



Nondecoupling in multi-Higgs doublet models

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Abstract We consider models with any number of Higgs doublets and study the conditions for decoupling. We show that, under very general circumstances, all the quadratic coefficients of the scalar potential must be present, except in special cases, which include terms related to directions of vanishing vacuum expectation values. We give a few examples. Moreover, we show that the decoupling of all charged scalars implies the decoupling of all extra neutral scalars and vanishing \mathcal{CP} violation in scalar-pseudoscalar mixing.

1 Introduction

It has been determined experimentally that there are four electroweak gauge bosons [1–4], as predicted by the $\text{SU}(2)_L \times \text{U}(1)_Y$ gauge group of the Standard Model (SM). The SM gauge structure does not predict the number of fermion families, which was established at LEP and SLD by the invisible width of the Z [5]. Neither does the SM structure predict the number of scalar doublet families. This is the most fundamental open question, and it is being actively pursued at LHC.

The ATLAS and CMS experiments have already identified a 125 GeV Higgs particle (h_{125}) [6, 7], as established in the (minimal $N = 1$ Higgs version of the) SM, and have checked that its couplings are consistent with those predicted, within errors of order 20% [8]. In N Higgs doublet models (NHDM) there are in general $2N - 1$ neutral scalars S_α^0 ($\alpha = 2, \dots, 2N$) and $N - 1$ charged scalar pairs S_a^\pm ($a = 2, \dots, N$).¹ States beyond the $S_2^0 = h_{125}$ scalar have not been found, nor its number limited. The reason is that the generic NHDM has

a decoupling limit, where the extra scalars have very high masses and the remaining h_{125} has basically the properties of the SM one.

There are, however, NHDMs in which the extra Higgses do not decouple. That is the case, for example, with the 2HDM with an exact Z_2 symmetry [9]. Nevertheless, Gunion and Haber [10] have shown that the decoupling limit is recovered by including in the potential a term of dimension two which breaks softly the Z_2 symmetry. Nondecoupling has also been analyzed by Nebot in the 2HDM with spontaneous \mathcal{CP} violation, both with and without soft symmetry breaking terms [11], with interesting physical consequences explored by Nierste, Tabet, and Ziegler [12]. One further example with 3HDM and a S_3 symmetry has been discussed by Bhattacharyya and Das [13]. In this article we discuss the situation in the NHDM. If these have an exact symmetry limiting the number of quadratic and quartic couplings, then one can expect that nondecoupling occurs. Conversely, as we show in this article, for the existence of a decoupling limit, all soft breaking terms must be included in the scalar potential, except when there are small mixing angles in the scalar sector, which includes directions with vanishing vacuum expectation values. We introduce our notation in Sect. 2, where we also present our first results. Section 3 is devoted to theorems valid when there are no vanishing vacuum expectation values (vev) and/or small mixing angles. We present a few 3HDM examples in Sect. 4, both of the theorems and of what happens when the assumptions of the theorems are violated. We discuss briefly \mathcal{CP} violation in Sect. 5. In Sect. 6 we derive one further theorem, valid when only one vev is nonzero, and we present a complete discussion of all symmetric 2HDM. We draw our conclusions in Sect. 7. Some detailed discussions have been relegated to the Appendices.

2 Notation and first results

Consider a $\text{SU}(2)_L \times \text{U}(1)_Y$ theory with N complex scalar doublets Φ_i with hypercharge $Y = 1/2$.

¹We keep the first entries $S_1^0 = G^0$ and $S_1^\pm = G^\pm$ for the would-be Goldstone bosons.

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2.1 The scalar potential

Following the notation in [14–16], we write the scalar potential as

$$V_H = Y_{ij} (\Phi_i^\dagger \Phi_j) + Z_{ij,kl} (\Phi_i^\dagger \Phi_j) (\Phi_k^\dagger \Phi_l), \quad (1)$$

whose hermiticity implies,

$$Y_{ij} = Y_{ji}^*, \quad Z_{ij,kl} = Z_{kl,ij}^* = Z_{ji,lk}^*. \quad (2)$$

Requiring a massless photon implies that the global minimum corresponds to vacuum expectation values (vev) which preserve the charge symmetry, $\mathbb{U}(1)_Q$, generated by $Q = T_3 + Y$. Expanding the fields around those vevs v_i , we write

$$\Phi_i = v_i + \varphi_i = \begin{pmatrix} 0 \\ v_i/\sqrt{2} \end{pmatrix} + \begin{pmatrix} \phi_i^+ \\ (\rho_i + i\chi_i)/\sqrt{2} \end{pmatrix}, \quad (3)$$

where each v_i is in general complex. The stationary conditions are given by

$$(Y_{ij} + Z_{ij,kl} v_k^* v_l) v_j = 0. \quad (4)$$

Implicit summation of repeated indices is used throughout, except where noted otherwise. The mass matrix for the charged scalars is,

$$(M_{\pm}^2)_{ij} = Y_{ij} + Z_{ij,kl} v_k^* v_l, \quad (5)$$

in terms of which the stationarity conditions in (4) may be written as

$$(M_{\pm}^2)_{ij} v_j = 0. \quad (6)$$

The mass matrix for the neutral scalars is,

$$M_0^2 = \begin{pmatrix} M_{\rho\rho}^2 & M_{\rho\chi}^2 \\ M_{\chi\rho}^2 & M_{\chi\chi}^2 \end{pmatrix}, \quad (7)$$

$$(M_{\rho\rho}^2)_{ij} = \text{Re} \{ Y_{ij} + Z_{ij,kl} v_k^* v_l + Z_{ik,lj} v_k v_l^* + Z_{ik,jl} v_k v_l \}, \quad (8)$$

$$(M_{\chi\chi}^2)_{ij} = \text{Re} \{ Y_{ij} + Z_{ij,kl} v_k^* v_l + Z_{ik,lj} v_k v_l^* - Z_{ik,jl} v_k v_l \}, \quad (9)$$

$$-(M_{\rho\chi}^2)_{ij} = \text{Im} \{ Y_{ij} + Z_{ij,kl} v_k^* v_l + Z_{ik,lj} v_k v_l^* - Z_{ik,jl} v_k v_l \}, \quad (10)$$

$$(M_{\chi\rho}^2)_{ij} = \text{Im} \{ Y_{ij} + Z_{ij,kl} v_k^* v_l + Z_{ik,lj} v_k v_l^* + Z_{ik,jl} v_k v_l \}, \quad (11)$$

where, under the canonical definition of \mathcal{CP} , $M_{\rho\rho}^2$ is the mass matrix of the \mathcal{CP} -even scalars, $M_{\chi\chi}^2$ of the \mathcal{CP} -odd scalars, and $M_{\rho\chi}^2 = (M_{\chi\rho}^2)^T$ gives the mixing between the \mathcal{CP} -even

and \mathcal{CP} -odd scalars. Substituting (5), we obtain

$$(M_{\rho\rho}^2)_{ij} = \text{Re} \left\{ (M_{\pm}^2)_{ij} + Z_{ik,lj} v_k v_l^* + Z_{ik,jl} v_k v_l \right\}, \quad (12)$$

$$(M_{\chi\chi}^2)_{ij} = \text{Re} \left\{ (M_{\pm}^2)_{ij} + Z_{ik,lj} v_k v_l^* - Z_{ik,jl} v_k v_l \right\}, \quad (13)$$

$$-(M_{\rho\chi}^2)_{ij} = \text{Im} \left\{ (M_{\pm}^2)_{ij} + Z_{ik,lj} v_k v_l^* - Z_{ik,jl} v_k v_l \right\}. \quad (14)$$

2.2 Basis freedom and symmetries

We may choose to describe the theory in terms of new fields Φ'_i , obtained through a basis transformation which leaves the kinetic terms unchanged

$$\Phi_i \rightarrow \Phi'_i = U_{ij} \Phi_j, \quad (15)$$

where U is an $N \times N$ unitary matrix. Since the theory is invariant under a global $\mathbb{U}(1)$, we may take U in $\mathbb{SU}(N)$. Under this basis change, the potential parameters and the vevs are transformed as,

$$Y_{ij} \rightarrow Y'_{ij} = U_{ik} Y_{kl} U_{jl}^*, \quad (16)$$

$$Z_{ij,kl} \rightarrow Z'_{ij,kl} = U_{im} U_{ko} Z_{mn,op} U_{jn}^* U_{lp}^*, \quad (17)$$

$$v_i \rightarrow v'_i = U_{ij} v_j. \quad (18)$$

This means that not all parameters have physical significance; only basis invariant combinations can be observed experimentally [14].

Still, any model with more than one scalar has many free parameters. Most often, these are curtailed by invoking some specific symmetry

$$\Phi_i \rightarrow \Phi_i^S = S_{ij} \Phi_j, \quad (19)$$

also given by a $\mathbb{U}(N)$ matrix, which imposes relations among the potential parameters,

$$Y_{ij} = Y_{ij}^S = S_{ik} Y_{kl} S_{jl}^*, \quad (20)$$

$$Z_{ij,kl} = Z_{ij,kl}^S = S_{im} S_{ko} Z_{mn,op} S_{jn}^* S_{lp}^*. \quad (21)$$

Recall that in a basis change the potential parameters do not remain the same, whereas under a symmetry these must remain invariant.² The symmetry may (or not) be spontaneously broken, according to whether (or not)

$$v_i = v_i^S = S_{ij} v_j. \quad (22)$$

² In both situations the scalar potential is unaffected.

Consider a theory in which V_H , when written in terms of the fields Φ_i , has the symmetry S . Now, perform the basis transformation in Eq. (15). When written in terms of the new fields Φ'_i , V_H is no longer invariant under S ; rather, it is now invariant under

$$S' = USU^\dagger. \quad (23)$$

As an example, the \mathbb{Z}_2 2HDM symmetry

$$\Phi_1 \rightarrow \Phi_1, \quad \Phi_2 \rightarrow -\Phi_2, \quad (24)$$

becomes

$$\Phi'_1 \leftrightarrow \Phi'_2, \quad (25)$$

in the basis

$$\Phi'_1 = \frac{1}{\sqrt{2}}(\Phi_1 + \Phi_2), \quad \Phi'_2 = \frac{1}{\sqrt{2}}(\Phi_1 - \Phi_2). \quad (26)$$

Equation (23) is a conjugacy relation within $\mathbb{U}(N)$. In a suitable basis, S may be brought to the form [17]

$$\begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_N} \end{pmatrix}, \quad (27)$$

or, using the invariance under global hypercharge,

$$\begin{pmatrix} 1 & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_N} \end{pmatrix}. \quad (28)$$

This choice makes the presence of the symmetry in the Higgs potential more transparent.

Strictly speaking, we have just described the situation where there is only one generator S .³ Imagine that there are two generators S_1 and S_2 . Then, it may be that one can bring both generators to diagonal form. In that case, the symmetry generated by both, $S = \{S_1, S_2\}$, is Abelian and easy to guess from the form of V_H when it is written in the basis where both generators are diagonal. If S_1 and S_2 do not commute, then $S = \{S_1, S_2\}$ is a non-Abelian subset of $\mathbb{U}(N)$.⁴ One can diagonalize S_1 , or S_2 , but not both.

2.3 The symmetry basis and the charged Higgs basis

The basis freedom in Eq. (15) may be used to study a given theory in a specific basis. There is always a specially simple basis, the so-called *Charged Higgs basis* (CH basis), in which

the $\mathbb{U}(N)$ basis freedom is used in order to diagonalize the mass matrix of the charged Higgs in Eq. (5) [16, 18]. In this basis, $v_1 = v$, $v_{i \neq 1} = 0$, and the fields may be parametrized as

$$\begin{aligned} \Phi_1^{\text{CH}} &= \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v + H^0 + iG^0) \end{pmatrix}, \\ \Phi_2^{\text{CH}} &= \begin{pmatrix} S_2^+ \\ \frac{1}{\sqrt{2}}\phi_2^{C0} \end{pmatrix}, \\ &\vdots \\ \Phi_N^{\text{CH}} &= \begin{pmatrix} S_N^+ \\ \frac{1}{\sqrt{2}}\phi_N^{C0} \end{pmatrix}, \end{aligned} \quad (29)$$

where S_2^+, \dots, S_N^+ are the physical charged Higgs mass eigenstate fields, with corresponding masses $m_{\pm,i}^2$, and

$$Y_{ij}^{\text{CH}} + Z_{ij,11}^{\text{CH}} v^2 = \delta_{ij} m_{\pm,i}^2 \quad (\text{no sum}). \quad (30)$$

$S_1^\pm = G^\pm$ is the would-be Goldstone boson with $m_{\pm,1}^2 = 0$. This basis is especially adapted to study the decoupling limit. In this limit, all (massive) charged Higgs ($S_{i \neq 1}^\pm$) should acquire a very large mass. Since perturbativity and unitarity constrains the quartic parameters to lie below some upper bound, which one may take to be of order $4\pi \sim O(10)$, we find

$$Y_{11}^{\text{CH}} + Z_{11,11}^{\text{CH}} v^2 = 0 \quad \Rightarrow \quad Y_{11}^{\text{CH}} \sim v^2, \quad (31)$$

$$Y_{i \neq 1}^{\text{CH}} + Z_{ij,11}^{\text{CH}} v^2 = 0 \quad \Rightarrow \quad Y_{i \neq 1}^{\text{CH}} \sim v^2, \quad (32)$$

$$Y_{j \neq 1}^{\text{CH}} + Z_{jj,11}^{\text{CH}} v^2 = m_{\pm,j}^2 \quad \Rightarrow \quad Y_{j \neq 1}^{\text{CH}} \sim m_{\pm,j \neq 1}^2 \gg v^2. \quad (33)$$

Substituting Eq. (30) in Eqs. (12)–(14), one obtains [16]

$$(M_{\rho\rho}^2)_{ij} = \delta_{ij} m_{\pm,i}^2 + v^2 \text{Re} \{Z_{i1,1j} + Z_{i1,j1}\}, \quad (34)$$

$$(M_{\chi\chi}^2)_{ij} = \delta_{ij} m_{\pm,i}^2 + v^2 \text{Re} \{Z_{i1,1j} - Z_{i1,j1}\}, \quad (35)$$

$$(M_{\rho\chi}^2)_{ij} = -v^2 \text{Im} \{Z_{i1,1j} - Z_{i1,j1}\}, \quad (36)$$

where no sum is implied. This leads to our first important results. First, as the charged Higgs masses become very large, all neutral particles also become very massive. This is easy to understand. As $Y_{i \neq 1}^{\text{CH}}$ becomes positive and very large,

$$\mathcal{M}_i^2 \equiv Y_{i \neq 1}^{\text{CH}} \gg v^2, \quad (37)$$

the whole doublet Φ_i^{CH} decouples from the rest. Thus, within the framework of effective field theory, \mathcal{M}_i^2 is the interesting measure of decoupling—see, for example [19]. Second, if all charged Higgs become very massive, then $M_{\rho\rho}^2$ and $M_{\chi\chi}^2$ become very large and almost diagonal, while $M_{\rho\chi}^2$

³ We are using here “generator” in the assertion often used in connection with the “presentation of a discrete group” and not in connection with the algebra of a continuous Lie group.

⁴ We leave a subtlety for Appendix A.

remains of (small) order v^2 . Thus, all \mathcal{CP} violation in scalar-pseudoscalar mixing vanishes. This could be viewed as our **Theorem 0**.

At first sight, this second result may seem puzzling. Consider two very heavy Higgs doublets. Can't there be scalar-pseudoscalar mixing among those two heavy doublets? No! To understand the reason, we notice that Eqs. (34)–(36) mean that, in order to have significant \mathcal{CP} violation in scalar-pseudoscalar mixing, the charged Higgs masses $m_{\pm, i \neq 1}^2$ must be of order $v^2 = |v_1|^2 \dots |v_N|^2$. Now, the vev of each doublet is bounded by $|v_j| \leq v$ and, thus, it cannot be very large. With vevs of order v , one is left with two options. Charged Higgs masses of order v and possible large scalar-pseudoscalar mixing, or alternatively, charged Higgs masses much larger than v and necessarily small scalar-pseudoscalar mixing. Ref. [16] showed how crucial the CH basis was in interpreting unitarity bounds. The two results presented above provide another striking example of the usefulness of this basis. Indeed, these general results would be very difficult to guess in a generic basis.

For the most general NHDM, the Y^{CH} and Z^{CH} parameters are free to take any value (consistent with perturbativity and unitarity). However, such models suffer from several problems. On the one hand, they have too many free parameters and their study (bounds from current experiments and proposed new signals) is effectively very difficult. On the other hand, such models tend to lead to very large scalar flavour changing neutral couplings (sFCNC) with fermions, which are very tightly bound by flavour experiments. There are three solutions: make the new scalar masses large (precisely the decoupling limit); take the sFCNC small; or, make the sFCNC exactly zero by enforcing some symmetry. We now focus on models with some symmetry S . As seen in Eqs. (20)–(21), such a symmetry imposes conditions on the parameters of V_H . Eq. (23) implies that the specific constraint depends on which representation is chosen for the symmetry. We consider some specific representation for the symmetry and name the basis where the symmetry has that particular form as the *Symmetry basis* (S basis). And, because it constrains the available parameter space, such a symmetry will have an impact on the nature of the parameters in the CH basis.

The CH Eqs. (30)–(33) and (37) can be written compactly as

$$Y_{ij}^{CH} = \mathcal{M}_i^2 (1 - \delta_{i1} \delta_{j1}) \delta_{ij} + \Omega_{ij}^{CH}, \quad (\text{no sum}) \quad (38)$$

where

$$\Omega_{ij}^{CH} = -Z_{ij,11}^{CH} v^2 (1 - \delta_{ij}) - Z_{11,11}^{CH} v^2 \delta_{i1} \delta_{j1}. \quad (\text{no sum}) \quad (39)$$

Recall that as \mathcal{M}_i^2 increases, it drives the decoupling and defines the energy scale of the states within the $\Phi_{k \neq 1}^{CH}$ doublet.

The CH basis and the S basis are related by

$$\Phi^{CH} = U^{CH} \Phi^S, \quad (40)$$

$$U_{1j}^{CH} = \frac{v_j^*}{v}, \quad (41)$$

$$Y_{ij}^S = U_{ki}^{CH*} Y_{kl}^{CH} U_{lj}^{CH}, \quad (42)$$

$$Z_{ij,kl}^{CH} = U_{im}^{CH} U_{ko}^{CH} Z_{mn,op}^S U_{jn}^{CH*} U_{lp}^{CH*}. \quad (43)$$

Equations (38) and (42) can be combined into

$$Y_{ij}^S = \sum_{k=2}^N U_{ki}^{CH*} \mathcal{M}_k^2 U_{kj}^{CH} + \sum_{k=1, l=1}^N U_{ki}^{CH*} \Omega_{kl}^{CH} U_{lj}^{CH}. \quad (44)$$

This equation shows explicitly how the quadratic parameters in the S basis depend on the decoupling parameters $\mathcal{M}_{k \neq 1}^2$. In order to have a decoupling limit with all $\mathcal{M}_{k \neq 1}^2 \gg v^2$ it is certainly sufficient to include all Y_{ij}^S . However, most symmetries preclude some of these quadratic coefficients, possibly precluding a decoupling limit. Our aim is to study when this can and when it cannot happen.

Recall that the matrix U^{CH} is the unitary transformation which diagonalizes the mass matrix of the charged scalars in Eq. (5), when written in the S basis. The first line of U^{CH} must have the form in Eq. (41) because, through Eq. (6), that guarantees that the first eigenvector has zero mass, corresponding to G^+ . Eq. (41) also guarantees that the vev of the first doublet in the CH basis is v , while all other doublets in the CH basis have zero vev. A generic $N \times N$ unitary matrix is defined by the $N(N-1)/2$ angles in an orthogonal $N \times N$ matrix, and by $N(N+1)/2$ phases. From the original $N(N-1)/2$ angles, $(N-1)$ are determined by the vevs, which appear in the first line of U^{CH} , c.f. Eq. (41). We denote such angles by β_i and, in our notation, they cannot be multiples of $\pi/2$ if we wish to keep all vevs different from zero. There remain $(N-1)(N-2)/2$ angles, which we denote by ω_i . In contrast with the β_i , these ω_i angles depend not only on the vevs, but also on the independent Y_{ij}^S and $Z_{mn,op}^S$. We may choose regions of parameter space such that some ω_i , and, thus, some entries in the U^{CH} matrix are zero.

Consider a real 3HDM, with real vevs. Then, the unitary transformation from the S basis to the CH basis can be written as,

$$U^{CH} = \frac{1}{vv_{12}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega) & \sin(\omega) \\ 0 & -\sin(\omega) & \cos(\omega) \end{pmatrix} \times \begin{pmatrix} v_{12} & 0 & v_3 \\ 0 & v & 0 \\ -v_3 & 0 & v_{12} \end{pmatrix} \begin{pmatrix} v_1 & v_2 & 0 \\ -v_2 & v_1 & 0 \\ 0 & 0 & v_{12} \end{pmatrix}. \quad (45)$$

Parametrizing the vevs as

$$\begin{aligned} v_1 &= v \sin(\beta_2) \cos(\beta_1), \\ v_2 &= v \sin(\beta_2) \sin(\beta_1), \\ v_3 &= v \cos(\beta_2), \\ v_{12} &= \sqrt{|v_1|^2 + |v_2|^2} = v \sin(\beta_2), \end{aligned} \quad (46)$$

one obtains

$$U^{CH} = \begin{pmatrix} s_2 c_1 & s_2 s_1 & c_2 \\ -c_1 c_2 s_\omega - s_1 c_\omega & c_1 c_\omega - s_1 c_2 s_\omega & s_2 s_\omega \\ s_1 s_\omega - c_1 c_2 c_\omega & -c_1 s_\omega - s_1 c_2 c_\omega & s_2 c_\omega \end{pmatrix}, \quad (47)$$

with $s_1 \equiv \sin \beta_1$, $c_1 \equiv \cos \beta_1$, $s_2 \equiv \sin \beta_2$, $c_2 \equiv \cos \beta_2$, $s_\omega \equiv \sin \omega$, $c_\omega \equiv \cos \omega$. As mentioned, the angles β_1 and β_2 are determined solely by the vevs, while the angle ω will depend on the quadratic and quartic parameters in the S basis in some complicated fashion.

One final notational issue must be addressed. If there are two doublets with the same group charge, then any basis change among those two doublets is allowed. For example, the 3HDM with the \mathbb{Z}_2 symmetry $S = \text{diag}(1, -1, -1)$ does not have a well defined symmetry basis, since one can mix at will the last two scalars. We will concentrate on models which do have a *well defined S basis*, such as the $\mathbb{Z}_2 \times \mathbb{Z}_2$ 3HDM generated by $S_1 = \text{diag}(1, -1, 1)$ and $S_2 = \text{diag}(1, 1, -1)$, or the \mathbb{Z}_3 3HDM generated by $S = \text{diag}(1, \omega, \omega^2)$, with $\omega^3 = 1$.

In summary, the S basis is useful to identify the set of independent parameters, while the CH basis is useful to discuss decoupling. Going from the former to the latter involves diagonalizing $N \times N$ (for the charged scalars) and $2N \times 2N$ matrices (for the neutral scalars), and it is basically only manageable for $N = 2$ or other exceedingly simple cases. We will now show that the converse procedure of starting from quadratic parameters in the CH basis and looking for generic properties of the quadratic parameters in the S basis can lead to results valid for any N .

3 Decoupling or nondecoupling

Consider a NHDM with some symmetry. It may decouple to the SM in two ways:

- NHDM \rightarrow SM, with one single scale $\mathcal{M}_{k \neq 1} = \mathcal{M} \gg v$;
- NHDM \rightarrow SM, with multiple scales $\mathcal{M}_k \gg \mathcal{M}_j \gg v$.

3.1 NHDM \rightarrow SM: single scale

Regarding the first possibility, we present our main

Theorem 1 *Barring some zero vevs ($v_i = 0$ for some i), and/or very small mixing angles in the matrix U^{CH} , then all*

Y_{ij}^S must be present in order for a decoupling limit NHDM \rightarrow SM with one single scale to exist.

In most cases, a symmetry S forces some Y_{ij}^S to vanish. Our claim is that, under the conditions of the theorem, such models will *not* have a decoupling limit. Conversely, in order to keep the decoupling limit, the way out is to break the symmetry softly by including *all* the quadratic terms. This applies to any N and any symmetry S .

The proof uses Eq. (44). Perturbativity implies that all $Z_{ij,kl}^{CH}$ must be smaller than about 4π . Thus, we conclude from Eq. (39) that each Ω_{kl}^{CH} must be smaller than $O(10)v^2$. Taking all doublets to share a decoupling scale

$$\mathcal{M}_{k \neq 1}^2 = \mathcal{M}^2 \gg v^2, \quad (48)$$

Equation (44) leads to

$$\begin{aligned} \frac{Y_{ij}^S}{\mathcal{M}^2} &\approx \sum_{k=2}^N U_{ki}^{CH*} U_{kj}^{CH} + O\left(\frac{v^2}{\mathcal{M}^2}\right) \\ &= \delta_{ij} - U_{li}^{CH*} U_{lj}^{CH} + O\left(\frac{v^2}{\mathcal{M}^2}\right). \end{aligned} \quad (49)$$

Thus,

$$\frac{Y_{ii}^S}{\mathcal{M}^2} \approx 1 - \frac{|v_i|^2}{v^2} + O\left(\frac{v^2}{\mathcal{M}^2}\right), \quad (50)$$

$$\frac{Y_{i \neq j}^S}{\mathcal{M}^2} \approx -\frac{v_i v_j^*}{v^2} + O\left(\frac{v^2}{\mathcal{M}^2}\right). \quad (51)$$

The right-hand side (RHS) of Eq. (50) must be of order one. Similarly, the RHS of Eq. (51) must be of order one, unless some vev is very small.

There are a few caveats. First, the theorem does not hold in directions with $v_i = 0$; ie, in inert models. Second, the theorem may also cease to hold if there are vevs of order v^2/\mathcal{M} , such that there is a cancellation between the two terms on the RHS of Eq. (51). In fact, in such circumstances, it could even happen that some Y_{ij}^S is exactly zero.⁵ Recall that, according to Eq. (41), if $v_k \sim v^2/\mathcal{M}$ for some k , then the corresponding angle in the first line entry U_{1k}^{CH} will be small, of order v/\mathcal{M} . Third, one could consider decoupling in multiple scales.

Notice that we followed an uncommon strategy. Usually, one discusses the decoupling limit by starting from the restricted set of parameters in the S basis and then finding how to rotate from this basis into a new basis. For example, Gunion and Haber [10] start from the S basis of the \mathbb{Z}_2 2HDM, construct the mass matrices in this basis, and then diagonalize them, transforming from this basis into the mass eigenstates directly. Later, the result was revisited by

⁵ See Eq. (78) below.

Bernon *et al.* [20] by starting again in the symmetry basis and going into the Higgs basis.⁶ Again, this requires that one minimizes the potential explicitly and finds the matrix going from the S basis into the Higgs basis, and then from this into the mass basis of the neutral scalars. This is easy for $N = 2$, but unmanageable for larger N . Here we follow the opposite strategy. We write the S basis in term of the CH basis, where decoupling is very easy to describe.

For our theorem, it was sufficient to use the approximate results in Eqs. (49)–(51). To compare with exact results in the literature, incorporating in the quartic couplings the constraints from the symmetry S , one can apply the following **Strategy**:

- find the constraints that the symmetry imposes on the quartic couplings in the S basis, $Z_{mn,op}^S$;
- write the quartic parameters in the CH basis, $Z_{ij,kl}^{CH}$, using Eq. (43). This guarantees that the quartic parameters in the CH basis already encode the constraints that the symmetry places on the quartic couplings;
- use Eq. (44) to see how the quadratic parameters in the S basis depend on the decoupling parameters \mathcal{M}_i^2 .

Taking as an example the softly broken \mathcal{CP} conserving \mathbb{Z}_2 2HDM,

$$V_H = Y_{11}^S |\Phi_1|^2 + Y_{22}^S |\Phi_2|^2 + Y_{12}^S [\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1] + \lambda_1 |\Phi_1|^4 + \lambda_2 |\Phi_2|^4 + \lambda_3 |\Phi_1|^2 |\Phi_2|^2 + \lambda_4 (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_1) + \lambda_5 [(\Phi_1^\dagger \Phi_2)^2 + (\Phi_2^\dagger \Phi_1)^2], \quad (52)$$

we find

$$Y_{12}^S = -\frac{s_{2\beta}}{2} \left\{ \mathcal{M}_2^2 + \frac{v^2}{2} \left[s_\beta^2 c_\beta^2 \lambda_{12} + (c_\beta^4 + s_\beta^4) \lambda_{345} \right] \right\}, \quad (53)$$

where $\tan \beta = v_2/v_1$, $\lambda_{12} = \lambda_1 + \lambda_2$, and $\lambda_{345} = \lambda_3 + \lambda_4 + \lambda_5$. One can explicitly see that with $\beta \neq 0$, $\pi/2$ it is not possible to simultaneously have $\mathcal{M}_2 \gg v$ and $Y_{12}^S = 0$. We recover the result that the exact \mathbb{Z}_2 symmetric 2HDM does not have a decoupling limit [10, 20]. Moreover, if $\beta = 0$, then we are in the case of the Inert Doublet Model, which can indeed have a decoupling limit while $Y_{12}^S = 0$.

3.2 NHDM \rightarrow SM: multiple scales

We now consider the possibility that there are multiple $Y_{kk}^{CH} \equiv \mathcal{M}_k^2 \gg v^2$ ($k \neq 1$) scales in the decoupling

⁶ There is no distinction between the Higgs basis and the CH basis when $N = 2$. See ref. [16] for details.

NHDM \rightarrow SM. For definiteness, one could imagine that $\mathcal{M}_3 \gg \mathcal{M}_2 \gg v$. We assume that the two scales are distinct, but also independent. For example, we exclude the possibility that $\mathcal{M}_3 = 100 \mathcal{M}_2$, both growing towards very high values in proportion to each other. We find the following

Theorem 2 *Barring some zero vevs ($v_i = 0$ for some i), and/or very small mixing angles in the matrix U^{CH} , in order for a multiple scale NHDM \rightarrow SM decoupling to exist:*

- all Y_{ii}^S must be present and large;*
- moreover, if there are no judicious cancellations, then all $Y_{i \neq j}^S$ must be present and with large magnitudes.*

The proof of a) is very similar to that in the previous section. Taking

$$\mathcal{M}_{k \neq 1}^2 \gg v^2, \quad (54)$$

Equation (44) leads to

$$Y_{aa}^S \approx \sum_{k=2}^N |U_{ka}^{CH}|^2 \mathcal{M}_k^2 + O(v^2). \quad (55)$$

For a fixed a , Y_{aa}^S can only be of order v^2 (or vanish) if $U_{ka}^{CH} = 0$ for all values of $k \neq 1$. But this is impossible. Indeed, if it were true, the unitarity condition

$$1 = \sum_{k=1}^N |U_{ka}^{CH}|^2 = |U_{1a}^{CH}|^2 = \frac{|v_1|^2}{v^2} \quad (56)$$

would force $|v_1| = v$ and all $v_{k \neq 1} = 0$, contradicting our hypothesis.

To prove b) we start from

$$Y_{ab}^S \approx \sum_{k=2}^N U_{ka}^{CH*} U_{kb}^{CH} \mathcal{M}_k^2 + O(v^2). \quad (57)$$

We assume that there is no judicious cancellation among large terms, such as

$$U_{2a}^{CH*} U_{2b}^{CH} \mathcal{M}_2^2 + U_{3a}^{CH*} U_{3b}^{CH} \mathcal{M}_3^2 = 0. \quad (58)$$

Such a cancellation could occur because one is looking at a particularly fine-tuned region of parameters.⁷ But it could also occur naturally due to the symmetry. Recall that, given some non-Abelian symmetry, one can diagonalize one generator, but not another. The latter will necessarily impose relations between parameters corresponding to different doublets

⁷ For example, in Eq. (68) below, valid for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ 3HDM, one sees that Y_{12}^S is proportional to $\mathcal{M}_3^2 - \mathcal{M}_2^2 + \frac{v^2}{2} (\lambda_{13} - \lambda_{23})$. One could then choose the fine-tuned parameter region $\mathcal{M}_3^2 = \mathcal{M}_2^2 + \frac{v^2}{2} (\lambda_{13} - \lambda_{23})$, thus allowing $Y_{12}^S = 0$. The same holds in Eq. (92) below, applicable to the \mathbb{Z}_3 3HDM. It is this type of situation, possible only when there are two mass scales, that the “no judicious cancellations” requirement precludes.

and, under those circumstances, an equation such as (58) cannot be excluded a priori. The theorem b) applies when there is no such cancellation. In addition, we are also excluding situations in which there are small entries $U_{ka}^{CH*}U_{kb}^{CH}$, such that the RHS of Eq. (57) has some term in \mathcal{M}_k^2 exactly canceled by some term of order v^2 . Excluding such judicious cancellations, for given a and $b \neq a$, Y_{ab}^S in Eq. (57) can only vanish if all combinations $U_{ka}^{CH*}U_{kb}^{CH} = 0$ for all values of $k \neq 1$. But this is impossible. Indeed, if it were true, the unitarity condition

$$0 = \sum_{k=1}^N U_{ka}^{CH*}U_{kb}^{CH} = U_{1a}^{CH*}U_{1b}^{CH} = \frac{v_a^*v_b}{v^2} \quad (59)$$

would force some $v_{i \neq 1} = 0$, contradicting our hypothesis.

Notice that, in both Theorems 1 and 2, the result that Y_{aa}^S must be present is not very useful in cases in which all generators of the symmetry are simultaneously diagonalizable and chosen to be represented as in Eq. (27). Indeed, in such cases, all Y_{aa}^S are allowed by the symmetry from the start. In contrast, in cases where the generators cannot be simultaneously diagonalized, such as the case mentioned in Eq. (A.1), then one or more Y_{aa}^S might be precluded by the exact symmetry. In that case, Theorem 1 and case a) of Theorem 2 assert that the symmetry must be softly violated by $Y_{aa}^S \neq 0$.

3.3 Decoupling of one doublet

It is interesting that a strong result is possible, even under the simple situation in which only one doublet decouples.

Theorem 3 *Barring some zero vevs ($v_i = 0$ for some i), for every doublet, $\Phi_{k \neq 1}^{CH}$, that decouples from the low energy theory, and barring judicious cancellations, at least three quadratic parameter in the Symmetry basis, $Y_{ab \neq a}^S$, Y_{aa}^S and Y_{bb}^S , will depend on $\mathcal{M}_{k \neq 1}^2$.*

The proof uses Eqs. (44), (41), and the fact that the matrix U^{CH} is unitary. The latter implies that the sum of the squares of the entries in a given row (column) must be unity. Since we are excluding judicious cancellations (that is, proportionality among) $\mathcal{M}_{k \neq 1}^2$ for different k , we can consider the impact of each one independently. Imagine that for a given $k \neq 1$, $\mathcal{M}_{k \neq 1}^2 \gg v^2$, and consider the k -th row of U^{CH} . Since the sums of squares along the row must add to one, there must be at least a column a such that $|U_{ka}| \neq 0$. However, we know that also $|U_{ka}| \neq 1$; otherwise, considering now the sum of all squares in column a , we would find $|U_{1a}| = 0$. That, according to Eq. (41), would imply $v_a = 0$, violating our hypothesis. So, there must be at least one other column b such that $|U_{kb}| \neq 0$. But, looking back at Eq. (44) we see that, as a result, $Y_{ab \neq a}^S$, Y_{aa}^S , and Y_{bb}^S grow with $\mathcal{M}_{k \neq 1}^2$.

It may help to visualize the argument made here to look back at Eq. (47). Imagine that we are taking $\mathcal{M}_2^2 \gg v^2$. One

can make $U_{23}^{CH} = 0$ by setting $\omega = 0$; the matrix simplifies into

$$U^{CH} = \begin{pmatrix} s_2 c_1 & s_2 s_1 & c_2 \\ -s_1 & c_1 & 0 \\ -c_1 c_2 & -s_1 c_2 & s_2 \end{pmatrix}. \quad (60)$$

Then, according to Eq. (44), none of Y_{i3}^S and Y_{3i}^S depend on \mathcal{M}_2^2 . Nevertheless, both U_{21}^{CH} and U_{22}^{CH} must be nonvanishing, or there would be one zero vev. As a result, even taking $\omega = 0$, Y_{12}^S , Y_{11}^S , and Y_{22}^S will grow with \mathcal{M}_2^2 .

4 Simple examples

To illustrate both the application of our theorems (decoupling) and some relevant violations of the assumptions (thus, nondecoupling or decoupling using fine tuned regions with vevs of order v^2/\mathcal{M}), we concentrate on 3HDM models with Abelian symmetries. These have been classified in [17] and [21]. The symmetries $\mathbb{U}(1) \times \mathbb{U}(1)$, $\mathbb{U}(1) \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, \mathbb{Z}_3 , and \mathbb{Z}_4 , yield a well defined symmetry basis. The corresponding potentials are shown in Appendix B. All these symmetries allow for quadratic diagonal coefficients Y_{ii}^S and preclude quadratic off-diagonal coefficients $Y_{ij \neq i}^S$. When off-diagonal quadratic couplings $Y_{ij \neq i}^S$ are needed for decoupling, then one must either give up decoupling or else break the symmetry softly.

4.1 The $\mathbb{Z}_2 \times \mathbb{Z}_2$ 3HDM

This model was suggested by Weinberg [22] and explored for spontaneous \mathcal{CP} violation with three quark families by Branco [23]. The potential is written in Eqs. (B.8) and (B.10). We now follow the strategy outlined at the end of Section 3.1.

To simplify the expressions, we take the vevs real and use Eq. (47) with $\omega = 0$, $\cos(\beta_2) \approx 1 - \beta_2^2/2$, $\sin(\beta_2) \approx \beta_2$, $\cos(\beta_1) \approx 1 - \beta_1^2/2$, $\sin(\beta_1) \approx \beta_1$. These conditions ensure that $v_3 \rightarrow v$ and that the contributions for the $Y_{ij \neq i}^S$ due to $\mathcal{M}_{k \neq 1}^2 \gg v^2$ are suppressed. Using Eqs. (38), (39), and (43), we find

$$Y_{11}^{CH} = -v^2 Z_{11,11}^{CH} \approx -v^2 \left[\frac{\lambda_{33}}{2} + \beta_2^2 (\lambda_{13} + \lambda'_{13} + \lambda_{13,13} - \lambda_{33}) \right], \quad (61)$$

$$Y_{12}^{CH} = -v^2 Z_{12,11}^{CH} \approx \frac{v^2}{2} \beta_1 \beta_2 (\lambda_{13} + \lambda'_{13} + \lambda_{13,13} - \lambda_{23} - \lambda'_{23} - \lambda_{23,23}), \quad (62)$$

$$Y_{13}^{CH} = -v^2 Z_{13,11}^{CH}$$

$$\approx \frac{v^2}{2} \beta_2 (\lambda_{13} + \lambda'_{13} + \lambda_{13,13} - \lambda_{33}), \quad (63)$$

$$Y_{22}^{CH} = \mathcal{M}_2^2, \quad (64)$$

$$Y_{23}^{CH} = -v^2 Z_{23,11}^{CH} \approx \frac{v^2}{2} \beta_1 (\lambda_{23} - \lambda_{13}), \quad (65)$$

$$Y_{33}^{CH} = \mathcal{M}_3^2. \quad (66)$$

Substituting in Eq. (42), we get the leading order terms

$$Y_{11}^S \approx \mathcal{M}_3^2 + \beta_1^2 \left[\mathcal{M}_2^2 - \mathcal{M}_3^2 - v^2 (\lambda_{13} - \lambda_{23}) \right] - \beta_2^2 \left[\mathcal{M}_3^2 + v^2 \left(\lambda_{13} + \lambda'_{13} + \lambda_{13,13} - \frac{\lambda_{33}}{2} \right) \right], \quad (67)$$

$$Y_{12}^S \approx \beta_1 \left[\mathcal{M}_3^2 - \mathcal{M}_2^2 + \frac{v^2}{2} (\lambda_{13} - \lambda_{23}) \right], \quad (68)$$

$$Y_{13}^S \approx -\beta_2 \left[\mathcal{M}_3^2 + \frac{v^2}{2} (\lambda_{13} + \lambda'_{13} + \lambda_{13,13}) \right], \quad (69)$$

$$Y_{22}^S \approx \mathcal{M}_2^2 + \beta_1^2 \left[\mathcal{M}_3^2 - \mathcal{M}_2^2 + v^2 (\lambda_{13} - \lambda_{23}) \right], \quad (70)$$

$$Y_{23}^S \approx -\beta_1 \beta_2 \left[\mathcal{M}_3^2 + \frac{v^2}{2} (\lambda_{13} + \lambda'_{23} + \lambda_{23,23}) \right], \quad (71)$$

$$Y_{33}^S \approx \mathcal{M}_3^2 \beta_2^2 - \frac{v^2}{2} \lambda_{33} (1 - \beta_2^2), \quad (72)$$

We start by noticing that, as proved in Sect. 3.3, Y_{12}^S , Y_{11}^S , and Y_{22}^S grow with \mathcal{M}_2^2 .

Let us now assume, as in Sect. 3.2, that there are two very different scales $\mathcal{M}_3 \gg \mathcal{M}_2 \gg v$ in the decoupling 3HDM \rightarrow SM. Then, all Y_{ij}^S must be present. However, as we see in Eq. (B.8), the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry precludes the $Y_{ij \neq i}^S$ terms. Thus, in this region of the parameter space, one either keeps the symmetry and loses the decoupling limit or, else, one must break the symmetry softly with all possible terms. It is true that one could (for example) make Y_{13}^S in Eq. (69) parametrically small by choosing fine tuned regions with $\beta_2 \sim v^2/\mathcal{M}_3^2$. Small but *not zero*. Again, in the particular region $\omega = 0$, $\beta_1, \beta_2 \ll 1$, for a 3HDM \rightarrow SM with several scales one must include *all* $Y_{ij \neq i}^S$ terms to have a decoupling limit. Notice that this holds despite the fact that in this particular case we are even allowing small vevs.

Now, we notice that the leading term for Y_{12}^S in Eq. (68) is proportional to $\mathcal{M}_3^2 - \mathcal{M}_2^2$. So, we consider the situation where there is only one scale $\mathcal{M} = \mathcal{M}_3 = \mathcal{M}_2 \gg v^2$ in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ 3HDM \rightarrow SM decoupling, as discussed in Sect. 3.1. One finds

$$Y_{11}^S \approx \mathcal{M}^2 - \beta_1^2 v^2 (\lambda_{13} - \lambda_{23}) - \beta_2^2 \left[\mathcal{M}^2 + v^2 \left(\lambda_{13} + \lambda'_{13} + \lambda_{13,13} - \frac{\lambda_{33}}{2} \right) \right], \quad (73)$$

$$Y_{13}^S \approx -\beta_2 \left[\mathcal{M}^2 + \frac{v^2}{2} (\lambda_{13} + \lambda'_{13} + \lambda_{13,13}) \right], \quad (74)$$

$$Y_{22}^S \approx \mathcal{M}^2 + \beta_1^2 v^2 (\lambda_{13} - \lambda_{23}), \quad (75)$$

$$Y_{23}^S \approx -\beta_1 \beta_2 \left[\mathcal{M}^2 + \frac{v^2}{2} (\lambda_{13} + \lambda'_{23} + \lambda_{23,23}) \right], \quad (76)$$

$$Y_{33}^S \approx \mathcal{M}^2 \beta_2^2 - \frac{v^2}{2} \lambda_{33} (1 - \beta_2^2), \quad (77)$$

and, taking Y_{12}^S to higher order in β 's,

$$Y_{12}^S \approx -\beta_1 \left[\mathcal{M}^2 \beta_2^2 - \frac{v^2}{2} (\lambda_{13} - \lambda_{23}) \right]. \quad (78)$$

Now it is possible to make Y_{12}^S *exactly zero*, by choosing fine tuned regions of $\lambda_{13} - \lambda_{23}$, \mathcal{M} (very large) and β_2 (correspondingly very small), such that the RHS of Eq. (78) vanishes. Notice that, according to eq. (46), $\beta_2 \rightarrow 0$ implies $v_1 \rightarrow 0$ and $v_2 \rightarrow 0$, in contradiction with the conditions of Theorem 1. For the general NHDM, we do not know how many $Y_{ij \neq i}^S$ need to be included in order to attain this parametric decoupling with $\mathcal{M}_{k \neq 1} = \mathcal{M} \gg v$. Nevertheless, we suspect that it may be possible to exclude $Y_{ab \neq a}^S$ when both $v_a \rightarrow 0$ and $v_b \rightarrow 0$; granted, using a very fine tuned choice of parameters.

Although simplified by taking $\omega = 0$ and small β_1 and β_2 , Eqs. (67)–(72) and their special case (73)–(78) illustrate both the theorems and their caveats. Eqs. (73)–(78) show that, barring β 's decreasing as some power of v/\mathcal{M} , decoupling implies that all quadratic couplings must be present, in accordance with Theorem 1. Similarly, Eqs. (67)–(72) illustrate Theorem 2. Finally, looking at Eqs. (67)–(72) by taking only $\mathcal{M}_3^2 \gg v^2$ we see that all quadratic couplings must be present in the S basis. Moreover, taking only $\mathcal{M}_2^2 \gg v^2$ we see that some quadratic couplings may be absent, but that there are at least three which must be present.⁸ This illustrates Theorem 3.

This section also illustrates the caveats in the theorems. For example, by taking small angles $\omega = 0$ and very fine tuned regions of parameter space where the vevs decrease with the decoupling scale (eg, $\beta_2 \sim v/\mathcal{M}$), Eq. (78) shows that we can set $Y_{12}^S = 0$ in the 3HDM \rightarrow SM decoupling with one single scale. Eqs. (67)–(72) show that that is not possible with multiple independent scales $\mathcal{M}_3^2 \gg \mathcal{M}_2^2 \gg v^2$ for this fine tuned region of parameter space where $\omega = 0$ while β_1 and β_2 are small.

However, in other fine tuned regions of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ 3HDM parameter space, it is indeed possible to remove some $Y_{ab}^S = 0$ for $a \neq b$, even with multiple energy scales. Consider the situation with $\omega = 0$, $\beta_2 = \pi/2 - \epsilon$, $\cos(\beta_2) \approx \epsilon$,

⁸ It was to illustrate the fact that we could allow some (but not all) quadratic parameters to be absent in some regions of parameter space, that we have taken $\omega = 0$.

$\sin(\beta_2) \approx 1 - \epsilon^2/2$, $\cos(\beta_1) \approx 1 - \beta_1^2/2$, and $\sin(\beta_1) \approx \beta_1$. Using Eqs. (38), (39), and (43), we find Y_{ij}^{CH} , which we then substitute in Eq. (42). The leading order terms are

$$Y_{11}^S \approx -\frac{1}{2}v^2\lambda_{11}(1 - \epsilon^2 - \beta_1^2) + \epsilon^2\mathcal{M}_3^2 + \beta_1^2\mathcal{M}_2^2, \quad (79)$$

$$Y_{12}^S \approx \beta_1 \left[\mathcal{M}_3^2\epsilon^2 - \mathcal{M}_2^2 - \frac{v^2}{2}(\lambda_{12} + \lambda'_{12} + \lambda_{12,12}) \right], \quad (80)$$

$$Y_{13}^S \approx -\epsilon \left[\mathcal{M}_3^2 + \frac{v^2}{2}(\lambda_{13} + \lambda'_{13} + \lambda_{13,13}) \right], \quad (81)$$

$$Y_{22}^S \approx \mathcal{M}_2^2(1 - \beta_1^2) + \beta_1^2v^2 \left(\frac{\lambda_{11}}{2} - \lambda_{12} - \lambda'_{12} - \lambda_{12,12} \right), \quad (82)$$

$$Y_{23}^S \approx -\epsilon\beta_1 \left[\mathcal{M}_3^2 + \frac{v^2}{2}(\lambda_{13} + \lambda'_{23} + \lambda_{23,23}) \right], \quad (83)$$

$$Y_{33}^S \approx \mathcal{M}_3^2(1 - \epsilon^2) + \epsilon^2v^2 \left(\frac{\lambda_{11}}{2} - \lambda_{13} - \lambda'_{13} - \lambda_{13,13} \right). \quad (84)$$

By choosing a fine tuned region of the parameter space where

$$\mathcal{M}_3^2 = \frac{1}{\epsilon^2} \left[\mathcal{M}_2^2 + \frac{v^2}{2}(\lambda_{12} + \lambda'_{12} + \lambda_{12,12}) \right], \quad (85)$$

one can now set $Y_{12}^S = 0$. Notice that in this situation it is possible to set $Y_{12}^S = 0$ as \mathcal{M}_3^2 takes increasingly larger values, regardless of \mathcal{M}_2^2 , as long as one remains in the fine tuned region of Eq. (85). However, one can never set $Y_{13}^S = 0$ for this region of parameter space.

4.2 The \mathbb{Z}_3 3HDM

To study the decoupling properties of the \mathbb{Z}_3 3HDM, we follow the strategy outlined at the end of Sect. 3.1. To simplify the expressions, we take again the vevs real, use Eq. (47) with $\omega = 0$, and expand in β_1 and β_2 . These conditions ensure that $v_3 \rightarrow v$ and that the contributions for the Y_{ij}^S due to $\mathcal{M}_{k \neq i}^2 \gg v^2$ are suppressed. Using Eqs. (38), (39), and (43), we find Y_{ij}^{CH} , which we then substitute in Eq. (42). The leading order terms are

$$Y_{11}^S \approx \mathcal{M}_3^2 + \beta_1^2 \left[\mathcal{M}_2^2 - v^2(\lambda_{13} - \lambda_{23}) \right] - v^2\beta_2 \left[\beta_2 \left(\lambda_{13} + \lambda'_{13} - \frac{\lambda_{33}}{2} \right) - \beta_1\lambda_{13,12} \right], \quad (86)$$

$$Y_{12}^S \approx \beta_1 \left[\mathcal{M}_3^2 - \mathcal{M}_2^2 + \frac{v^2}{2}(\lambda_{13} - \lambda_{23} - \beta_2\lambda_{12,32}) \right] - \frac{v^2}{2}\beta_2\lambda_{13,12}, \quad (87)$$

$$Y_{13}^S \approx -\beta_2 \left[\mathcal{M}_3^2 + \frac{v^2}{2}(\lambda_{13} + \lambda'_{13} + \beta_1\lambda_{13,23}) \right], \quad (88)$$

$$Y_{22}^S \approx \mathcal{M}_2^2 + \beta_1 \left[\beta_1\mathcal{M}_3^2 + v^2\beta_1(\lambda_{13} - \lambda_{23}) - v^2\beta_2\lambda_{13,12} \right], \quad (89)$$

$$Y_{23}^S \approx -\beta_2 \left[\beta_1\mathcal{M}_3^2 + \frac{v^2}{2}\beta_1(\lambda_{13} + \lambda'_{23}) + \frac{v^2}{2}\lambda_{13,23} \right], \quad (90)$$

$$Y_{33}^S \approx \beta_2^2\mathcal{M}_3^2 - \frac{1}{2}v^2\lambda_{33}. \quad (91)$$

Here one can see that one can set $Y_{12}^S = Y_{23}^S = 0$, by choosing a fine tuned region of parameter space with decoupling energy scales given to order $\beta_i v^2$ by

$$\mathcal{M}_2^2 - \mathcal{M}_3^2 \approx -\frac{v^2}{2}\frac{\beta_2}{\beta_1}\lambda_{13,12} + \frac{v^2}{2}(\lambda_{13} - \lambda_{23} - \beta_2\lambda_{12,32}), \quad (92)$$

$$\mathcal{M}_3^2 \approx -\frac{v^2}{2}\frac{1}{\beta_1}\lambda_{13,23} - \frac{v^2}{2}(\lambda_{13} + \lambda'_{23}). \quad (93)$$

One concludes that when $\beta_1 \rightarrow 0$ with β_2 fixed, the decoupling energy scales can still be larger than the electroweak scale without including all quadratic parameters. Through this procedure, it is possible to decouple a 3HDM \rightarrow SM with two scales which become larger and farther apart as $\beta_1 \rightarrow 0$.

4.3 Lessons from the stationarity equations

In all cases illustrated in Sect. 4.1, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ 3HDM can have decoupling if and only if there is at least some off-diagonal term breaking the symmetry softly. It is easy to see that, indeed, this is a general feature of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ 3HDM by writing the stationarity equations in the symmetry basis:

$$\begin{aligned} m_{11}^2 &= -\frac{1}{2} \left[v_1^2\lambda_{11} + v_2^2(\lambda_{12} + \lambda'_{12} + \lambda_{12,12}) + v_3^2(\lambda_{13} + \lambda'_{13} + \lambda_{13,13}) \right], \\ m_{22}^2 &= -\frac{1}{2} \left[v_1^2(\lambda_{12} + \lambda'_{12} + \lambda_{12,12}) + v_2^2\lambda_{22} + v_3^2(\lambda_{23} + \lambda'_{23} + \lambda_{23,23}) \right], \\ m_{33}^2 &= -\frac{1}{2} \left[v_1^2(\lambda_{13} + \lambda'_{13} + \lambda_{13,13}) + v_2^2(\lambda_{23} + \lambda'_{23} + \lambda_{23,23}) + v_3^2\lambda_{33} \right]. \end{aligned} \quad (94)$$

We see that, in the exact $\mathbb{Z}_2 \times \mathbb{Z}_2$ 3HDM (that is, with no Y_{ij}^S terms), all diagonal quadratic couplings $Y_{ii}^S = m_{ii}^2$ are of order v^2 , and thus there is no decoupling limit. Hence, the stationarity conditions of the exact $\mathbb{Z}_2 \times \mathbb{Z}_2$ 3HDM in the

symmetry basis are enough to see that one can only make the diagonal $Y_{ii}^S = m_{ii}^2$ large by including some off diagonal $Y_{ij \neq i}^S$ term.

This is no longer the case for the \mathbb{Z}_3 3HDM, whose potential is written in Eqs. (B.8) and (B.11). Indeed, the stationarity conditions for the real \mathbb{Z}_3 3HDM,

$$\begin{aligned} m_{11}^2 &= -\frac{1}{2} \left[v_1^2 \lambda_{11} + v_2^2 \left(\lambda_{12} + \lambda'_{12} + \frac{v_3}{v_1} \lambda_{12,32} \right) \right. \\ &\quad \left. + v_3^2 \left(\lambda_{13} + \lambda'_{13} + \frac{v_2}{v_1} \lambda_{13,23} \right) + 2v_2 v_3 \lambda_{13,12} \right], \\ m_{22}^2 &= -\frac{1}{2} \left[v_1^2 \left(\lambda_{12} + \lambda'_{12} + \frac{v_3}{v_2} \lambda_{13,12} \right) + v_2^2 \lambda_{22} \right. \\ &\quad \left. + v_3^2 \left(\lambda_{23} + \lambda'_{23} + \frac{v_1}{v_2} \lambda_{13,23} \right) + 2v_1 v_3 \lambda_{12,32} \right], \\ m_{33}^2 &= -\frac{1}{2} \left[v_1^2 \left(\lambda_{13} + \lambda'_{13} + \frac{v_2}{v_3} \lambda_{13,12} \right) \right. \\ &\quad \left. + v_2^2 \left(\lambda_{23} + \lambda'_{23} + \frac{v_1}{v_3} \lambda_{12,32} \right) \right. \\ &\quad \left. + v_3^2 \lambda_{33} + 2v_1 v_2 \lambda_{13,23} \right], \end{aligned} \quad (95)$$

contain ratios of vevs. To be specific, take the expression for m_{22}^2 . Now one can have $m_{22}^2 \gg v^2$ by taking $v_2 \rightarrow 0$ and $v_1, v_3 \rightarrow 0$.

It is interesting to interpret the difference between the stationarity conditions in this case, Eqs. (95), and in the previous case, Eqs. (94), in the following way. The decoupling limit is easy to interpret in the CH basis. It says that two parameters must become very large. And, using Eq. (42) and the fact that all $|U_{ij}^{CH}| < 1$, we conclude that some entries of Y^S must be large. Now, the stationarity conditions of the general NHDM in the S basis are

$$Y_{ii}^S = - \sum_{j=1, j \neq i}^N \left(Y_{ij}^S \frac{v_j}{v_i} + Z_{ij,kl}^S \frac{v_k^* v_l v_j}{v_i} \right), \quad (96)$$

Equation (96) shows that there are two ways to make the diagonal elements Y_{ii}^S large. First, one can make the off-diagonal elements in the first term on the RHS of Eq. (96) large

$$Y_{ii}^S + \sum_{j=1, j \neq i}^N \left(Y_{ij}^S \frac{v_j}{v_i} \right) = O(v^2). \quad (97)$$

Or, one can use the ratios of vevs in the second term on the RHS of Eq. (96) and make those large

$$\left| Z_{ij,kl}^S \frac{v_k^* v_l v_j}{v_i} \right| \gg v^2 \quad (\text{no sum}). \quad (98)$$

We conjecture that this difference might have a physical impact. We have found numerical regions of parameter space where an exact \mathbb{Z}_3 3HDM (with $Y_{ab}^S = 0$ for all $a \neq b$) could decouple one scalar, leaving a 2HDM. We have found no such situation for an exact $\mathbb{Z}_2 \times \mathbb{Z}_2$ 3HDM.

5 A note on \mathcal{CP} violation

The examples above focused on models with real vevs in a real symmetry basis. Then \mathcal{CP} is conserved, and only the magnitudes of the quadratic parameters are relevant for decoupling. The same type of arguments can be used to study cases where there is \mathcal{CP} violation. This is beyond the scope of this article, but we make a few simple comments here.

Let us consider cases in which the potential is invariant under the canonical definition of \mathcal{CP} :

$$\Phi_k \rightarrow \Phi_k^*. \quad (99)$$

Invariance of the potential (1) under this definition of \mathcal{CP} implies that

$$Y_{ij} = Y_{ij}^*, \quad Z_{ij,kl} = Z_{ij,kl}^*, \quad (100)$$

and all coefficients are real. Then one may have spontaneous \mathcal{CP} violation if some vevs have a relative phase. Under this definition of \mathcal{CP} , the fields ρ_k and χ_k in Eq. (3) are \mathcal{CP} -even and \mathcal{CP} -odd, respectively. In that case, \mathcal{CP} violation in scalar-pseudoscalar mixing is described by $(M_{\rho\chi}^2)_{ij}$ in Eq. (14), which, since it is at most of order v^2 , becomes irrelevant as all charged Higgs become very massive.

There are several issues that complicate a general analysis. First, the definition of \mathcal{CP} changes with basis transformations. For example, it is true that under the \mathcal{CP} transformation (99), ρ_k and χ_k in Eq. (3) are \mathcal{CP} -even and \mathcal{CP} -odd. However, the simple rephasing $\Phi_k \rightarrow \Phi'_k = i\Phi_k$ means that, under the same definition of \mathcal{CP} , it is now χ'_k which is \mathcal{CP} -even and ρ'_k which is \mathcal{CP} -odd.⁹ Second, when searching for \mathcal{CP} violation one must study all possible definitions of \mathcal{CP} – for an introduction to these problems, see for example Ref. [25]. Finally, one may even have the odd situation that there is \mathcal{CP} conservation even though there are irremovable complex couplings in the scalar potential. The first and simplest example is the so-called CP4 3HDM [30]. For these reasons, a complete analysis of the relation between decoupling and \mathcal{CP} violation (explicit or spontaneous) lies beyond the scope of this article.

Still, the result discussed below Eq. (37) is completely general. Indeed, if all charged Higgs become very massive, then Eqs. (34)–(36) imply that $M_{\rho\rho}^2$ and $M_{\chi\chi}^2$ become very large and almost diagonal, while $M_{\rho\chi}^2$ remains of (small) order v^2 . Thus, all \mathcal{CP} violation in scalar-pseudoscalar mixing vanishes, regardless of the precise details of \mathcal{CP} .

We can also recover a recent result on the 2HDM [11], because the model is very simple. Consider a 2HDM with a softly broken \mathbb{Z}_2 symmetry and complex vevs. As noted,

⁹ In fact, it was this precise problem which prompted the study of basis invariants quantities in the scalar sector in [24] and [14], later explored extensively in [25–29].

the symmetry basis is defined up to rephasing of its doublets. Then one can without loss of generality choose λ_5 to be real. In this basis m_{12}^2 can either be real or complex. If it is real, there can be [31] (or not) spontaneous \mathcal{CP} violation, and if it is complex, there is explicit \mathcal{CP} violation.¹⁰ By parameterising the vevs as $v_1 = v c_\beta$ and $v_2 = v s_\beta e^{i\theta}$, the transformation matrix from the S basis with real λ_5 , to the Higgs basis is given by,

$$\begin{pmatrix} \Phi_1^H \\ \Phi_2^H \end{pmatrix} = \frac{1}{v} \begin{pmatrix} v_1 & v_2^* \\ -v_2^* & v_1 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \quad (101)$$

To obtain the decoupling limit conditions we repeat our procedure of setting $Y^{CH} \equiv \mathcal{M}_2^2$ and writing Y_{11}^{CH}, Y_{12}^{CH} as a function of the S basis quartic parameters. Then we write the S basis quadratic parameters as a function of the CH basis quadratic parameters. Note that both Y_{12}^{CH} and U^{CH} are now complex. Then,

$$Y^S = \mathcal{M}_2^2 \begin{pmatrix} s_\beta^2 & -c_\beta s_\beta e^{-i\theta} \\ -c_\beta s_\beta e^{i\theta} & c_\beta^2 \end{pmatrix} + O(v^2), \quad (102)$$

and the soft breaking term will be given by

$$Y_{12}^S = -\frac{s_{2\beta}}{2} [\mathcal{M}_2^2 + O(v^2)] e^{-i\theta}. \quad (103)$$

Here, one can see that it is not possible to have a decoupling limit and spontaneous \mathcal{CP} violation in a 2HDM with a \mathbb{Z}_2 symmetry softly broken (by a real parameter). Y_{12}^S must be complex for a decoupling to exist, which in turn explicitly breaks the \mathcal{CP} symmetry.

6 On the hypothesis of the theorems

The previous theorems in this article assumed hypothesis ensuring that the vevs are neither zero nor very small. Without this, the various scenarios must be analyzed on a case-by-case basis. But there is one further situation in which one can make a statement valid for any NHDM.

Theorem 4 *If the symmetry imposes $Y_{ab}^S = 0$ for all $a \neq b$, and all Y_{aa}^S ($a \neq 1$) are independent of Y_{11}^S , then the inert vacuum $v_1 = v$, $v_{k \neq 1} = 0$ allows for a decoupling limit.*

Indeed, when $v_1 = v$ and $v_{k \neq 1} = 0$, the stationary conditions do not impose any restriction on the values of Y_{kk}^S for all $k \neq 1$. Moreover, we are assuming that Y_{kk}^S ($k \neq 1$) do not get restricted by Y_{11}^S .¹¹ Finally, if $Y_{ab}^S = 0$ for all $a \neq b$,

then $Y_{kk}^S \approx Y_{kk}^{CH} = \mathcal{M}_k^2$ as $Y_{kk}^S \rightarrow \infty$, and there is a valid decoupling limit.

This illustrates the difficulty one has with vanishing vevs; one must then look in detail at the stationarity conditions to see which vacuum solutions are valid and then find the various physical masses for that specific vacuum. In addition, some vacua may be solutions of the stationarity equations but never constitute a valid global minimum for the potential. There is no known solution for these problems in a general NHDM.

Indeed, even simple questions in 3HDM have no known answer. For example, the full list of discrete symmetry groups allowed in the 3HDM scalar sector was presented in [33, 34]. But this has never been attempted for the 4HDM and, even for the 3HDM, there is still no complete description of global minima or bounded from below conditions. For example, the necessary and sufficient bounded from below conditions for the famous $Z_2 \times Z_2$ 3HDM Weinberg model of 1976 [22] are not yet known fully, as explained recently in [35].

These difficulties highlight the power of the results presented here; under the assumed hypothesis, they are valid in complete generality for any NHDM. The difficulties also explain why we cannot give a full answer to the issue we turn to next.

As noted at the end of Sect. 3.1, the exact Z_2 2HDM with the Z_2 symmetry breaking vev $v_1 \neq 0$ and $v_2 \neq 0$ does not have a decoupling limit. In contrast, the exact Z_2 2HDM with the Z_2 symmetry preserving vev $(v_1, v_2) = (v, 0)$ —the so-called Inert Doublet Model—does have a decoupling limit. Similarly, at the end of the previous Sect. 5, we recovered the result of [11]: the exact CP conserving 2HDM with the CP violating vev does not have a decoupling limit; the exact CP conserving 2HDM with the CP conserving vev does have a decoupling limit. One may wonder whether this is a remark of general validity.¹² That is, whether a symmetric NHDM will have a decoupling limit if and only if the vacuum does not violate the symmetry. Or, maybe, in which set of cases could such a statement hold. The difficulties mentioned above explain why it is not easy to find the answer to this question in the most general NHDM.

But one can say something within the 2HDM, since all symmetries are known and there are relatively few parameters. The set of all possible symmetries which can be implemented in the 2HDM was determined in [36], and further explained in [37]. They were dubbed in [37] as Z_2 , $U(1)$, $U(2)$, CP1, CP2, and CP3.¹³

¹⁰ This holds, except in an exceptional region of parameter space [32].

¹¹ For example, in the CP2, CP3 and $U(2)$ 2HDM cases discussed below, $m_{22}^2 = m_{11}^2$. Combined with the fact that the stationarity equation for the inert vacuum, $2m_{11}^2 = -\lambda_1 v^2$, both are forced to be of order v^2 and there is no parameter to drive decoupling. This explains the “ Y_{aa}^S ($a \neq 1$) are independent of Y_{11}^S ” caveat of the theorem.

¹² We are very grateful to the anonymous referee who asked the question which led to this section.

¹³ The “CP conserving 2HDM” mentioned in the previous paragraph, really means the most general CP conserving 2HDM, dubbed CP1 in [37].

The $U(1)$ 2HDM can be found from the Z_2 2HDM by setting $\lambda_5 = 0$. Here, the vev $v_1 \neq 0$ and $v_2 \neq 0$ breaks the symmetry. And, since it is a continuous symmetry, it implies that there is a massless Goldstone boson; in the usual notation $m_A = 0$. This is as serious a nondecoupling limit as one could have. In contrast, the vev $(v_1, v_2) = (v, 0)$ does not break $U(1)$ and does have a decoupling limit, with the masses of the new scalar m_H , the new pseudoscalar m_A , and the new charged scalar m_{H^\pm} driven by m_{22}^2 (which is otherwise unconstrained). For such an inert vacuum, the only consequence of the increased $U(1)$ symmetry with respect to the inert Z_2 model is that the neutral pseudoscalar becomes degenerate with one of the neutral scalars: $m_A = m_H$.

All nonzero vacua violate CP2, CP3 and $U(2)$. Thus, Theorem 1 states that there will be nondecoupling if the vacuum has $v_1 \neq 0$ and $v_2 \neq 0$. What happens in the inert minima $(v_1, v_2) = (v, 0)$? we start by noting that, in the CP2, CP3 and $U(2)$ 2HDM, a basis can be chosen where the parameters obey $m_{22}^2 = m_{11}^2$, $m_{12}^2 = 0$, $\lambda_6 = \lambda_7 = 0$.¹⁴ Moreover, for these parameters, the stationarity condition forces $2m_{11}^2 = -\lambda_1 v^2$. Since $m_{22}^2 = m_{11}^2$ and $m_{12}^2 = 0$, there is no quadratic term to drive the decoupling. Therefore, the exact CP2, CP3 and $U(2)$ 2HDM will always have nondecoupling.

We have performed a complete analysis of all symmetric 2HDM, where indeed a symmetric 2HDM will have a decoupling limit if and only if the vacuum does not violate the symmetry. One might conjecture whether this is valid for all symmetric NHDM. Lacking a classification of all symmetries in NHDM, we have found no way to prove or disprove this conjecture.

7 Conclusions

We have investigated the situations under which a generic NHDM might have a decoupling limit. This is important for model building, since one might wish to gain intuition on a complicated model by studying it analytically when it decouples into a simpler theory. It is also very convenient when simulating numerically a complicated model, by debugging with simulations of numerical limits when it reduces effectively to a simpler model.

We have found that, under the assumptions that the vevs and the mixing angles are not parametrically small, *all* quadratic couplings must be included in order for decoupling to occur. For the most part, this means that either one has nondecoupling or, else, one must break the symmetry softly. In addition, we showed that there is no \mathcal{CP} violation in scalar-pseudoscalar mixing as the charged Higgs masses approach decoupling.

¹⁴ This is more difficult to see in the CP2 2HDM, but it is shown in [26, 36, 38].

Our theorems were illustrated with a number of special examples. These were also used to explore violations of the assumptions and to discuss what form of decoupling can occur in the absence of some Y_{ij}^S , as long as one goes to a very fine tuned region of parameter space, where some vevs decrease as v^2/\mathcal{M} .

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Appendix A: projective special unitary group

Imagine that a group has two generators S_1 and S_2 which do not commute. Then $\mathcal{S} = \{S_1, S_2\}$ is a non-Abelian subset of $\mathbb{U}(N)$. But the relevant group of transformations of the scalar potential, the so-called projective group, could be Abelian. In fact, as pointed out in detail in [21, 33, 34], the group of physically distinct unitary reparametrization transformations is not $\mathbb{SU}(N)$, but rather the projective special unitary group $\mathbb{PSU}(N) \simeq \mathbb{SU}(N)/\mathbb{Z}_N \simeq \mathbb{U}(N)/\mathbb{U}(1)$. As an example, take a 3HDM with the symmetry $\Delta(27)$ generated by

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (\text{A.1})$$

with $\gamma = \exp(2i\pi/3)$. This is a non-Abelian subgroup of $\mathbb{SU}(3)$. Nevertheless, the relevant structure to describe the invariances of the Higgs potential is not $\mathbb{SU}(3)$ but rather $\mathbb{PSU}(3) \simeq \mathbb{SU}(3)/\mathbb{Z}_3$. And, since

$$a b a^{-1} b^{-1} = \omega^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\text{A.2})$$

the relevant group is rather $\mathbb{Z}_3 \times \mathbb{Z}_3$, which is Abelian [33]. To make things slightly more complicated, it turns out that creating a 3HDM invariant under $\mathbb{Z}_3 \times \mathbb{Z}_3$ yields a scalar potential invariant under the larger group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \simeq \Sigma(54)/\mathbb{Z}_3$. Further details can be found in ref. [33].

Appendix B: 3HDMs with a well-defined symmetry basis

Here we shall explicitly write the 3HDM scalar potential for abelian symmetries that yield a well-defined Symmetry basis. These are the $\mathbb{U}(1) \times \mathbb{U}(1)$, $\mathbb{U}(1) \times \mathbb{Z}_2$, \mathbb{Z}_3 and \mathbb{Z}_4 , found in [17, 21], and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ put forward by Weinberg [22]. Such symmetries are realisable through the following representations,

$$S_{\mathbb{U}(1) \times \mathbb{U}(1)} = \text{diag}(1, e^{i\alpha}, e^{i\beta}), \quad (\text{B.3})$$

$$S_{\mathbb{U}(1) \times \mathbb{Z}_2} = \text{diag}(1, -1, e^{i\alpha}), \quad (\text{B.4})$$

$$S_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \{\text{diag}(1, -1, 1), \text{diag}(-1, 1, 1)\}, \quad (\text{B.5})$$

$$S_{\mathbb{Z}_3} = \text{diag}(1, e^{i2\pi/3}, e^{-i2\pi/3}), \quad (\text{B.6})$$

$$S_{\mathbb{Z}_4} = \text{diag}(1, e^{i\pi}, e^{-i\pi/2}). \quad (\text{B.7})$$

Here one can explicitly see that every doublet has a different group charge, so that these symmetry groups have a well-defined Symmetry basis.

By writing the general $\mathbb{U}(3)$ matrix in a suitable diagonal basis, we obtain the generator of the $\mathbb{U}(1) \times \mathbb{U}(1)$ symmetry in (B.3). Then the parameters that are invariant under (B.3) will also be invariant under abelian symmetries whose generator is written in a diagonal basis. The $\mathbb{U}(1) \times \mathbb{U}(1)$ symmetric 3HDM can be parameterised as [21],

$$\begin{aligned} V_0 = & m_{11}^2 (\phi_1^\dagger \phi_1) + m_{22}^2 (\phi_2^\dagger \phi_2) + m_{33}^2 (\phi_3^\dagger \phi_3) \\ & + \frac{\lambda_{11}}{2} (\phi_1^\dagger \phi_1)^2 + \frac{\lambda_{22}}{2} (\phi_2^\dagger \phi_2)^2 + \frac{\lambda_{33}}{2} (\phi_3^\dagger \phi_3)^2 \\ & + \lambda_{12} (\phi_1^\dagger \phi_1) (\phi_2^\dagger \phi_2) + \lambda_{13} (\phi_1^\dagger \phi_1) (\phi_3^\dagger \phi_3) \\ & + \lambda_{23} (\phi_2^\dagger \phi_2) (\phi_3^\dagger \phi_3) + \lambda'_{12} |\phi_1^\dagger \phi_2|^2 \\ & + \lambda'_{13} |\phi_1^\dagger \phi_3|^2 + \lambda'_{23} |\phi_2^\dagger \phi_3|^2. \end{aligned} \quad (\text{B.8})$$

The invariant potentials under each of the symmetry group generators in (B.4)–(B.7) can be parameterised as [21],

$$V_{\mathbb{U}(1) \times \mathbb{Z}_2} = V_0 + \frac{1}{2} \left[\lambda_{12,12} (\phi_1^\dagger \phi_2)^2 + \text{h.c.} \right], \quad (\text{B.9})$$

$$\begin{aligned} V_{\mathbb{Z}_2 \times \mathbb{Z}_2} = & V_0 + \frac{1}{2} \left[\lambda_{12,12} (\phi_1^\dagger \phi_2)^2 + \lambda_{13,13} (\phi_1^\dagger \phi_3)^2 \right. \\ & \left. + \lambda_{23,23} (\phi_2^\dagger \phi_3)^2 + \text{h.c.} \right], \end{aligned} \quad (\text{B.10})$$

$$V_{\mathbb{Z}_3} = V_0 + \left[\lambda_{21,31} (\phi_2^\dagger \phi_1) (\phi_3^\dagger \phi_1) \right.$$

$$\left. + \lambda_{12,32} (\phi_1^\dagger \phi_2) (\phi_3^\dagger \phi_2) + \lambda_{13,23} (\phi_1^\dagger \phi_3) (\phi_2^\dagger \phi_3) + \text{h.c.} \right], \quad (\text{B.11})$$

$$\begin{aligned} V_{\mathbb{Z}_4} = & V_0 + \left[\lambda_{13,23} (\phi_1^\dagger \phi_3) (\phi_2^\dagger \phi_3) \right. \\ & \left. + \frac{1}{2} \lambda_{12,12} (\phi_1^\dagger \phi_2)^2 + \text{h.c.} \right]. \end{aligned} \quad (\text{B.12})$$

In practice, to make calculations with these symmetric potentials it is convenient to use the tensorial parameterisation of the scalar potential, with $Z_{ij,kl}^S = \lambda_{ij,kl}/2$.

In the notation of ref. [17], $\lambda_{11} = 2r_1$, $\lambda_{22} = 2r_2$, $\lambda_{33} = 2r_3$, $\lambda_{12} = 2r_4$, $\lambda_{13} = 2r_5$, $\lambda_{23} = 2r_6$, $\lambda'_{12} = 2r_7$, $\lambda'_{13} = 2r_8$, $\lambda'_{23} = 2r_9$, $\lambda_{12,12} = 2c_3$, $\lambda_{13,13} = 2c_5$, and $\lambda_{23,23} = 2c_{17}$.

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