M_\ell$  are Hermitian and their eigenvalues directly yield the fermion masses. Discarding that rather trivial case from consideration, we find that Eqs. (58) and (59) lead to

$$0 = r^2[1 + \cos(2\theta)] \left[ \rho_d - \frac{1}{4}(x_1^2 - \sigma_d)^2 + 2x_1(s_3 - z_3) \right] \\ - [1 + \cos(2\psi)] \left[ \rho_\ell - \frac{1}{4}(x_1^2 - \sigma_\ell)^2 + 2x_1(s_3 - r^2 z_3) \right]. \quad (60)$$

On the other hand, since  $s_3$  and  $z_3$  are real when the matrices  $S$  and  $A$  are real, Eqs. (20) and (21) read in that case

$$s_3^2 + z_3^2 + 2\cos(2\psi)s_3 z_3 = \pi_d, \\ s_3^2 + r^4 z_3^2 + 2r^2 \cos(2\theta)s_3 z_3 = \pi_\ell. \quad (61)$$

Defining

$$f_1 = 1 - r^4, \quad (62)$$

$$f_2 = \cos(2\psi) - r^2 \cos(2\theta), \quad (63)$$

$$f_3 = r^4 \cos(2\psi) - r^2 \cos(2\theta), \quad (64)$$

$$f_4 = \pi_d - \pi_\ell, \quad (65)$$

$$f_5 = r^4 \pi_d - \pi_\ell, \quad (66)$$

$$f_6 = r^2 \cos(2\theta)\pi_d - \cos(2\psi)\pi_\ell, \quad (67)$$

the system of Eqs. (61) has solutions given by

$$s_3^2 = \frac{-f_1 f_5 - 2f_3 f_6 \pm 2f_3 \sqrt{f_6^2 - f_4 f_5}}{f_1^2 + 4f_2 f_3}, \\ z_3^2 = \frac{f_1 f_4 - 2f_2 f_6 \mp 2f_2 \sqrt{f_6^2 - f_4 f_5}}{f_1^2 + 4f_2 f_3}, \\ s_3 z_3 = \frac{f_2 f_5 + f_3 f_4 \pm f_1 \sqrt{f_6^2 - f_4 f_5}}{f_1^2 + 4f_2 f_3}. \quad (68)$$

We use as input the three charged-lepton masses, the three down-type-quark masses, and also  $r$ ,  $\cos(2\theta)$ , and  $\cos(2\psi)$ . Eq. (68) allow us to compute  $s_3$  and  $z_3$  from that input. Inserting those values of  $s_3$  and  $z_3$  in Eq. (60), we obtain a quartic equation for  $x_1$ , which may be analytically solved. The quantities  $s_2$  and  $a_2$  are then computed as

$$s_2 = \frac{x_1^2}{2} + \frac{r^2\sigma_d - \sigma_\ell}{2(1-r^2)}, \quad (69)$$

$$a_2 = \frac{\sigma_d - \sigma_\ell}{2(1-r^2)}. \quad (70)$$

Finally,  $\bar{z}_4$  is computed from either Eq. (58) or Eq. (59), and  $z_4$  is obtained from Eq. (53). All the invariants pertaining to the matrices  $S$  and  $A$  are thus analytically computed from the input.

One must, yet, take into account the fact that those invariants must satisfy several inequalities. In the weak basis (5),

$$x_1 = a + b + c, \quad (71)$$

$$x_2 = ab + bc + ca, \quad (72)$$

$$s_3 = abc. \quad (73)$$

The numbers  $a$ ,  $b$ , and  $c$  are real, and they may be negative, see footnote 2. The quantity

$$\Delta \equiv x_1^2 x_2^2 + 18x_1 x_2 s_3 - 4x_2^3 - 4x_1^3 s_3 - 27s_3^2 \quad (74)$$

$$= [(a-b)(b-c)(c-a)]^2 \quad (75)$$

must therefore be nonnegative. Further nonnegative quantities may be conveniently derived by using the weak basis (6) and deriving, in that basis, the values of  $a$ ,  $b$ ,  $c$ ,  $d^2$ , and  $f^2$  from the invariants. From the condition that  $f^2$  must be nonnegative one obtains

$$\Sigma \equiv a_2 z_4 - z_3^2 \quad (76)$$

$$\geq 0. \quad (77)$$

From the condition that  $d^2$  must be nonnegative one obtains

$$\Psi \equiv -z_4^3 + z_4^2(2x_1 z_3 - x_2 a_2) + z_4[a_2 z_3(3s_3 + x_1 x_2) - x_1 s_3 a_2^2 - (x_1^2 + x_2)z_3^2] \\ + z_3^3(x_1 x_2 - s_3) - a_2 z_3^2(x_2^2 + x_1 s_3) + 2x_2 s_3 a_2^2 z_3 - s_3^2 a_2^3 \quad (78)$$

$$\geq 0. \quad (79)$$

The conditions that  $\Delta$ ,  $\Sigma$ , and  $\Psi$  be nonnegative constitute a severe constraint on the inputted values of the charged-fermion masses and of  $r$ ,  $\theta$ ,  $\psi$ .

After having computed the invariants, one may further input the three up-type-quark masses and therefrom derive the values of  $p^2$ ,  $q^2$ , and  $\cos(2\xi)$ . In practice, this involves solving a cubic equation, and thereafter imposing the constraints  $p^2 \geq 0$ ,  $q^2 \geq 0$ , and  $|\cos(2\xi)| \leq 1$ . This obviously translates into constraints on the inputted up-type-quark masses.

In the way delineated in this section, one may analytically solve the case of real  $S$  and  $A$  matrices, by inputting the charged-fermion masses and therefrom deriving  $S$  and  $A$ , without having to have recourse to fits. In practice, however, doing things the other way round—trying to fit the charged-fermion masses numerically from some inputted values of  $S$ ,  $A$ , and the other parameters—proves more effective. We turn to that procedure in the next section.

## 6. The fitting procedure

In order to check whether the mass matrices (3) allow to reproduce the masses and CKM mixing angles at the GUT scale, we use a  $\chi^2$  analysis, as was previously applied for instance in [23,24]. As for the masses, the  $\chi^2$ -function is given by

$$\chi_{\text{masses}}^2 = \chi_d^2 + \chi_\ell^2 + \chi_u^2, \quad (80)$$

where

$$\chi_d^2 = \sum_{i=d,s,b} \left( \frac{m_i(x) - \bar{m}_i}{\delta m_i} \right)^2, \quad (81)$$

and analogously for  $\chi_{\ell,u}^2$ . The masses at the GUT scale are  $\bar{m}_i \pm \delta m_i$ , whereas the  $m_i(x)$  are the masses calculated from Eqs. (3) as functions of the parameter set  $x = \{S, A, p, q, r, \psi, \theta, \xi\}$  (see Section 2 for the distinction between the “real” and the “complex” cases). The total  $\chi^2$ -function is the sum

$$\chi_{\text{total}}^2 = \chi_{\text{masses}}^2 + \chi_{\text{CKM}}^2, \quad (82)$$

with

$$\chi_{\text{CKM}}^2 = \sum_{i=12,13,23} \left( \frac{\sin \theta_i(x) - \sin \bar{\theta}_i}{\delta \sin \theta_i} \right)^2. \quad (83)$$

We take the masses  $\bar{m}_i$  at the GUT scale, and their errors  $\delta m_i$ , from Table II of [21]; those masses refer to the MSSM with  $\tan \beta = 10$  and a GUT scale of  $2 \times 10^{16}$  GeV and have been obtained through the renormalization-group evolution of the masses given in [22] at the  $Z^0$ -mass scale. As for sines of the CKM angles,  $\sin \bar{\theta}_i \pm \delta \sin \theta_i$ , we use Table 1 in [11]. We do not take into account the CKM phase in our fitting procedure; this omission will be justified later.

In order to get a better understanding of our mass matrices, we perform separate minimizations of  $\chi_{\text{masses}}^2$  and of  $\chi_{\text{total}}^2$ . We also test the “real” versus the “complex” case.

For the numerical multi-dimensional minimization of the  $\chi^2$ -functions we employ the downhill simplex method [25]. Because the problem is highly nonlinear, we expect the existence of many local minima.<sup>3</sup> We start with randomly generated initial simplices. At the points where the numerical algorithm stops, we iterate the procedure with random perturbations in order to find a lower  $\chi^2$ . In this way we can be fairly certain about the distribution of the local minima and about the position of the global minimum.

In the description of the fits, the concept of “pull” with respect to an observable  $O$  is useful. The pull of  $O$  is defined as

$$\text{pull}(O) = \frac{O(\hat{x}) - \bar{O}}{\delta O}, \quad (84)$$

where the experimental value of the observable is  $\bar{O} \pm \delta O$ , while  $O(x)$  is the theoretical prediction of  $O$ , given as a function of the parameter set  $x$ ;  $\hat{x}$  is the parameter set at a local minimum

<sup>3</sup> The concept “local minimum” is not understood in a strict mathematical sense, rather it refers to a point where the minimization algorithm successfully stops.

of  $\chi^2$ . Thus,

$$\min \chi^2(x) \equiv \chi^2(\hat{x}) = \sum_O [\text{pull}(O)]^2. \quad (85)$$

## 7. Results

We have performed all fits and tests of our scenario separately for real and complex coupling matrices  $S$  and  $A$ —see Section 2. It turns out that there are no significant numerical differences between the two cases. The extra two phases in the “complex” case are unable to significantly improve our fits. Therefore, for simplicity in the following we confine ourselves to the “real” case.

### 7.1. Fits of the masses alone

Firstly we omit the CKM angles and test whether, with the mass matrices (3), we are able to fit the charged-fermion mass values at the SUSY GUT scale given in [21]. In Fig. 1 we show the distribution, in the  $\chi_d^2 - \chi_u^2$  plane, of the local minima of  $\chi_{\text{masses}}^2$  for which  $\min \chi_{\text{masses}}^2 \leq 40$ . Though the density of points in that figure depends sensitively on the number of random perturbations and on the number of restarts of the downhill simplex procedure, the overall picture is clear. The absolute minimum of  $\chi_{\text{masses}}^2$  is located at  $\chi_u^2 \simeq 0$ ,  $\chi_d^2 \simeq 23.3$ ; the corresponding fit masses, and the pulls, are given in Table 1. For comparison, we also show in Table 1 the central mass values of [21]. Looking at the pulls, we see that this mass fit fails only in the mass of the

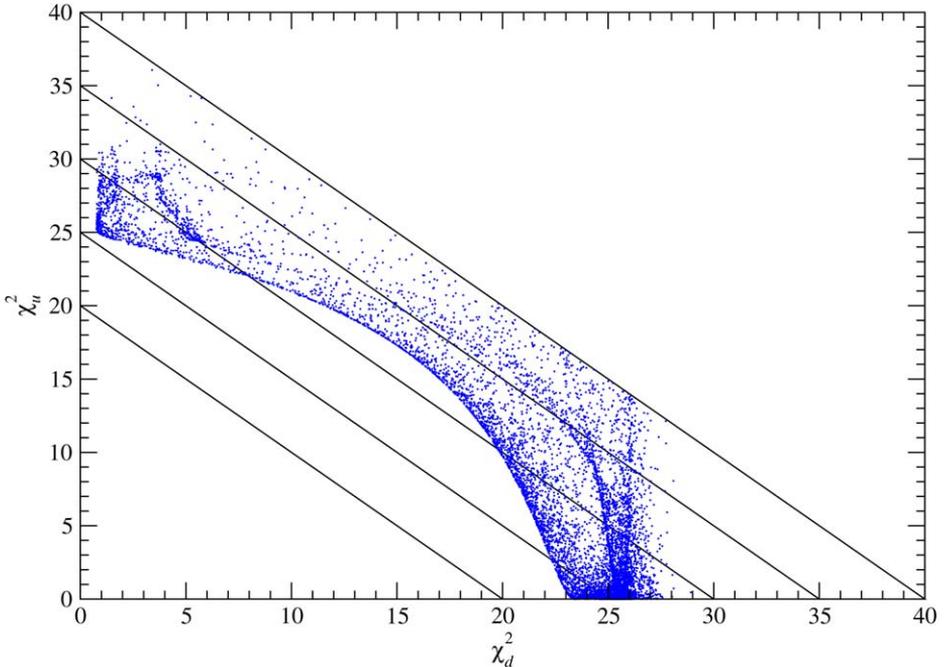


Fig. 1. The distribution of local minima of  $\chi_{\text{masses}}^2 \simeq \chi_d^2 + \chi_u^2$  in the  $\chi_d^2 - \chi_u^2$  plane (we force  $\chi_\ell^2$  to be always negligibly small). The straight lines refer to constant values of  $\chi_{\text{masses}}^2$ . This figure refers to the “real” case.

Table 1

Results of the fit for the “real” case without the CKM angles. The values of the masses in the second column, and the corresponding errors  $\delta m$  for the calculation of the pulls, have been taken from [21]. The third column gives our best fit and the fourth column displays the corresponding pulls. The fifth and sixth columns refer to best fit in the region of small  $\chi_d^2$ , i.e., the region in Fig. 1 with  $\chi_d^2 \simeq 1$  and  $\chi_u^2 \simeq 25$ . All the masses are in units of MeV

	$\bar{m}$	$\chi_{\text{masses}}^2 = 23.3$		$\chi_{\text{masses}}^2 = 25.8$	
		$m(\hat{x})$	pull( $m$ )	$m(\hat{x})$	pull( $m$ )
$m_e$	0.3585	0.3585	$6 \times 10^{-3}$	0.3585	$6 \times 10^{-4}$
$m_\mu$	75.67	75.67	$-4 \times 10^{-4}$	75.67	$-9 \times 10^{-4}$
$m_\tau$	1292.2	1292.2	$-4 \times 10^{-3}$	1292.2	$-4 \times 10^{-3}$
$m_d$	1.504	0.4112	-4.74	1.430	-0.321
$m_s$	29.95	29.54	-0.090	29.35	-0.132
$m_b$	1063.6	1187.0	0.873	1188.2	0.882
$m_u$	0.7328	0.7249	$8 \times 10^{-3}$	0.7321	0.061
$m_c$	210.33	212.47	0.113	214.66	0.228
$m_t$	82433	78466	-0.269	8778	-4.99

down quark; that particular pull is responsible for almost the complete  $\chi_{\text{masses}}^2 = 23.3$ . A glance at Fig. 1 also reveals that there are local minima with  $\chi_u^2 \simeq 25$  and  $\chi_d^2 \simeq 1$ ; those minima, the best of which is also displayed in Table 1, give rather good fits for *all* the down-type-quark masses, but fail severely in fitting the top-quark mass: the fit value is about one order of magnitude, or five  $\sigma$ , smaller than the experimental value—see Table 1.

Thus, with our scenario we cannot even fit all the charged-fermion masses. However, as stressed in the introduction, our scenario is extreme in that it allows only for the Yukawa couplings of one  $\mathbf{10} \oplus \mathbf{120}$  scalar representation. If we allow for small perturbations of the mass matrices, there are several ways out: there could be contributions from Yukawa couplings of one  $\mathbf{126}$  [17], several  $\mathbf{10}$  and/or  $\mathbf{120}$  of scalars, radiative corrections, or nonrenormalizable terms. Consequently, the absolute minimum of  $\chi_{\text{masses}}^2$  can be considered acceptable, since it fails only for  $m_d$ , which is small anyway. On the other hand, our philosophy of small perturbations forces us to discard the local minimum where the fit value of  $m_t$  is one order of magnitude too small.

## 7.2. Fits with masses and CKM angles

Fig. 2 shows the distribution in the  $\chi_d^2 - \chi_u^2$  plane, for the “real” case, of the local minima of  $\chi_{\text{total}}^2$  which have  $\min \chi_{\text{total}}^2 \leq 50$ . We see that the gross feature—the lower left corner is devoid of local minima—is the same as in the fit without CKM angles. The previous local minimum at  $\chi_u^2 \simeq 25$  and  $\chi_d^2 \simeq 1$  is now the absolute minimum. That absolute minimum is given in detail in Table 2. We see that the fit of  $m_t$  is as unacceptably bad as before, but the CKM angles are well reproduced.

Moving to the zone of  $\chi_u^2 \lesssim 1$  in Fig. 2, where the top-quark mass is well reproduced, we find the following characteristic features:

- In that zone the best fit has  $\chi_{\text{total}}^2 \simeq 45.4$ . The corresponding fit values and pulls are shown in columns five and six of Table 2, respectively.
- Varying  $\chi_{\text{total}}^2$  between 45.4 and 49, we find that the pull of  $m_d$  changes roughly from  $-2$  to  $-3$ .

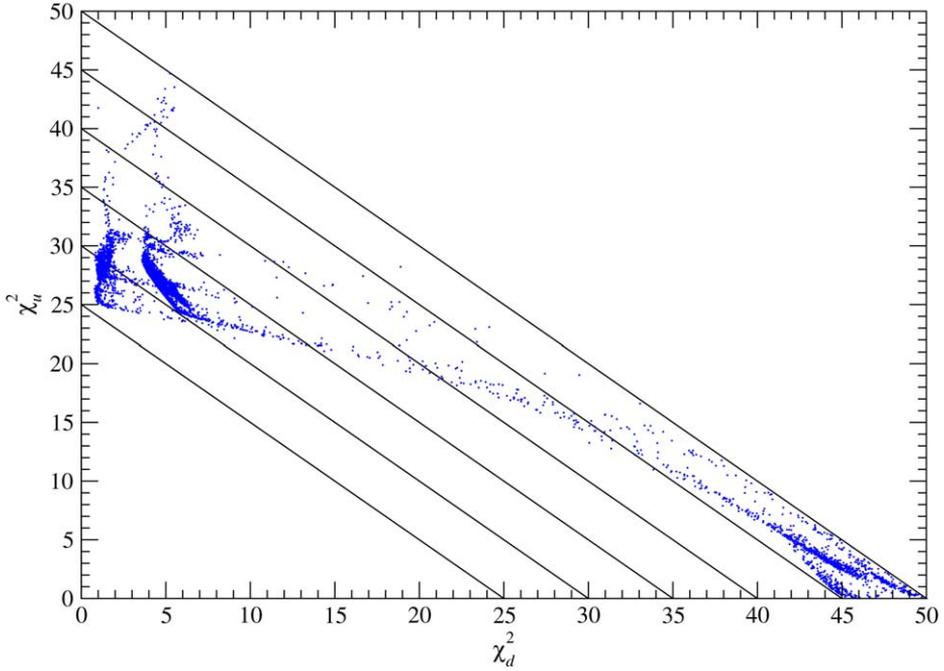


Fig. 2. The distribution of local minima of  $\chi_{\text{total}}^2$  (in the “real” case) in the  $\chi_d^2$ – $\chi_u^2$  plane. The straight lines refer to constant  $\chi_d^2 + \chi_u^2 \simeq \chi_{\text{total}}^2$ .

Table 2

Results of the fit for the “real” case including the CKM angles. The best fit is described by columns three and four: this point lies at  $\chi_d^2 \simeq 1$ ,  $\chi_u^2 \simeq 25$ . Columns five and six refer to the best fit in the region of small  $\chi_u^2$

	$\bar{m}$	$\chi_{\text{total}}^2 = 26.9$		$\chi_{\text{total}}^2 = 45.4$	
		$m(\hat{x})$	pull( $m$ )	$m(\hat{x})$	pull( $m$ )
$m_e$	0.3585	0.3585	$-1 \times 10^{-3}$	0.3585	$7 \times 10^{-3}$
$m_\mu$	75.67	75.67	$2 \times 10^{-3}$	75.67	$4 \times 10^{-3}$
$m_\tau$	1292.2	1292.2	$-4 \times 10^{-3}$	1292.2	$-3 \times 10^{-3}$
$m_d$	1.504	1.563	0.141	1.044	–1.993
$m_s$	29.95	28.24	–0.376	1.36	–6.29
$m_b$	1063.6	1191.0	0.903	1225.4	1.145
$m_u$	0.7238	0.7243	$3 \times 10^{-3}$	0.7279	0.030
$m_c$	210.33	215.35	0.264	216.83	0.342
$m_t$	82433	8179	–5.03	74145	–0.56
	$\sin \bar{\theta}$	$\sin \theta(\hat{x})$	pull( $\sin \theta$ )	$\sin \theta(\hat{x})$	pull( $\sin \theta$ )
$\sin \theta_{12}$	0.2243	0.2242	–0.047	0.2243	$2 \times 10^{-4}$
$\sin \theta_{23}$	0.0351	0.0348	–0.208	0.0352	0.093
$\sin \theta_{13}$	0.0032	0.0036	0.740	0.0034	0.318

- For that range of  $\chi_{\text{total}}^2$ , the pull of  $m_s$  remains close to  $-6$ , i.e., the fit value of  $m_s$  is one order of magnitude lower than the experimental value—indeed,  $m_s$  turns out hardly larger than  $m_d$ ! This is the main reason why  $\chi_{\text{total}}^2$  is so bad in the region of low  $\chi_u^2$ .
- The pull of  $m_b$  is always about  $+1$ .

Since we are unable to reproduce well all the quark masses, we cannot expect to obtain a realistic CKM phase, and we have not included it in our fit.

In view of our philosophy, we have also tried a fit of the masses and of the CKM mixing angles while allowing for artificially large errors in the light-fermion masses. Taking for instance  $\delta m_i = 5$  MeV for  $i = e, d, u$ , we are able to achieve  $\chi_{\text{total}}^2 \simeq 25.4$ . This is not really an improvement when compared to the best fit in columns 3 and 4 of Table 2. However, the characteristics of this fit are different from those of that best fit: the pulls of  $m_b$  and of  $m_t$  are approximately  $+1$  and  $-1$ , respectively, whereas the fit value of  $m_s$  is  $4.65\sigma$  too low. Thus, the fit with drastically increased errors in the light-fermion masses rather resembles the fit of columns 5 and 6 of Table 2.

### 7.3. A numerical test of $b$ - $\tau$ unification

In Section 4 we have traced  $b$ - $\tau$  unification to some inequalities involving the charged-lepton and the down-type-quark masses; those inequalities are conditions on the masses necessary for Eqs. (9), (10), (14), and (15) to have a solution. Neglecting the masses of the first generation, i.e., setting  $m_d \simeq m_e \simeq 0$ , those conditions are reformulated as [18]

$$1 - \frac{m_\mu + m_s}{m_\tau} \lesssim \frac{m_b}{m_\tau} \lesssim 1 + \frac{m_\mu + m_s}{m_\tau}. \quad (86)$$

Using the values  $\bar{m}_i$  for the masses in [21], this reads

$$0.92 \lesssim \frac{m_b}{m_\tau} \lesssim 1.08. \quad (87)$$

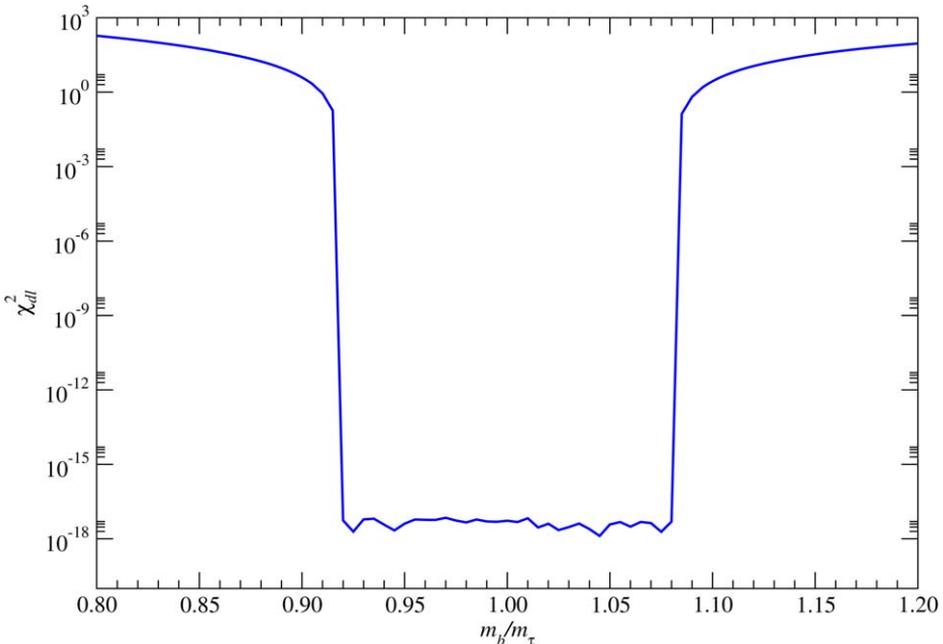


Fig. 3. The minimum of the  $\chi^2$ -function of Eq. (89) as a function of  $m_b$ .

We have performed a  $\chi^2$  analysis to check the inequalities (86). For this purpose, we consider

$$\chi_{d\ell}^2(x, m_b) \equiv \sum_{i=e,\mu,\tau,d,s} \left( \frac{m_i(x) - \bar{m}_i}{\delta m_i} \right)^2 + \left( \frac{m_b(x) - m_b}{0.01 m_b} \right)^2. \quad (88)$$

This  $\chi^2$ -function is identical with  $\chi_d^2 + \chi_\ell^2$  apart from the term corresponding to the bottom-quark mass, wherein we leave  $m_b$  free and assign to it a very small error bar of 1%. Then we define a minimal  $\chi^2$  as a function of  $m_b$ :

$$\chi_{d\ell}^2(m_b) \equiv \min_x \chi_{d\ell}^2(x, m_b). \quad (89)$$

This function allows one to test the down-type-quark and charged-lepton mass fits with respect to variations of  $m_b$  [24]. In Fig. 3 we have plotted  $\chi_{d\ell}^2(m_b)$  against  $m_b/m_\tau$ ; we have used the mean value  $m_\tau = 1292.2$  MeV given in [21]. We see that exactly in the range of Eq. (87) the minimum of  $\chi_{d\ell}^2(m_b)$  is, for all practical purposes, zero. This confirms our analytic derivation of  $b$ - $\tau$  unification.

We can also test Eq. (46) against our numerics. We find that equation is reproduced fairly well whenever the fit of  $m_t$  is good.

## 8. Summary

In this paper we have investigated a SUSY  $SO(10)$  scenario in which the charged-fermion masses are generated exclusively by the renormalizable Yukawa couplings of the fermions to one representation  $\mathbf{10} \oplus \mathbf{120}$  of scalars. We have studied the three-generations case, confirming the  $b$ - $\tau$  unification which had previously been proved for two generations [18]. However, our tests of this scenario against the charged-fermion masses and against the CKM mixing angles at the GUT scale show that it is not satisfactory: the fit value of  $m_t$  comes out much too low for the best fit; allowing for a larger  $\chi_{\text{total}}^2$ , we are able to obtain a good fit of  $m_t$ , at the price of  $m_s$  turning out one order of magnitude too low and of  $m_d$  also being too small. We thus find that the scenario investigated here is too restrictive: an additional mechanism for charged-fermion mass generation is required.

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