

## Highest states in light-cone $AdS_5 \times S^5$ superstring

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**ABSTRACT:** We study the highest states in the compact rank-1 sectors of the  $AdS_5 \times S^5$  superstring in the framework of the recently proposed light cone Bethe Ansatz equations. In the  $\mathfrak{su}(1|1)$  sector we present strong coupling expansions in the two limits  $L, \lambda \rightarrow \infty$  (expanding in power of  $\lambda^{-1/4}$  with fixed large  $L$ ) and  $\lambda, L \rightarrow \infty$  (expanding in power of  $1/L$  with fixed large  $\lambda$ ) where  $\lambda$  is the 't Hooft coupling and  $L$  is the number of Bethe momenta. The two limits do not commute apart from the leading term which reproduces the result obtained with the Arutyunov-Frolov-Staudacher phase in the  $\lambda, L \rightarrow \infty$  limit. In the  $\mathfrak{su}(2)$  sector we perform the strong coupling expansions in the  $L \rightarrow \infty$  limit up to  $\mathcal{O}(\lambda^{-1/4})$ , and our result is in agreement with previous String Bethe Ansatz analysis.

**KEYWORDS:** Bethe Ansatz, AdS-CFT Correspondence.

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## 1. Introduction

The verification of the conjectured AdS/CFT correspondence [1–5] is a non-trivial interpolation problem between a pair of quite different theories, string theory on  $AdS_5 \times S^5$  and the maximally supersymmetric four dimensional  $\mathcal{N} = 4$  super Yang-Mills  $SU(N)$  gauge theory (SYM). The duality predicts that certain SYM gauge invariant operators have anomalous dimensions equal to the energy of dual massive string states. Even though a drastic simplification is achieved in the planar limit  $N \rightarrow \infty$  with fixed 't Hooft coupling  $\lambda = g_{\text{YM}}^2 N$ , both quantities can not be simultaneously computed in the full range of the 't Hooft coupling.

On the gauge side, weak coupling perturbation theory provides explicit results at a *relatively* low loop order [6, 7]. Assuming quantum integrability, it is possible to identify the dilatation operator with an integrable quantum Hamiltonian and write its Bethe Ansatz (BA) equations [8]. They are conjectured to reproduce the full weak coupling expansion of anomalous dimensions in various closed sectors of the full  $PSU(2, 2|4)$  symmetry group. This is true up to wrapping problems which occur at order  $\mathcal{O}(\lambda^L)$  where  $L$  is the classical dimension of the composite operator. This is an essential limitation to the possibility of computing the all-order weak coupling expansion of operators with fixed dimension. Some investigations suggest that wrapping problems could be overcome in the fermionic approach based on the Hubbard model [9–13]. However, even if wrapping problems could be solved, it would be non-trivial to extrapolate the BA predictions to strong coupling. It would also be necessary to prove that eventual non-perturbative effects are captured by the gauge BA equations.

On the string side, the exact quantization of type IIB superstring on  $AdS_5 \times S^5$  is not known. In the usual approach, one starts with an exact supergravity solution at large  $\lambda$  and computes perturbative  $\sigma$ -model corrections. The accessible regions in the gauge and string theories are apparently disjoint.

An overlap window opens as soon as BMN-like scaling limits are taken [14]. One considers classical solutions with large angular momenta  $J$  on  $S^5$  (and/or spin on  $AdS_5$  in more general cases). On the gauge side  $J$  is the  $R$ -charge of the dual composite operator. In the strict BMN limit,  $J \rightarrow \infty$  with fixed  $\lambda' = \lambda/J^2$ , or in the near-BMN corrections suppressed by  $1/J$  factors, the two calculations can be compared because  $\lambda$  is large while the gauge theory effective coupling  $\lambda'$  can be small. The comparison reveals a typical three loop disagreement (see for instance [15] for a review). Understanding the precise mechanism behind this discrepancy is a main open problem in AdS/CFT. It has been suggested that it arises because the string and gauge calculations are performed in the double limit  $J \rightarrow \infty$  and  $\lambda' \rightarrow 0$  taken in opposite order [8]. This is an essential obstruction. To match string calculations one should at least resum the weak coupling perturbative series before taking the  $J \rightarrow \infty$  limit, something which is forbidden by the wrapping problems. Unfortunately, we lack the necessary technical tools to perform such resummations,

For these reasons, it is sensible to try to look for string BA equations encoding the  $\sigma$ -model corrections. At the classical level, the  $AdS_5 \times S^5$  superstring [16] is integrable [17]. Assuming that the integrable structure can be maintained at the quantum level, string BA

equations (SBA) have been proposed in [20]. They are similar in structure to the gauge BA equations, but are modified by a non trivial dressing phase [18, 19].

To fix the dressing phase, the SBA equations have been deeply tested by comparing their predictions with the semiclassical quantization of pp-wave states and spinning string solutions where available (see for instance [21]). For pp-wave states, explicit string theory calculations including curvature corrections to the flat space background predict anomalous dimensions with the typical form [22]

$$\Delta^{\text{PP}} - J = \Delta_0^{\text{PP}}(\lambda') + \frac{1}{J} \Delta_1^{\text{PP}}(\lambda') + \dots \quad (1.1)$$

Both the thermodynamical limit  $\Delta_0^{\text{PP}}(\lambda')$  (independent on the dressing phase) and the first quantum correction  $\Delta_1^{\text{PP}}(\lambda')$  are in full agreement with the explicit string calculation. For spinning string states the comparison is more problematic. In this case, it is customary to introduce  $\mathcal{J} = J/\sqrt{\lambda}$  and the typical prediction for the semiclassical energy is

$$\Delta^{\text{FT}} = \sqrt{\lambda} \Delta_0^{\text{FT}}(\mathcal{J}) + \Delta_1^{\text{FT}}(\mathcal{J}) + \dots, \quad (1.2)$$

(with FT standing for Frolov-Tseytlin). Now, the agreement with explicit string calculations is perfect for  $\Delta_0^{\text{FT}}$  where it holds by construction, but only partial in  $\Delta_1^{\text{FT}}$ . The problem has been recently clarified in [23] with an explicit comparison in the case of the  $\mathfrak{sl}(2)$  spinning string [24] and using the full available information about the dressing phase [25–27]. At large  $\mathcal{J}$  the exact string calculation admits the expansion

$$\Delta_1^{\text{FT}}(\mathcal{J}) = \sum_{\ell \geq 2} \frac{f_\ell}{\mathcal{J}^\ell} + \sum_{s \geq 0} a_s e^{-2\pi s \mathcal{J}}. \quad (1.3)$$

The SBA equations are known to reproduce the full power series, but not the exponentially suppressed terms. The reason behind this failure in capturing non-perturbative finite size corrections is a fundamental limitation of any approach based on the thermodynamical classical Bethe Ansatz. It is not clear whether this problem could be solved by resumming the conjectured all-order strong coupling series for the dressing factor [28, 29]. The resolution of this discrepancy could require the introduction of new degrees of freedom in the quantum Bethe Ansatz as pointed out very clearly in [23].

An outcome of the above discussion is that it is definitely very important to test the SBA equations in all possible ways. In this spirit, apart from states admitting BMN-like limits, another important structural test of the SBA equations is the ability of reproducing the Gubser-Klebanov-Polyakov (GKP) prediction

$$E \sim 2\sqrt{n} \lambda^{1/4}, \quad (1.4)$$

for the energy of level  $n$  massive string states as  $\lambda \rightarrow \infty$  [2]. The SBA equations are known to generically agree with the GKP law, at least under mild reasonable assumptions on the asymptotic behavior of Bethe momenta as  $\lambda \rightarrow \infty$  at finite  $J$  [20]. Nevertheless, the results are reliable for large  $J$  only which is the limit where the SBA equations have been derived.

The easiest cases where the GKP law can be explicitly investigated (determining also the dual level  $n$ ) are the highest states in the compact rank-1 subsectors  $\mathfrak{su}(2)$  and

$\mathfrak{su}(1|1)$  [30–32]. For notational purposes we shall call these states *antiferromagnetic* (AF) borrowing the wording from the  $\mathfrak{su}(2)$  case. In a recent paper [32], two of us proved that the SBA equations predict the following result

$$\begin{aligned} \frac{\Delta_{\mathfrak{su}(2)}^{\text{SBA}}(L, \lambda)}{2L} &= \frac{1}{2} \lambda^{1/4} + \mathcal{O}(\lambda^0), \\ \frac{\Delta_{\mathfrak{su}(1|1)}^{\text{SBA}}(L, \lambda)}{L} &= \frac{1}{\sqrt{2}} \left(1 - \frac{1}{L^2}\right)^{1/2} \lambda^{1/4} + \mathcal{O}(\lambda^0), \end{aligned} \quad (1.5)$$

where  $L$  denotes the number of Bethe momenta in both sectors. Unfortunately, it is not easy to compare these results with string theory calculations since the state dual to the AF operators is not known. The only exception is the proposal [33] in the  $\mathfrak{su}(2)$  sector which, however, does not agree with the GPK law.

The result eq. (1.5) is obtained at fixed  $L$  and studying the suitable solution of the SBA equations as  $\lambda \rightarrow \infty$ . Since the SBA equations are valid for large  $L$ , one would like to know how many terms in the  $1/L$  expansion of eq. (1.5) are correct. Indeed, the correct procedure would require to take the large  $L$  limit of the SBA equations obtaining the functions  $\Delta_{0,1}^{\text{AF}}(\lambda)$  appearing in the expansion

$$\frac{\Delta^{\text{SBA}}(L, \lambda)}{L} = \Delta_0^{\text{AF}}(\lambda) + \frac{1}{L} \Delta_1^{\text{AF}}(\lambda) + \dots \quad (1.6)$$

Then, one could safely take the large  $\lambda$  limit of each term. This is what we denote the  $L, \lambda \rightarrow \infty$  limit. Unfortunately, it is not known how to solve the integral equations for the Bethe roots distribution at large  $L$  in neither sector. Also, their strong coupling expansion is ambiguous and only the  $\lambda, L \rightarrow \infty$  limit, i.e. expanding in power of  $\lambda^{-1/4}$  with fixed  $L$  and eventually expanding in  $1/L$ , is currently calculable.

In principle, the SBA equations are only one possible discretization of the classical string Bethe equations. Also, different gauge-fixed formulation can lead to equivalent equations, although with their special technical features. A remarkable example is indeed described in [34] where quantum SBA equations are derived starting from the string action in the so-called uniform light-cone gauge. This is the generalization of the usual flat space light-cone gauge to the  $AdS_5 \times S^5$  case [35–41, 34, 42]. Again, the equations are obtained starting from the leading thermodynamical term in a (suitable light-cone)  $1/J$  expansion and discretising. In [34] the equations are matched to the near-BMN corrections to pp-wave states fixing the leading dressing phase. Remarkably, a compact set of equations is obtained where the dressing phase is somewhat reabsorbed.

The light-cone Bethe Ansatz equations (LCBA) recast the spectral problem in an intriguing way and deserve in our opinion further investigation. In this paper, we analyze them working on the AF states at large  $\lambda$ . We indeed show that the calculation in [32] can be repeated in the LCBA framework achieving much more insight. In particular, in the  $\mathfrak{su}(1|1)$  sector, we are able to solve them in the safe  $L, \lambda \rightarrow \infty$  limit clarifying the accuracy of our previous calculation eq. (1.5).

## 2. The light-cone Bethe Ansatz

We briefly review the LCBA equations derived in [34] to setup the notation. The uniform light-cone gauge is based on the introduction of light-cone variables

$$X_{\pm} = \frac{1}{2}(\varphi \pm t), \tag{2.1}$$

where  $\varphi$  is an angle on  $S^5$  conjugate to the angular momentum  $J$  and  $t$  is the global time on  $AdS_5$  conjugate to the energy  $E$ . The gauge is fixed by the choice

$$X_+ = \tau, \quad p_+ = P_+ = \text{const}, \tag{2.2}$$

where  $p_+$  is conjugate to  $x_-$ . The world-sheet light-cone Hamiltonian is

$$H_{\text{lc}} = -P_-, \tag{2.3}$$

and is a function of  $P_+$ . Expanding at large  $P_+$  with  $\lambda/P_+^2$  fixed one recovers the BMN and near-BMN limit suitable to study the pp-wave states. The two equations

$$E - J = H_{\text{lc}}(P_+), \tag{2.4}$$

$$E + J = P_+, \tag{2.5}$$

lead to the following equation determining  $E$

$$E = J + H_{\text{lc}}(E + J). \tag{2.6}$$

The results for pp-wave states in all rank-1 sectors (including the non compact  $\mathfrak{sl}(2)$  case) are consistent at  $\mathcal{O}(1/P_+)$  with the discrete equations

$$\exp\left(i p_k \frac{P_+ + \mathfrak{s} M}{2}\right) = \prod_{\substack{j=1 \\ j \neq k}}^M \left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\right)^{\mathfrak{s}}, \tag{2.7}$$

where  $\mathfrak{s} = -1, 0, 1$  in  $\mathfrak{sl}(2)$ ,  $\mathfrak{su}(1|1)$ , and  $\mathfrak{su}(2)$ . The variables  $x^{\pm}$  are

$$x^{\pm}(p) = \frac{1}{4} \left( \cot \frac{p}{2} \pm i \right) (1 + H_{\text{lc}}(p)), \tag{2.8}$$

where

$$H_{\text{lc}}(p) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}. \tag{2.9}$$

In the next sections we shall analyze in details the properties of the highest energy solution of the above equation in the two cases  $\mathfrak{s} = 0, 1$ . Of course, the energy  $E$  must be identified with the anomalous dimension  $\Delta$  of the dual gauge invariant operators.

### 3. $\mathfrak{su}(1|1)$ sector

#### 3.1 General features of the LCBA equations

The light cone Bethe equations are particularly simple in the  $\mathfrak{su}(1|1)$  sector and read

$$\exp\left(i \frac{P_+}{2} p_k\right) = 1, \tag{3.1}$$

$$P_- = \sum_{k=1}^M \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}}. \tag{3.2}$$

where  $P_{\pm} = \Delta \pm J$ . To study the highest state and match the notation in [32] we consider

$$J = \frac{L}{2}, \quad M = L, \quad L \in 2\mathbb{N} + 1. \tag{3.3}$$

The equation for the Bethe momenta can be solved immediately and gives

$$p_k = \frac{4\pi}{\Delta + L/2} n_k, \quad n_k \in \mathbb{Z}. \tag{3.4}$$

The remaining equation determines  $\Delta_L(\lambda)$

$$\Delta_L(\lambda) = \frac{L}{2} + \sum_{k=1}^L \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{2\pi n_k}{\Delta_L(\lambda) + L/2}}. \tag{3.5}$$

We solve this equation with  $\{n_k\}$  in the symmetric range

$$\{n_k\} = \left\{ -\frac{L-1}{2}, \dots, 0, \dots, \frac{L-1}{2} \right\}, \tag{3.6}$$

which uniquely selects the highest state [31, 32]. In appendix (A) we prove that the above equation admits a unique solution  $\Delta_L(\lambda)$  at fixed  $L$ .

The LCBA being derived at strong coupling, they should not be trusted to yield a correct solution for  $\Delta(\lambda)$  at weak coupling; however, in the same spirit of [31], we present in appendix (B) some results for its weak coupling expansion which could be useful to compare with those from future improved LCBA equations with a refined dressing.

#### 3.2 Strong coupling expansion in the $\lambda \rightarrow \infty$ limit at fixed $L$

As explained in the Introduction, we begin our analysis of the LCBA equations by studying the  $\lambda, L \rightarrow \infty$  limit. In other words, we fix  $L$  and take the large  $\lambda$  limit, eventually expanding in  $1/L$ .

Due to the simplicity of the equations, we can prove analyticity at strong coupling, i.e. exclude non-analytic corrections to the above relation as well as prove its convergence in a suitable neighborhood of  $\lambda = +\infty$ . This is non trivial and indeed is false in the  $L \rightarrow \infty$  limit as we shall discuss later. The proof of analyticity is reported in appendix (C).

We can now systematically evaluate the perturbative strong coupling coefficients. We denote the large  $\lambda$  expansion coefficients as

$$\frac{\Delta_L(\lambda)}{L} = c_L \lambda^{1/4} + d_L + e_L \lambda^{-1/4} + \dots \quad (3.7)$$

We easily find the leading term (from the theorem in appendix (C), see also eq. (8.14) of [34])

$$c_L = \frac{2}{L} \left( \sum_{n_k > 0} n_k \right)^{1/2} = \frac{2}{L} \sqrt{\frac{1}{2} \frac{L-1}{2} \frac{L+1}{2}} = \frac{1}{\sqrt{2}} \left( 1 - \frac{1}{L^2} \right)^{1/2}. \quad (3.8)$$

in perfect agreement with [32].

The NLO is also easy. The expansion of momenta is

$$p_k = \frac{\alpha_k}{\lambda^{1/4}} + \frac{\beta_k}{\lambda^{1/2}} + \dots \quad (3.9)$$

where

$$\alpha_k = \frac{4\pi n_k}{L c_L}, \quad \beta_k = -\frac{4\pi n_k}{L c_L^2} \left( d_L + \frac{1}{2} \right). \quad (3.10)$$

On the other hand when  $p \neq 0$  we have

$$\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}} = \lambda^{1/4} \frac{1}{2\pi} |\alpha_k| + \frac{1}{2\pi} \beta_k \text{sign } \alpha_k. \quad (3.11)$$

Taking into account the term with  $p = 0$  we obtain

$$\frac{\Delta_L(\lambda)}{L} = \lambda^{1/4} \frac{1}{2\pi L} \sum_k |\alpha_k| + \frac{1}{2\pi L} \sum_k \beta_k \text{sign } \alpha_k + \frac{1}{2} + \frac{1}{L} + \dots \quad (3.12)$$

Consistency requires as before

$$\sum_k |n_k| = \frac{L^2}{2} c_L^2, \quad (3.13)$$

but also

$$\begin{aligned} d_L &= \frac{1}{2} + \frac{1}{L} + \frac{1}{2\pi L} \sum_k \frac{-4\pi n_k}{L c_L^2} \left( d_L + \frac{1}{2} \right) \text{sign } n_k = \\ &= \frac{1}{2} + \frac{1}{L} - \frac{2}{L^2 c_L^2} \sum_k |n_k| \left( d_L + \frac{1}{2} \right) = \frac{1}{L} - d_L. \end{aligned} \quad (3.14)$$

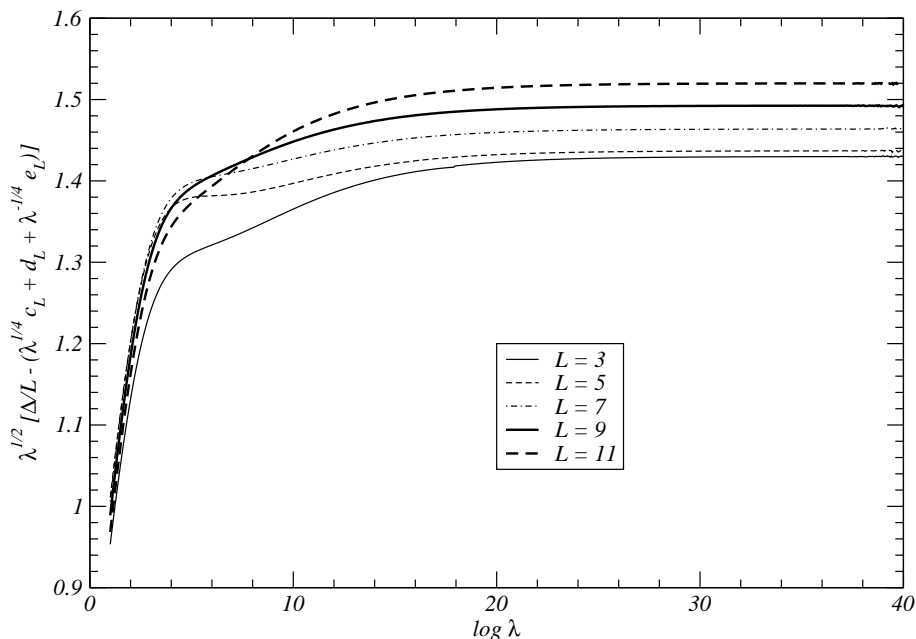
Hence,

$$d_L = \frac{1}{2L}. \quad (3.15)$$

The NNLO is more involved. After some calculations it reads

$$e_L = \frac{1}{4\sqrt{2}} \left( 1 + \frac{1}{L} \right)^2 \left( 1 - \frac{1}{L^2} \right)^{-1/2} - \frac{\pi^2}{12\sqrt{2}} \left( 1 - \frac{1}{L^2} \right)^{1/2} + \frac{1}{4\sqrt{2}} \left( 1 - \frac{1}{L^2} \right)^{1/2} \sum_{k=1}^{\frac{L-1}{2}} \frac{1}{k}. \quad (3.16)$$





**Figure 1:** Numerical check of the NNLO strong coupling expansion of  $\Delta_L(\lambda)(\lambda)$  in the  $\mathfrak{su}(1|1)$  sector.

At large  $L$ ,  $e_L \sim \ln L/(4\sqrt{2})$  and does not admit a finite limit as  $L \rightarrow \infty$ . In the next section we shall discuss this important point. It is worthwhile to emphasize that the NLO contributions to the dressing factor can in principle modify this term. Hence, it could be correct if the LCBA equations turned out to reabsorb the full dressing phase or, what is more natural, it would not be reliable if the LCBA equations required additional corrections.

As a consistency check of the calculation, the above expansion is confirmed by the numerical solution of the equation for  $\Delta$  as illustrated in figure (1) where we show the constant values approached at large  $\lambda$  by the difference

$$\lambda^{1/2} \left( \frac{\Delta_L(\lambda)}{L} - c_L \lambda^{1/4} - d_L - e_L \lambda^{-1/4} \right), \quad (3.17)$$

at various  $L$ .

To summarize, the main result of this section is the expansion

$$\lambda \rightarrow \infty, \quad \frac{\Delta_L(\lambda)}{L} = \frac{1}{\sqrt{2}} \left( 1 - \frac{1}{L^2} \right)^{1/2} \lambda^{1/4} + \frac{1}{2L} + e_L \lambda^{-1/4} + \dots \quad (3.18)$$

If one expands in  $1/L$ , then only the  $\mathcal{O}(L^0)$  and  $\mathcal{O}(L^{-1})$  terms are reliable because the LCBA equations are derived at first order in  $1/P_+$ . However,  $e_L$  has not a finite limit as  $L \rightarrow \infty$  as a hint of the fact that the physically meaningful limit is the opposite one  $L, \lambda \rightarrow \infty$ . In the next section, we shall confirm the calculation leading to (the first two terms of) eq. (3.18) by an independent calculation using the SBA equations in the same limit. Later, we shall discuss the opposite  $L, \lambda \rightarrow \infty$  case comparing it with eq. (3.18).

#### 4. Improved calculation in the SBA framework

As an independent check of eq. (3.18), we can repeat the calculation within the SBA equations. We briefly recall some information about the strong coupling expansion of the dressing factor. Then, we show that the leading term is enough to reproduce the first two terms in eq. (3.18). Finally we do the computation, finding full agreement.

##### 4.1 The dressing factor at strong coupling

The quantum string Bethe Ansatz equations can be written [20]

$$e^{i p_i L} = \prod_{j \neq i}^M S_{ij}, \quad S_{ij} = \left( \frac{x_i^+ - x_j^-}{x_i^- - x_j^+} \right)^{\mathfrak{s}} \frac{1 - \frac{\lambda}{16\pi^2} \frac{1}{x_i^+ x_j^-}}{1 - \frac{\lambda}{16\pi^2} \frac{1}{x_i^- x_j^+}} e^{i \vartheta_{ij}}, \quad (4.1)$$

where  $\vartheta$  is the universal dressing factor. In the various rank-1 sectors we have

$$\mathfrak{s} = \begin{matrix} 1 & 0 & -1 \\ \{\mathfrak{su}(2) & \mathfrak{su}(1|1) & \mathfrak{sl}(2)\} \end{matrix}, \quad L = J + \frac{\mathfrak{s} + 1}{2} M. \quad (4.2)$$

The variables  $x^\pm$  are again

$$x^\pm = \frac{e^{\pm i p/2}}{4 \sin(p/2)} \left( 1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \right). \quad (4.3)$$

We now introduce the variables

$$\zeta = \frac{2\pi}{\sqrt{\lambda}}, \quad \tilde{x}^\pm = 2 \zeta x^\pm. \quad (4.4)$$

The scattering phase can be written [26]

$$\vartheta_{ij} = \frac{1}{\zeta} \sum_{r \geq 2} \sum_{n \geq 0} c_{r,r+1+2n}(\zeta) (q_r(\tilde{x}_i) q_{r+1+2n}(\tilde{x}_j) - (i \leftrightarrow j)), \quad (4.5)$$

where the local charges are

$$q_r(\tilde{x}) = \frac{i}{r-1} \left( \frac{1}{(\tilde{x}^+)^{r-1}} - \frac{1}{(\tilde{x}^-)^{r-1}} \right). \quad (4.6)$$

The first two terms in the  $\zeta$ -expansion of  $c_{r,s}$  are

$$c_{r,s} = \delta_{r+1,s} - \zeta \frac{4}{\pi} \frac{(r-1)(s-1)}{(r+s-2)(s-r)} + \mathcal{O}(\zeta^2). \quad (4.7)$$

The scattering phase can be organized as

$$\vartheta_{ij} = \frac{1}{\zeta} [\chi_{ij}^{--} - \chi_{ij}^{-+} - \chi_{ij}^{+-} + \chi_{ij}^{++} - (i \leftrightarrow j)], \quad (4.8)$$

$$\chi_{ij}^{\sigma\sigma'} = \chi \left( \frac{4\pi}{\sqrt{\lambda}} x_i^\sigma, \frac{4\pi}{\sqrt{\lambda}} x_j^{\sigma'} \right). \quad (4.9)$$

The function  $\chi$  is expanded at strong coupling as:

$$\chi = \sum_{n \geq 0} \chi_n \zeta^n, \tag{4.10}$$

and the first two terms  $\chi_{0,1}$  can be given in closed form [26]

$$\chi_0(x, y) = -\frac{1}{y} - \frac{xy - 1}{y} \log \frac{xy - 1}{xy}, \tag{4.11}$$

$$\begin{aligned} \chi_1(x, y) = \frac{1}{\pi} & \left[ \log \frac{y - 1}{y + 1} \log \frac{x - 1/y}{x - y} + \right. \\ & + \text{Li}_2 \left( \frac{\sqrt{y} - 1/\sqrt{y}}{\sqrt{y} - \sqrt{x}} \right) - \text{Li}_2 \left( \frac{\sqrt{y} + 1/\sqrt{y}}{\sqrt{y} - \sqrt{x}} \right) \\ & \left. + \text{Li}_2 \left( \frac{\sqrt{y} - 1/\sqrt{y}}{\sqrt{y} + \sqrt{x}} \right) - \text{Li}_2 \left( \frac{\sqrt{y} + 1/\sqrt{y}}{\sqrt{y} + \sqrt{x}} \right) \right] \end{aligned} \tag{4.12}$$

Notice we cannot read trivially the powers of  $\zeta$  from eq. (4.10) because  $\lambda$  appears non trivially in the arguments of  $\chi_n$  as well as in the expression of  $x^\pm$ .

## 4.2 Subleading corrections to the SBA equations

Let us expand at large  $\lambda$  the Bethe momenta

$$p_k = \frac{\alpha_k}{\lambda^{1/4}} + \frac{\alpha'_k}{\lambda^{1/2}} + \dots \tag{4.13}$$

The dressing phase at LO and NLO can also be expanded (for  $\alpha_k > 0$ ) with the result

$$\vartheta_{kj}^{\text{LO}} = \frac{\alpha_k \alpha_j}{2\pi} + \frac{1}{\lambda^{1/4}} \left[ \alpha_k - \frac{2}{3} \alpha_j + \frac{\alpha'_k \alpha_j}{3\pi} + \frac{|\alpha_j|}{3} - \frac{\alpha'_k |\alpha_j|}{6\pi} + \frac{\alpha_k \alpha'_j}{2\pi} \right], \tag{4.14}$$

$$\vartheta_{kj}^{\text{NLO}} = \mathcal{O}(\lambda^{-1/2}). \tag{4.15}$$

This means that the coefficients  $\alpha'$  can be determined from the LO only, i.e. without involving  $\chi_1$  and its complicated analytic structure.

Let us now compute the subleading corrections in the SBA equations that explicitly involve the LO dressing factor. The SBA equations in logarithmic form are

$$p_k L = 2\pi n_k + \sum_{j \neq k} \left( \vartheta_{jk} - i \log \frac{1 - \frac{\lambda}{16\pi^2} \frac{1}{x_j^+ x_k^-}}{\lambda \frac{1}{16\pi^2} \frac{1}{x_j^- x_k^+}} \right) \tag{4.16}$$

With the previous notation for the sums and exploiting antisymmetry of the term in brackets we find

$$\sum_{k \in P} p_k L = 2\pi \sum_{k \in P} n_k + \sum_{k \in P} \sum_{j \in M} \left( \vartheta_{jk} - i \log \frac{1 - \frac{\lambda}{16\pi^2} \frac{1}{x_j^+ x_k^-}}{\lambda \frac{1}{16\pi^2} \frac{1}{x_j^- x_k^+}} \right) \tag{4.17}$$

For  $k \in P$  and  $j \in M$  we have at large  $\lambda$

$$-i \log \frac{1 - \frac{\lambda}{16\pi^2} \frac{1}{x_j^+ x_k^-}}{1 - \frac{\lambda}{16\pi^2} \frac{1}{x_j^- x_k^+}} = \frac{1}{2}(\alpha_j - \alpha_k) \frac{1}{\lambda^{1/4}} + \dots \quad (4.18)$$

At leading order we insert the first term of the expansion of  $\vartheta^{\text{LO}}$  and obtain

$$0 = 2\pi \sum_{k \in P} n_k + \frac{1}{2\pi} \sum_{k \in P} \sum_{j \in M} \alpha_k \alpha_j. \quad (4.19)$$

Using parity invariance we recover the known result

$$S \equiv \sum_{k \in P} \alpha_k = \frac{\pi}{\sqrt{2}} (L^2 - 1)^{1/2}. \quad (4.20)$$

The next correction is determined from the  $\mathcal{O}(\lambda^{-1/4})$  terms. The equation is

$$L \sum_{k \in P} \alpha_k = \sum_{k \in P} \sum_{j \in M} \left\{ \frac{1}{2}(\alpha_j - \alpha_k) + \alpha_k - \frac{2}{3}\alpha_j + \frac{\alpha'_k \alpha_j}{3\pi} + \frac{|\alpha_j|}{3} - \frac{\alpha'_k |\alpha_j|}{6\pi} + \frac{\alpha_k \alpha'_j}{2\pi} \right\} \quad (4.21)$$

Evaluating the sums and defining also

$$S' = \sum_{k \in P} \alpha'_k, \quad (4.22)$$

we easily obtain (again exploiting parity invariance of the Bethe momenta)

$$L S = \frac{L-1}{2} S - \frac{1}{\pi} S S' \quad \longrightarrow \quad S' = -\pi \frac{L+1}{2}. \quad (4.23)$$

The asymptotic expansion of the anomalous dimension is

$$\Delta_L(\lambda) = \frac{L}{2} + 1 + 2 \sum_{k \in P} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}} \quad (4.24)$$

$$\longrightarrow \frac{\lambda^{1/4}}{\pi} S + \frac{1}{\pi} S' + \frac{L+2}{2} + \mathcal{O}(\lambda^{-1/4}). \quad (4.25)$$

Replacing the values of  $S$  and  $S'$  we obtain

$$\frac{\Delta_L(\lambda)}{L} = \frac{1}{\sqrt{2}} \left(1 - \frac{1}{L^2}\right)^{1/2} \lambda^{1/4} + \frac{1}{2L} + \mathcal{O}(\lambda^{-1/4}), \quad (4.26)$$

in full agreement with the first two terms in eq. (3.18).

### 5. The $\mathfrak{su}(1|1)$ sector at $L \rightarrow \infty$ limit at fixed $\lambda$

In the previous sections we have established eq. (3.18) which is the  $\lambda, L \rightarrow \infty$  strong coupling expansion of the anomalous dimension. In this section, we discuss the  $1/L$  expansion of the equation determining  $\Delta$  as well as its (correct)  $L, \lambda \rightarrow \infty$  limit. The main point is that in the  $\mathfrak{su}(1|1)$  sector the LCBA equations for the Bethe momenta are immediately solved by eq. (3.4). Thus, we do not need any integral equation for the Bethe root distribution and we simply have to take the  $1/L$  expansion of a transcendental equation for  $\Delta$  itself.

We start with the LCBA equation that we write as

$$\Delta_L(\lambda) = \frac{L}{2} + 1 + 2 \sum_{k=1}^{\frac{L-1}{2}} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{2\pi k}{\Delta_L(\lambda) + \frac{L}{2}}}. \quad (5.1)$$

The sum in the r.h.s. can be evaluated by applying the Euler-MacLaurin summation formula ( $B_k$  are Bernoulli numbers)

$$\sum_{k=1}^{N-1} f(k) = \int_0^N f(x) dx - \frac{1}{2} [f(0) + f(N)] + \sum_{k \geq 1} \frac{B_{2k}}{(2k)!} [f^{(2k-1)}(N) - f^{(2k-1)}(0)], \quad (5.2)$$

where we notice that in our case  $f^{(2k-1)}(0) = 0$ . The Euler-MacLaurin formula provides an asymptotic expansion in powers of  $1/L$  because the  $k$ -th term in the last contribution to eq. (5.2) scales like  $1/L^{2k-1}$ . Our strategy will be that of solving the LCBA equation order by order in  $1/L$  in this asymptotic expansion. As discussed in appendix (D) the expansion is expected to be only asymptotic in the Poincaré sense. This is not surprising since the  $1/L$  expansion computes the various loop corrections in the  $\sigma$ -model and zero radius of convergence is a common feature in perturbation theory of non-trivial field theories.

Writing

$$\Delta_L(\lambda) = L \left( u - \frac{1}{2} \right) + z_1 + \frac{1}{L} z_2 + \dots \quad (5.3)$$

and expanding, we obtain the leading order result

$$u = 1 + \frac{u}{\pi} E \left( \frac{\pi}{u}, -\frac{\lambda}{\pi^2} \right), \quad (5.4)$$

where  $E(z, m)$  is the standard incomplete elliptic integral of the second kind

$$E(z, m) = \int_0^z \sqrt{1 - m \sin^2 \theta} d\theta. \quad (5.5)$$

The NLO and NNLO corrections are determined by  $u$  and are given by

$$z_1 = 0, \quad (5.6)$$

$$z_2 = -\frac{\lambda}{12\pi} \sin \frac{2\pi}{u} \frac{1}{\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{\pi}{u}} (1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{\pi}{u}})}. \quad (5.7)$$

Remarkably, the  $\mathcal{O}(1/L)$  correction vanishes. We can now expand  $u(\lambda)$  at large  $\lambda$ . As a consistency check, we also report in appendix (E) the expansion at small  $\lambda$ .

Setting

$$x = \frac{\pi}{u}, \quad t = \frac{\lambda}{\pi^2}, \quad (5.8)$$

we have to solve for  $t \rightarrow \infty$  the equation

$$\pi = x + E(x, -t). \quad (5.9)$$

The expansion of this equation at  $t \rightarrow +\infty$  and  $x = \mathcal{O}(t^{-1/4})$  is not at all trivial. It is worked out in appendix (F) with the result

$$E(x, -t) = (1 - \cos x)\sqrt{t} + \frac{1}{4\sqrt{t}} \left( 1 + \log(16t) + 2 \log \tan \frac{x}{2} \right) + \dots \quad (5.10)$$

Using this expansion, the solution of the above equation turns out to be

$$x(t) = \sqrt{2\pi}t^{-1/4} + \frac{1}{24\sqrt{2\pi}}t^{-3/4}(-3 \log t - 6 \log(8\pi) + 6 + 4\pi^2) + \dots \quad (5.11)$$

Replacing in

$$\lim_{L \rightarrow \infty} \frac{\Delta_L(\lambda)}{L} = \frac{\pi}{x} - \frac{1}{2}, \quad (5.12)$$

we obtain

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{\Delta_L(\lambda)}{L} &= u - \frac{1}{2} \\ &= \frac{1}{\sqrt{2}}\lambda^{1/4} + \frac{1}{48\sqrt{2}}\lambda^{-1/4}(3 \log \lambda + 18 \log 2 + 18 - 4\pi^2) + \mathcal{O}(\lambda^{-1/2} \log \lambda). \end{aligned} \quad (5.13)$$

Replacing the strong coupling expansion of  $u$  we obtain

$$z_2 = -\frac{1}{6\sqrt{2}}\lambda^{1/4} + \mathcal{O}(\lambda^{-1/4} \log \lambda). \quad (5.14)$$

In summary, we have found in the  $L, \lambda \rightarrow \infty$  limit:

$$\begin{aligned} \frac{\Delta_L(\lambda)}{L} &= \frac{1}{\sqrt{2}}\lambda^{1/4} + \frac{1}{48\sqrt{2}}\lambda^{-1/4}(3 \log \lambda + 18 \log 2 + 18 - 4\pi^2) + \mathcal{O}(\lambda^{-1/2} \log \lambda) + \\ &+ \frac{1}{L^2} \left( -\frac{1}{6\sqrt{2}}\lambda^{1/4} + \mathcal{O}(\lambda^{-1/4} \log \lambda) \right) + \dots \end{aligned} \quad (5.15)$$

A comparison with eq. (3.18) shows that the two limits in  $\lambda$  and  $L$  do not commute. The equations are valid in the order  $L, \lambda \rightarrow \infty$  where the result is eq. (5.15). *Only the leading term both in  $L$  and  $\lambda$  is independent on the order.* Similar results in the Hubbard model formulation of the gauge BA are illustrated in [13].

Actually, eq. (5.15) contains an additional information beyond this term. The second line is in principle affected by the NLO strong coupling terms in the dressing factor that, honestly, is not expected to be taken into account in the LCBA equations. Also, the  $1/L^2$

term in the third line is beyond the validity of the equations that are fixed by looking at  $\mathcal{O}(1/P_+)$  corrections. Nevertheless, we have proved that  $z_1 = 0$ . Thus, our result reads

$$L, \lambda \rightarrow \infty, \quad \frac{\Delta_L(\lambda)}{L} = \frac{1}{\sqrt{2}} \lambda^{1/4} + \mathcal{O}\left(\frac{\lambda^{-1/4}}{L}\right) + \mathcal{O}\left(\frac{\lambda^{1/4}}{L^2}\right). \quad (5.16)$$

Within this precision, the discrepancy with the  $\lambda, L \rightarrow \infty$  limit is localized in the  $1/(2L)$  term appearing in eq. (3.18).

## 6. The $\mathfrak{su}(2)$ sector

### 6.1 General features of the LCBA equations

The LCBA equations read in this sector

$$\exp\left(ip_k \frac{P_+ + M}{2}\right) = \prod_{j \neq k} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \quad (6.1)$$

where

$$x^\pm = \frac{1}{4} \left( \cot \frac{p}{2} \pm i \right) \left( 1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \right), \quad (6.2)$$

where again

$$P_+ = \Delta + J, \quad (6.3)$$

$$P_- = \Delta - J = \sum_{k=1}^M \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}}. \quad (6.4)$$

We are interested in the sector of operators with  $2L$  fields and zero angular momentum. So  $M = L$ ,  $J = 2L - M = L$ .

#### 6.1.1 $\lambda = 0$

Let us first discuss the case  $\lambda = 0$ . In this limit

$$\frac{1}{2}(P_+ + M) = J + M = 2L. \quad (6.5)$$

We define

$$u = \frac{1}{2} \cot \frac{p}{2}, \quad (6.6)$$

$$p = 2 \arctan \frac{1}{2u}. \quad (6.7)$$

The map is 1-1 with  $u \in \mathbb{R}$  and  $p \in (-\pi, \pi)$ . By standard manipulations we arrive at

$$2 \sum_{j \neq k} \arctan(u_k - u_j) - 4L \arctan(2u_k) = 2\pi J_k, \quad (6.8)$$

where the correct choice of Bethe quantum numbers for the AF state is

$$J_k = \left\{ -\frac{L-1}{2}, -\frac{L-3}{2}, \dots, \frac{L-3}{2}, \frac{L-1}{2} \right\}. \quad (6.9)$$

The associated solution has the first  $L/2$   $u > 0$  and the other negative. This Bethe equation can be recast in terms of the  $p$  variables and reads

$$2 \sum_{j \neq k} \arctan \left( \frac{1}{2} \cot \frac{p_k}{2} - \frac{1}{2} \cot \frac{p_j}{2} \right) + 2L p_k = 2\pi R_k \quad (6.10)$$

with

$$R_k = \left\{ \frac{L+1}{2}, \frac{L+3}{2}, \dots, L - \frac{1}{2} \right\} \cup (\text{the opposite list}). \quad (6.11)$$

### 6.1.2 $\lambda \neq 0$

We now take  $\lambda > 0$ . We simply have to replace

$$2L \longrightarrow \frac{\Delta_L(\lambda) + 2L}{2} \quad (6.12)$$

and use the more complicated form of  $x^\pm$ . The result is

$$2 \sum_{j \neq k} \arctan(X_{kj}) + \frac{\Delta_L(\lambda) + 2L}{2} p_k = 2\pi R_k, \quad (6.13)$$

$$\Delta_L(\lambda) = L + \sum_{k=1}^L \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}} \quad (6.14)$$

with the above  $\{R_k\}$  and where

$$X_{kj} = \frac{\cot \frac{p_k}{2} (1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}}) - \cot \frac{p_j}{2} (1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}})}{2 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}} + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}}}. \quad (6.15)$$

### 6.2 The $\lambda \rightarrow \infty$ limit at fixed $L$

The numerical solution of the LCBA equations at fixed  $L$  by means of the Newton algorithm [43] is perfectly feasible as discussed in full details in [32]. In figure (2), we show the case  $L = 10$  and the scaled momenta  $\lambda^{1/4} p_k$ . In the same figure, we have also shown the analytical prediction for the asymptotic  $p$  as derived below.

We can assume  $p_k \sim \alpha_k \lambda^{-1/4}$ . If  $\alpha_i > 0$  and  $\alpha_j < 0$  we have at large  $\lambda$

$$X_{ij} \rightarrow \frac{4\lambda^{1/4}}{\alpha_i - \alpha_j} \rightarrow +\infty \quad (6.16)$$

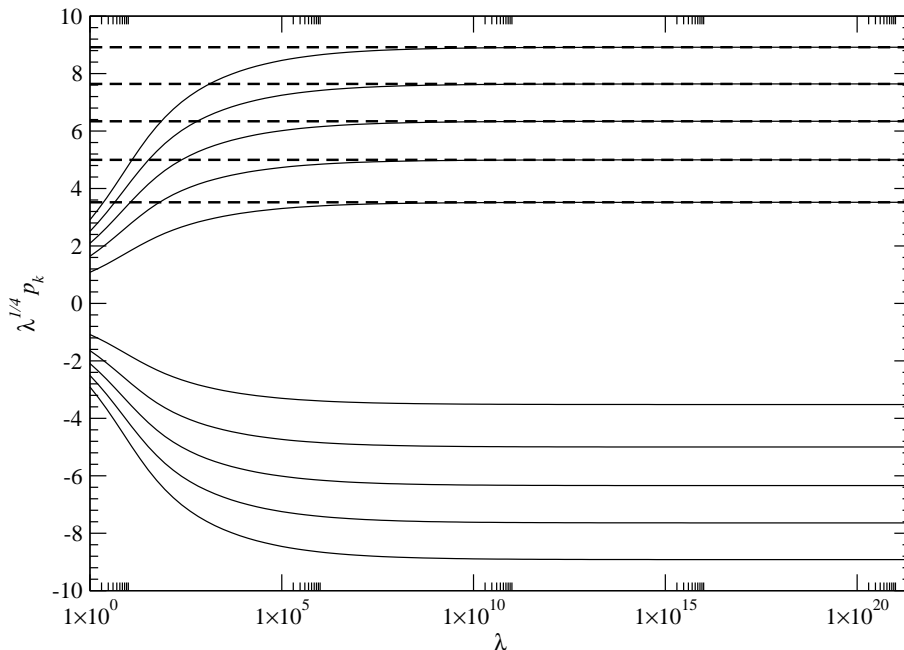
If instead  $\alpha_i, \alpha_j > 0$  we find

$$X_{ij} \rightarrow -\frac{4\pi}{\alpha_i \alpha_j} \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j}. \quad (6.17)$$

The Bethe equations reduce to

$$\frac{\pi L}{2} - 2 \sum_{\alpha_j > 0, j \neq i} \arctan \left( \frac{4\pi}{\alpha_i \alpha_j} \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \right) + \frac{1}{2\pi} \alpha_i \sum_{\alpha_j > 0} \alpha_j = 2\pi R_i. \quad (6.18)$$





**Figure 2:** Scaled Bethe momenta for the AF state in the  $\mathfrak{su}(2)$  sector. Here  $L = 10$ .

This equation determines the  $\alpha_i > 0$ . For instance, for  $L = 10$  we find

$$\begin{aligned}
 \alpha_1 &= 3.51948258944006628701335069602, \\
 \alpha_2 &= 4.99794847442355681996436161771, \\
 \alpha_3 &= 6.34122560474632117268892612476, \\
 \alpha_4 &= 7.63958729635505788529227272867, \\
 \alpha_5 &= 8.91768257093293021966752266564,
 \end{aligned}
 \tag{6.19}$$

which are the values appearing in the previous figure. The distribution of positive  $\alpha_k$  for  $L = 150$  is shown in figure (3).

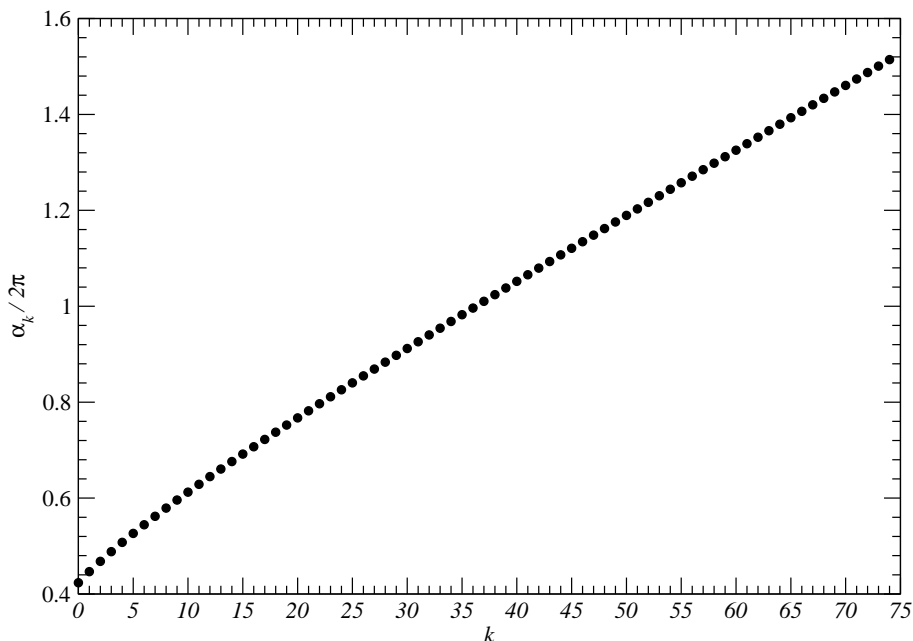
The strong coupling expansion of  $\Delta_L(\lambda)$  is thus again

$$\frac{\Delta_L(\lambda)}{2L} = c_L \lambda^{1/4} + d_L + \mathcal{O}(\lambda^{-1/4})
 \tag{6.20}$$

The analogous expansion for the Bethe momenta is

$$p_k = \frac{\alpha_k}{\lambda^{1/4}} + \frac{\beta_k}{\lambda^{1/2}} + \dots
 \tag{6.21}$$

and note that there is symmetry  $p \rightarrow -p$  in the solution for the highest state. Expanding,



**Figure 3:** Distribution of  $\alpha_k$  for  $L = 150$ .

we find ( $\epsilon_x = \text{sign } x$ )

$$\begin{aligned}
 \frac{\Delta_L(\lambda)}{2L} &= \frac{1}{2} + \frac{1}{2L} \sum_{k=1}^M \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \\
 &= \frac{\lambda^{1/4}}{4\pi L} \sum_k |\alpha_k| + \frac{1}{4\pi L} \sum_k \epsilon_{\alpha_k} \beta_k + \frac{1}{2} + \dots \\
 &= \frac{\lambda^{1/4}}{2\pi L} \sum_{k \in P} \alpha_k + \frac{1}{2\pi L} \sum_{k \in P} \beta_k + \frac{1}{2} + \dots,
 \end{aligned}
 \tag{6.22}$$

where  $P$  is the set of  $k$  such that  $\alpha_k > 0$ .

The Bethe equation can be written in the  $\lambda \rightarrow \infty$  limit and for  $i \in P$

$$\begin{aligned}
 2 \sum_{j \in P} \arctan X_{ij} + \sum_{j \notin P} \left( \pi - \frac{1}{2}(\alpha_i - \alpha_j) \lambda^{-1/4} + \dots \right) + \frac{\alpha_i}{4\pi} \sum_j |\alpha_j| + \\
 + \frac{1}{2} \left\{ \frac{\beta_i}{2\pi} \sum_j |\alpha_j| + 3L \alpha_i + \frac{\alpha_i}{2\pi} \sum_j \epsilon_{\alpha_j} \beta_j \right\} \lambda^{-1/4} = 2\pi R_i.
 \end{aligned}
 \tag{6.23}$$

and using the parity symmetry

$$\begin{aligned}
 2 \sum_{j \in P} \arctan X_{ij} + \sum_{j \in P} \left( \pi - \frac{1}{2}(\alpha_i + \alpha_j) \lambda^{-1/4} + \dots \right) + \frac{\alpha_i}{2\pi} \sum_{j \in P} \alpha_j + \\
 + \frac{1}{2} \left\{ \frac{\beta_i}{\pi} \sum_{j \in P} \alpha_j + 3L \alpha_i + \frac{\alpha_i}{\pi} \sum_{j \in P} \beta_j \right\} \lambda^{-1/4} = 2\pi R_i.
 \end{aligned}
 \tag{6.24}$$

We now sum over  $i \in P$  and due to

$$\sum_{i,j \in P} \arctan X_{ij} = 0, \tag{6.25}$$

we obtain

$$\begin{aligned} \frac{\pi L^2}{4} - \frac{L}{2} \sum_{i \in P} \alpha_i \lambda^{-1/4} + \frac{1}{2\pi} \left( \sum_{i \in P} \alpha_i \right)^2 \\ + \left\{ \frac{1}{\pi} \sum_{i \in P} \beta_i \sum_{j \in P} \alpha_j + \frac{3}{2} L \sum_{i \in P} \alpha_i \right\} \lambda^{-1/4} = 2\pi \sum_{R_i > 0} R_i. \end{aligned} \tag{6.26}$$

Collecting terms, we find the leading order

$$\frac{1}{2\pi} \left( \sum_{\alpha_i > 0} \alpha_i \right)^2 = 2\pi \sum_{R_i > 0} R_i - \frac{\pi L^2}{4} = 2\pi \frac{3L^2}{8} - \frac{\pi L^2}{4} = \frac{\pi L^2}{2} \tag{6.27}$$

Hence,

$$\sum_{\alpha_i > 0} \alpha_i = \pi L, \quad \longrightarrow \quad c_L = \frac{1}{2}. \tag{6.28}$$

Also, the NLO terms give

$$\frac{1}{\pi} \sum_{i \in P} \beta_i = -L, \quad \longrightarrow \quad d_L = 0. \tag{6.29}$$

In summary,

$$\lambda \rightarrow \infty, \quad \frac{\Delta_L(\lambda)}{2L} = \frac{1}{2} \lambda^{1/4} + 0 + \mathcal{O}(\lambda^{-1/4}). \tag{6.30}$$

This result can also be obtained in the SBA framework by repeating the calculation we did in the  $\mathfrak{su}(1|1)$  sector. Unfortunately, here we are not able to find the strong coupling limit of the LCBA equations at large  $L$ , exactly as with the SBA equations. From our experience in the  $\mathfrak{su}(1|1)$  it seems very reasonable to claim that the leading term  $1/2 \lambda^{1/4}$  is independent on the order of limits.

## 7. Conclusions

The story of AdS/CFT duality is vexed by discrepancies related to the different limits in which calculations can be performed under control on the two sides of the correspondence. This is usually considered a weak coupling problem. In BMN limits, one takes a large  $R$ -charge  $J$  and 't Hooft coupling  $\lambda$  to control the string side. When going to small  $\lambda' = \lambda/J^2$  it is possible to compare with gauge theory perturbative calculations, but this is just one of the infinite directions along which  $\lambda$  and  $J$  can grow.

In this paper, we have considered these problems from another perspective wondering whether the large  $\lambda$  and  $L$  region is free of ambiguities. We have shown that this is true only at leading order. Actually, this is a problem which is not immediately related to the

AdS/CFT correspondence. Instead, it seems to be a genuine feature of the string Bethe Ansatz equations which are derived not only assuming that both  $\lambda$  and  $L$  are large, but also taking the two limits in a precise order. We have shown that the anomalous dimensions of the highest states in the compact rank-1 sectors do depend on the order of limits beyond the leading term. This is not at all surprising, but seems to us an important warning.

Our results have been possible due to the particular simplicity of the light-cone string Bethe Ansatz equations in the fermionic  $\mathfrak{su}(1|1)$  sector where the Bethe roots distribution is trivial for all  $L$ . Gauge independence of the anomalous dimensions suggests that the result should hold also for the standard equations with the AFS phase, although we could not prove this statement in that context.

In conclusion, we remark that although rather special, highest states appears to be an interesting island in the moduli space of the  $AdS_5 \times S^5$  superstring, complementary to pp-wave and spinning string states. Indeed, our limited investigation has revealed some subtleties in the structural properties of its quantum Bethe Ansatz equations enlightening with explicit calculations the detailed way in which the Gubser-Klebanov-Polyakov law is reproduced.

### Acknowledgments

We would like to thank A. Tseytlin for drawing our attention to the LCBA.

### A. Existence and unicity of $\Delta_L(\lambda)$ in the $\mathfrak{su}(1|1)$ sector

**Theorem 1.** *The light cone equation eq. (3.5) for  $\Delta_L(\lambda)$  at fixed  $L$  admits a unique solution.*

*Proof:* We write the equation in the form

$$\Delta_L(\lambda) = \frac{L}{2} + 1 + 2 \sum_{k=1}^{\frac{L-1}{2}} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{2\pi k}{\Delta_L(\lambda) + L/2}} \tag{A.1}$$

The equation implies

$$\Delta_L(\lambda) \geq \frac{L}{2} + 1 + 2 \frac{L-1}{2} = \frac{3L}{2}. \tag{A.2}$$

The derivative of the square root is

$$\frac{d}{d\Delta_L(\lambda)} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{2\pi k}{\Delta_L(\lambda) + L/2}} = -\frac{k\lambda}{\pi} \frac{\sin \frac{4\pi k}{\Delta_L(\lambda) + L/2}}{(\Delta_L(\lambda) + L/2)^2 \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{2\pi k}{\Delta_L(\lambda) + L/2}}} \tag{A.3}$$

When  $\Delta_L(\lambda) \geq 3L/2$  and  $1 \leq k \leq (L-1)/2$ , the above expression is negative. Hence the right hand side of eq. A.1 is monotonically decreasing with  $\Delta_L(\lambda)$ . We conclude that there is always a unique intersection with the left hand side.

□

## B. Weak coupling expansion in the $\mathfrak{su}(1|1)$ sector

It is clear that the function  $\Delta_L(\lambda)$  (at fixed  $L$ ) is holomorphic in  $\lambda$  in a neighborhood of  $\lambda = 0$  by the analytic implicit function theorem [44]. We then expand

$$\frac{\Delta_L(\lambda)}{L} = \sum_{n=0}^{\infty} \gamma_L^{(n)} \left(\frac{\lambda}{\pi^2}\right)^n \tag{B.1}$$

With some effort, the various coefficients can be evaluated analytically. The first is trivial

$$\gamma_L^{(0)} = \frac{3}{2}. \tag{B.2}$$

The next coefficient is

$$\gamma_L^{(1)} = \frac{1}{L} \sum_{n=0}^{\frac{L-1}{2}} \sin^2 \frac{n\pi}{L} = \frac{1}{4}, \quad (L \in 2\mathbb{N} + 1) \tag{B.3}$$

We have computed the analytical expression of the next two coefficients and it reads

$$\gamma_L^{(2)} = -\frac{1}{64} \left( 3 + \frac{2\pi}{L \sin \frac{\pi}{L}} \right), \tag{B.4}$$

$$\gamma_L^{(3)} = \frac{1}{1024 L^2} \frac{1}{\cos^2 \frac{\pi}{2L}} \left[ 20 L^2 \cos^2 \frac{\pi}{2L} + \pi L \cot \frac{\pi}{2L} \left( 9 + \frac{1}{\cos \frac{\pi}{L}} \right) + 2\pi^2 \right]. \tag{B.5}$$

Their expansion at large  $L$  is

$$\begin{aligned} \gamma_L^{(2)} &= -\frac{5}{64} - \frac{1}{192} \frac{\pi^2}{L^2} + \dots, \\ \gamma_L^{(3)} &= \frac{5}{128} + \frac{19}{3072} \frac{\pi^2}{L^2} + \dots \end{aligned} \tag{B.6}$$

The other coefficients  $\{\gamma_L^{(n)}\}_{n \geq 4}$  are more and more involved functions of  $L$ . Their expression is not enlightening.

Starting from  $\gamma_L^{(2)}$  the expansion coefficients depend on  $L$ . This is in sharp contrast with what is obtained in the usual conformal gauge. There, all coefficients  $\gamma_L^{(n)}$  are  $L$ -independent for a suitably large (but finite)  $L$  [31, 32]. Beside and more remarkably, the disagreement with perturbative gauge theory starts at two loops. This is a simple fact that in our opinion suggest that future improvement of the LCBA equations will be needed to match the genuine weak coupling region.

## C. Analyticity of $\Delta_L(\lambda)$ in the $\mathfrak{su}(1|1)$ sector at large $\lambda$

**Theorem 2.** *The solution of the light cone equation eq. (3.5) for  $\Delta_L(\lambda)$  at fixed  $L$  admits an analytic expansion at large  $\lambda$  of the form*

$$\Delta_L(\lambda) = \sum_{n=0}^{\infty} a_n (\lambda^{1/4})^{1-n}, \tag{C.1}$$

with a finite radius of convergence.

*Proof:* we start again from

$$\Delta_L(\lambda) = \frac{L}{2} + \sum_{k=1}^L \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{2\pi n_k}{\Delta_L(\lambda) + L/2}}, \quad (\text{C.2})$$

$$\{n_k\} = \left\{ -\frac{L-1}{2}, \dots, 0, \dots, \frac{L-1}{2} \right\}. \quad (\text{C.3})$$

We set

$$w = \frac{2\pi}{\Delta_L(\lambda) + L/2}, \quad x^4 = \frac{\pi^2}{\lambda}. \quad (\text{C.4})$$

The equation becomes

$$\frac{2\pi}{w} - L = 1 + 2 \sum_{k=1}^{\frac{L-1}{2}} \sqrt{1 + \frac{1}{x^4} \sin^2(kw)}, \quad (\text{C.5})$$

or, equivalently:

$$2\pi x^2 - (L+1)x^2 w - 2w \sum_{k=1}^{\frac{L-1}{2}} \sin(kw) \sqrt{1 + \frac{x^4}{\sin^2(kw)}} = 0. \quad (\text{C.6})$$

We scale  $z = x/w$  and obtain

$$\Phi(z, w) = 0, \quad (\text{C.7})$$

where

$$\Phi(z, w) = 2\pi z^2 - (L+1)z^2 w - 2 \sum_{k=1}^{\frac{L-1}{2}} \frac{\sin(kw)}{w} \sqrt{1 + z^4 \frac{w^4}{\sin^2(kw)}} \quad (\text{C.8})$$

The equation

$$\Phi(z_0, 0) = 0, \quad (\text{C.9})$$

has the solution

$$z_0^2 = \frac{1}{\pi} \sum_{k=1}^{\frac{L-1}{2}} k = \frac{L^2 - 1}{8\pi}. \quad (\text{C.10})$$

This corresponds to the asymptotic term

$$\frac{\Delta_L(\lambda)}{L} \sim \frac{1}{\sqrt{2}} \left(1 - \frac{1}{L^2}\right)^{1/2} \lambda^{1/4}. \quad (\text{C.11})$$

Now, the function  $\Phi(z, w)$  is holomorphic in a neighborhood of  $(z_0, 0)$  and its partial derivatives are non vanishing at that point since

$$\left. \frac{\partial \Phi}{\partial z} \right|_{(z_0, 0)} = 4\pi z_0, \quad \left. \frac{\partial \Phi}{\partial w} \right|_{(z_0, 0)} = -(L+1)z_0^2. \quad (\text{C.12})$$

Therefore, by exploiting once again the analytic implicit function theorem [44], we conclude that both  $z(w)$  and  $w(z)$  are holomorphic functions and also that  $w$  is an analytic function of  $x = \sqrt{\pi} \lambda^{-1/4}$  in a neighbourhood of  $x = 0$ , which is our thesis.  $\square$

### D. Convergence properties of the $1/L$ expansion of the gap equation in the $\mathfrak{su}(1|1)$ sector

We discuss in some details the convergence properties of the  $1/L$  expansion of the gap equation in the  $\mathfrak{su}(1|1)$  sector. The difficult piece is the finite sum appearing in the LCBA

$$h_L = \sum_{k=1}^{\frac{L-1}{2}} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{2\pi k}{L u}} \equiv \sum_{k=1}^{\frac{L-1}{2}} f\left(\frac{k}{L}\right). \quad (\text{D.1})$$

We study this sum treating  $\lambda$  and  $\Delta/L$  as fixed parameters and discussing the convergence of the Euler-MacLaurin summation at large  $L$ . As a little simplification, we keep the leading term in  $\Delta$  and therefore set  $\Delta/L = u$ , with a fixed  $u \geq 2$ .

The Euler-MacLaurin summation formula gives

$$h_L = \int_0^{\frac{L+1}{2L}} f(x) dx - \frac{1}{2} \left[ f(0) + f\left(\frac{L+1}{2L}\right) \right] + e_L, \quad (\text{D.2})$$

where

$$e_L = \sum_{p \geq 1} \frac{B_{2p}}{(2p)! L^{2p-1}} f^{(2p-1)}\left(\frac{L+1}{2L}\right). \quad (\text{D.3})$$

This is known to be an asymptotic expansion of the Poincaré type, not necessarily convergent. To investigate the convergence properties of eq. (D.3), we exploit an exact representation of  $e_L$  at finite  $L$  provided by the Abel-Plana formula [45]

$$e_L = -i \int_0^\infty \frac{1}{e^{2\pi\rho} - 1} \left[ f\left(\frac{L+1+2i\rho}{2L}\right) - f\left(\frac{L+1-2i\rho}{2L}\right) \right] d\rho \quad (\text{D.4})$$

The advantage of this formula is that it can be analytically continued in the  $L$  variable in order to study its analytic structure. From the formula and the specific form of  $f(z)$ , we see that there is a cut extending up to  $L \rightarrow \infty$  forbidding analyticity. As a check, we have evaluated several hundreds of terms in the series

$$e_L = \sum_{k \geq 0} \frac{c_k}{L^k}. \quad (\text{D.5})$$

By the way, this can be done quite efficiently by expanding the integrand of the Abel-Plana formula and integrating term by term. We have performed the computation for generic values of  $\lambda, u$  as well as for the pair  $(\lambda, u(\lambda))$  solving eq. (5.4), one easily always find that the successive odd coefficients  $d_k = c_{2k+1}$  have the leading behavior

$$|d_k| \sim a b^k k^c k^{dk}, \quad (\text{D.6})$$

with  $d > 0$  and suitable  $a, b, c$ . The convergence radius of the expansion is therefore confirmed to be zero.

As a toy computation explaining the precise origin of this non-analyticity, one can consider the following simpler integral having the same analytic structure of the Abel-Plana formula for our problem,

$$I(z) = \int_0^\infty e^{-\rho} \sqrt{1 + \rho z}. \quad (\text{D.7})$$

The exact integral can be evaluated and its imaginary part is indeed discontinuous at  $\arg z = \pi$  for any radius  $|z|$  showing the presence of a cut branching from  $z = 0$ . If we expand the integrand in powers of  $z$  and integrate each term, we obtain the asymptotic expansion

$$I(z) = \sum_{k \geq 0} c_k z^k, \quad c_k = \binom{\frac{1}{2}}{k} \Gamma(k+1). \quad (\text{D.8})$$

Using the expansions at large  $k$

$$\binom{\frac{1}{2}}{k} \sim \frac{2}{\sqrt{\pi}} \sin[\pi(k-1/2)] k^{-3/2}, \quad (\text{D.9})$$

$$\Gamma(k) \sim \sqrt{2\pi} k^{k-1/2} e^{-k}, \quad (\text{D.10})$$

we obtain

$$|c_k| \sim \frac{1}{\sqrt{2}} e^{-k} k^{k-1}, \quad (\text{D.11})$$

which has the same form as eq. (D.6).

### E. Weak coupling expansion of $\Delta_L(\lambda)_{\text{su}(1|1)}$ in the $L \rightarrow \infty$ limit

The weak coupling expansion of the eq. (5.4) is straightforward and we find

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{\Delta_L(\lambda)}{L} = u - \frac{1}{2} = \frac{3}{2} + \frac{1}{4} \frac{\lambda}{\pi^2} - \frac{5}{64} \left(\frac{\lambda}{\pi^2}\right)^2 + \frac{5}{128} \left(\frac{\lambda}{\pi^2}\right)^3 + \\ + \frac{4\pi^2 - 1179}{49152} \left(\frac{\lambda}{\pi^2}\right)^4 + \frac{3240 - 29\pi^2}{196608} \left(\frac{\lambda}{\pi^2}\right)^5 + \mathcal{O}(\lambda^6). \end{aligned} \quad (\text{E.1})$$

Replacing  $u$  in  $z_2$  we also obtain

$$z_2 = \pi^2 \left( -\frac{1}{192} \left(\frac{\lambda}{\pi^2}\right)^2 + \frac{19}{3072} \left(\frac{\lambda}{\pi^2}\right)^3 + \frac{\pi^2 - 462}{73728} \left(\frac{\lambda}{\pi^2}\right)^4 + \dots \right). \quad (\text{E.2})$$

The agreement with our previous results eq. (B.6) is complete.

### F. The expansion of $E(x, -t)$ for $t \rightarrow +\infty$

**Theorem 3.** *The incomplete elliptic integral  $E(x, -t)$  with fixed  $x > 0$ , admits the expansion for  $t \rightarrow +\infty$*

$$E(x, -t) = h_0(x)\sqrt{t} + \sum_{n=1}^{\infty} \frac{h_n(x) + c_n \log(16t)}{t^{n-\frac{1}{2}}}, \quad (\text{F.1})$$

where the first terms of the expansion are

$$\begin{aligned} E(x, -t) = (1 - \cos x)\sqrt{t} + \frac{1}{4\sqrt{t}} \left( 1 + \log(16t) + 2 \log \tan \frac{x}{2} \right) + \\ + \frac{1}{64t^{3/2}} \left( 3 - 2 \log(16t) - 4 \log \tan \frac{x}{2} + 4 \frac{\cos x}{\sin^2 x} \right) + \dots \end{aligned} \quad (\text{F.2})$$

and the other are explicitly constructed in the proof.



*Proof:* First we split

$$E(x, -t) = E(-t) - \int_x^{\pi/2} \sqrt{1 + t \sin^2 \theta} d\theta, \tag{F.3}$$

where  $E(-t)$  is the complete elliptic integral of the second kind

$$E(-t) = \int_0^{\pi/2} \sqrt{1 + t \sin^2 \theta} d\theta = \frac{\pi}{2} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1, -t\right). \tag{F.4}$$

Its expansion for large  $t$  is non trivial since in the integral the quantity  $t \sin^2 \theta$  is not large when  $\theta \rightarrow 0$ .

The asymptotic expansion can be derived by using the formula

$$\begin{aligned} {}_2F_1(a, a + m, c, z) &= \frac{\Gamma(c)(-z)^{-a-m}}{\Gamma(a+m)\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{(a)_{n+m}(1-c+a)_{n+m}}{n!(n+m)!} z^{-n} [\log(-z) + \\ &+ \psi(1+m+n) + \psi(1+n) - \psi(a+m+n) - \psi(c-a-m-n)] + \\ &+ (-z)^a \frac{\Gamma(c)}{\Gamma(a+m)} \sum_{n=0}^{m-1} \frac{\Gamma(m-n)(a)_n}{n!\Gamma(c-a-n)} z^{-n} \end{aligned} \tag{F.5}$$

which is valid for  $|\arg(-z)| < \pi$ ,  $|z| > 1$  and  $c - a \notin \mathbb{Z}$ . We are interested in the case  $a = -1/2$ ,  $m = 1$  and  $c = 1$  that gives

$$\begin{aligned} E(-t) &= \frac{\pi}{2} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1, -t\right) \\ &= \sqrt{t} + \frac{1}{\sqrt{t}} \left[ \frac{1 + \log 16t}{4} + \frac{3 - 2 \log 16t}{64t} + \frac{3(\log 16t - 2)}{256t^2} + \frac{5(133 - 60 \log 16t)}{49152t^3} + \dots \right] \end{aligned} \tag{F.6}$$

This expansion can be applied for  $t > 1$ .

The second integral in eq. (F.3) can be expanded at large  $t$  provided  $t \sin^2 x > 1$  as follows

$$\begin{aligned} \int_x^{\pi/2} \sqrt{1 + t \sin^2 \theta} d\theta &= \sum_{k=0}^{\infty} \binom{1/2}{k} t^{\frac{1}{2}-k} \int_x^{\pi/2} \frac{1}{\sin^{2k-1} \theta} d\theta = \\ &= \sqrt{t} \cos x - \sum_{k=1}^{\infty} \binom{1/2}{k} t^{\frac{1}{2}-k} I_k(\cos x), \end{aligned} \tag{F.7}$$

where

$$I_k(a) = \int_0^a \frac{1}{(1-u^2)^k} du \tag{F.8}$$

This integral is elementary and reads

$$I_k(a) = \frac{\Gamma(k-1/2)}{2\sqrt{\pi}\Gamma(k)} \log \frac{1+a}{1-a} + \frac{P_k(a)}{(1-a^2)^{k-1}}, \tag{F.9}$$

where the polynomials  $P_k(a)$  are defined by

$$P_1(a) = 0, \tag{F.10}$$

$$P_{k+1}(a) = \left(1 - \frac{1}{2k}\right) (1 - a^2) P_k(a) + \frac{a}{2k}. \tag{F.11}$$

The first cases are

$$P_2(a) = \frac{a}{2}, \tag{F.12}$$

$$P_3(a) = -\frac{a}{8}(3a^2 - 5), \tag{F.13}$$

$$P_4(a) = \frac{a}{48}(15a^4 - 40a^2 + 33). \tag{F.14}$$

Collecting these results, we obtain

$$\int_x^{\pi/2} \sqrt{1 + t \sin^2 \theta} d\theta = \sqrt{t} \cos x - \frac{1}{2\sqrt{t}} \log \tan \frac{x}{2} + \frac{1}{16t^{3/2}} \left( \log \tan \frac{x}{2} - \frac{\cos x}{\sin^2 x} \right) + \dots \tag{F.15}$$

Combining our results we prove the thesis. □

Remark: from the proof, we see that the expansion is valid if  $t \sin^2 x > 1$ . For our application we have  $x \sim t^{-1/4}$  and the expansion can be applied for large  $t$ .

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