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Unitarity bounds for all symmetry-constrained 3HDMs

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Abstract: Models with three Higgs doublets (3HDM) are the source of much recent activity, for they are key components of many solutions to the problems of the Standard Model; from extra sources of CP violation to Dark Matter candidates. We compute explicitly the theoretical bounds for all symmetry-constrained 3HDM arising from the perturbative unitarity of two-to-two scattering amplitudes. In addition, we propose a method based on principal minors that foregoes diagonalization and which is preferable in all models (not only 3HDM) dealing with large scattering matrices.

Keywords: Discrete Symmetries, Multi-Higgs Models

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JHEP08(2022)273

Contents

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1 Introduction

The discovery at LHC [1, 2] of a scalar particle with 125 GeV has inaugurated the era of experimental exploration of the spontaneous symmetry breaking (SSB) mechanism. Questions which are being addressed include the following. Is there only one scalar particle? Since there are multiple fermion families, perhaps there are also more scalar families, naturally urging one to study N Higgs doublet models (NHDM) — for reviews see, for example, [3–5]. Such models, besides more scalars, usually involve also couplings of the 125 GeV scalar to gauge bosons and to fermions at odds with the Standard Model (SM). How close are the measured couplings from those SM values? Can NHDM fix problems currently unsolved by the SM? Indeed, new sources of CP violation in the scalar sector can explain the observed baryon asymmetry in the universe, which cannot be accommodated in the SM. Moreover, many NHDM can accommodate one (or more) dark matter particles.

NHDM usually involve a very large parameter space. It is customary to reduce the number of parameters through the use of symmetries acting on the space of scalar fields. This is done for several reasons. First, such symmetries reduce the number of independent parameters, making it easier to explore the range of possibilities in a given model. Second, when extended to the fermion sector, NHDM usually have flavour changing neutral scalar interactions, which are severely constrained by experiments in flavour physics. Some family symmetries set these flavour changing neutral scalar coupling to zero in a natural way. The most well known case is the preclusion of such couplings via a $\mathbb{Z}_2$ symmetry in the 2HDM [6, 7]. Finally, when both the Lagrangian and the vacuum respect a given symmetry, the particle spectrum has the same symmetry; by setting all SM particles in a sector with no "charge" under the discrete symmetry, a neutral lightest particle in a sector "charged" under the discrete symmetry is a candidate for dark matter. The classification of all symmetry-constrained 2HDM can be found in [8] and for the 3HDM in [9–12]. This is summarized in section 2. The full classification has not yet been achieved for NHDM with $N \geq 4$.

JHEP08(2022)273

The large parameter space of NHDM is further reduced by constraints of a theoretical nature, including conditions for bounded from below potential [13–17], for the chosen vacuum to constitute indeed the absolute minimum of the theory [14, 17, 18], and for the scattering matrices to exhibit perturbative unitarity. These constrains are required in order for the theory and any phenomenology consequences derived therefrom to be consistent with both unitarity and perturbativity. This article is dedicated to the study of perturbative unitarity for all symmetry-constrained 3HDM. In section 5, we write explicitly all scattering sub-matrices, except for the $\mathbb{Z}_2^{(\mathrm{CP})}$ symmetric 3HDM, which involves a 9×9 scattering matrix. We also present, in section 3, several techniques which are applicable to matrices of arbitrary dimension, involving the study of principal minors, and which enable faster numerical studies, when compared with the numerical determination of the eigenvalues. The important results of section 3, are illustrated in section 4 with applications based on some of the matrices obtained in section 5. In conjunction, we cover all symmetry-constrained 3HDM.

Perturbative unitarity has been thoroughly studied in the context of the Standard Model in a method championed by Lee, Quigg and Thacker [19, 20]. In the 2HDM, it was computed for a model with $\mathbb{Z}_2$ symmetry [21, 22] and, later, for the general case [23, 24]. In the 3HDM, it has been studied with an $S_3 \times \mathbb{Z}_2^{(\mathrm{CP})}$ symmetry [25], $CP4$ and $\mathbb{Z}_3$ symmetries [26] and in the case of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{(\mathrm{CP})}$ [27]. In the former and latter cases, the authors started from a Higgs family and then imposed that all complex coefficients are real, effectively enlarging the symmetry group.

Concentrating on special cases, refs. [21, 27] explored the use of both the electric charge and the Abelian charges of the discrete symmetries to classify the scattering matrices. Here, we use both the hypercharge $\mathcal{Y}$ and electric charge Q, following [26]. We combine this with a simple algorithm to block diagonalize the matrices with permutations, presented in appendix C. With this algorithm, we automatically separate the Abelian charges of the global symmetries that are imposed. Thus, we often obtain the minimal form for the scattering matrices, for every possible symmetry. We show in appendix B that some scattering matrices always coincide, thus simplifying the analysis.

We include the simplest explicit formulae for any particular symmetry-constrained 3HDM, despite the fact that some models can be obtained as limits of models with a smaller symmetry. We do this for three reasons. First, the reader can simply concentrate on the particular model of interest and its notation, without having to set, sometimes error-prone limits (see reason three). Second, higher symmetries usually turn a large matrix into its smaller blocks, where exact formulae for the eigenvalues then become possible. Third, consider a subgroup G' of a larger symmetry G. It is often the case that the potential invariant under G' is simpler to see (or more commonly studied in the literature) in a basis where the extension to G becomes quite complicated. Said otherwise, the natural basis to study the G-invariant potential and the natural basis to study the G'-invariant potential are often at odds with each other. This problem is discussed in detail in appendix D.

Throughout the paper, we will use the notation of [28], which denotes the real (complex) coefficients by r_i (c_i). We summarize the notations in appendix A, by stating some common alternatives.

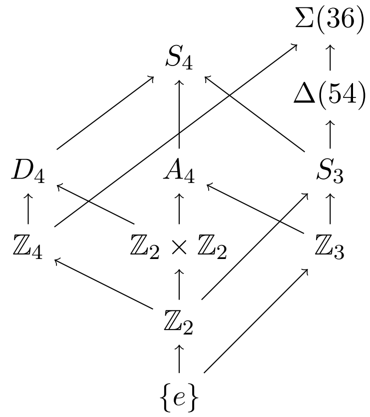

Figure 1. Tree of finite realizable groups of Higgs-family transformations in 3HDM.

2 Symmetry-constrained 3HDMs

The scalar potential of the most general 3HDM is given by

$$V_H = \mu_{ij}(\Phi_i^\dagger \Phi_j) + z_{ij,kl}(\Phi_i^\dagger \Phi_j)(\Phi_k^\dagger \Phi_l) = -\mathcal{L}_{\text{Higgs}}\,, \tag{2.1}$$

with i, j, k, l running from 1 to 3. Before using the freedom to perform unitary transformations in the space of scalar fields, one has the following independent parameters [28] in the potential (2.1): μ_{ij} has 3 real and 3 complex (6 magnitudes and 3 phases); $z_{ij,kl}$ has 9 real and 18 complex (27 magnitudes and 18 phases). The first counting is trivial, since μ_{ij} is a 3×3 Hermitian matrix, while that for $z_{ij,kl}$ is easily seen from the parametrization in (A.1). Thus, the most general 3HDM has 12 real and 21 complex (33 magnitudes and 21 phases) parameters. However, one can choose a different parametrization for the scalar fields, using a 3×3 unitary transformation, which keeps the kinetic terms invariant. Such a transformation can be used to take out 3 magnitudes and 5 phases from the parameters of V_H (one further overall phase in the unitary transformation has no impact on V_H). This leaves 30 independent magnitudes and 16 independent phases in V_H.

In the 3HDM, many symmetries may be imposed on the potential as to prevent flavour changing neutral currents (FCNC), model dark matter or impose CP properties in the theory.

The study of symmetries in the 3HDM has been thoroughly performed in [9–12]. In figure 1, we illustrate the map of realizable discrete Higgs-family symmetries obtained in [9].[1] By "realizable" symmetry we mean a symmetry which, when imposed on the potential, does not yield a potential with a larger symmetry. To be specific, consider the 2HDM. Imposing $\mathbb{Z}_3$ on the 2HDM scalar potential, it becomes immediately invariant under the full Peccei-Quinn U(1) symmetry. Thus, there is *no* realizable $\mathbb{Z}_3$ 2HDM. In contrast, imposing $\mathbb{Z}_3$ on the 3HDM scalar potential does not lead to a potential invariant under a larger symmetry. Thus, *there exists* a realizable $\mathbb{Z}_3$ 3HDM. The full list of realizable

[1] We note that throughout this paper we distinguish the semidirect and direct products by commutativity of the involved symmetries. Thus, $A \rtimes B$ and $A \times B$ are the same if the generators of A and B commute.

JHEP08(2022)273

Discrete symmetries in the 3HDM	
Unitary	$\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, S_3, D_4, A_4, S_4, \Delta(54), \Sigma(36)$
Anti-unitary (GCP)	$\mathbb{Z}_2^{(\mathrm{CP})}, \mathbb{Z}_2 \times \mathbb{Z}_2^{(\mathrm{CP})}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{(\mathrm{CP})}, CP4, \mathbb{Z}_3 \rtimes \mathbb{Z}_2^{(\mathrm{CP})}, S_3 \times \mathbb{Z}_2^{(\mathrm{CP})}, \Delta(54) \rtimes \mathbb{Z}_2^{(\mathrm{CP})}$

Table 1. Full list of discrete symmetries in the 3HDM, where $\mathbb{Z}_2^{(\mathrm{CP})}$ stands for the usual CP.

Continuous symmetries in the 3HDM	
Abelian	$\mathrm{U}(1)_1, \mathrm{U}(1)_2, \mathrm{U}(1)_2 \times \mathbb{Z}_2, \mathrm{U}(1) \times \mathrm{U}(1)$
Non-abelian	$\mathrm{U}(2), \mathrm{O}(2), \mathrm{SU}(3), \mathrm{SO}(3)$

Table 2. List of continuous symmetries in the 3HDM.

discrete symmetries in the 3HDM was composed in [9] which we summarize in table 1. In table 2 we summarize the continuous groups in the 3HDM.

3 Optimized unitarity bounds

In this section we will provide for both necessary conditions for unitarity in any theory and a procedure that greatly improves the usual method.

In the literature, the standard route for unitarity bounds (and the one we will pursue later, in section 5) is to build the scattering matrices and diagonalize them. Then, one proceeds to impose a bound on the eigenvalues such that for a scattering matrix A, its eigenvalues λ_i are bounded by $|\lambda_i| < 8\pi$. This method was spearheaded in [19, 20].

3.1 Unitarity bounds without diagonalization

As stated before, the standard method relies heavily on diagonalization, which (barring an explicit formula, impossible for matrices larger than 4×4) has to be performed numerically for each point in the parameter space of the model. But, this is not the most efficient method. In this section we propose an approach which is based on lemma 10.4.1 of [29]. If A is an Hermitian matrix with eigenvalues λ_i, then $A + cI$ has eigenvalues $\lambda_i + c$. Then, we can use this simple statement and Sylvester's criterion involving principal minors to state the following remark.[2]

Remark. *Let A be an $n \times n$ Hermitian matrix and λ_i its eigenvalues. Then the following statements are equivalent:*

1. *The eigenvalues are bounded as $|\lambda_i| < c$;*

[2]Principal minors have also been used by [16] in the different context of searching for bounded from below conditions in scalar potentials.

JHEP08(2022)273

2. *The determinants of all the upper left k-by-k submatrices of $A + cI$ and $cI - A$ are positive;*

3. *The leading principal minors $D_k(A + cI)$ and $D_k(cI - A)$ are positive.*

Thus, if A is a scattering matrix with bounds on its eigenvalues $|\lambda_i| < 8\pi$, then

$$D_k(A + 8\pi I) > 0 \quad \text{and} \quad D_k(8\pi I - A) > 0\,, \tag{3.1}$$

such that

$$D_k(A + 8\pi I) = \begin{vmatrix} A_{11} + 8\pi & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} + 8\pi \end{vmatrix} . \tag{3.2}$$

In particular, $D_1(A + 8\pi I) = A_{11} + 8\pi$ and $D_n(A + 8\pi I) = \det(A + 8\pi I)$.

Although not needed, one may also add further conditions. Specifically, if the leading principal minors $D_k(A + cI)$ and $D_k(cI - A)$ are positive, then all of its principal minors, not just the leading ones, are positive. A direct consequence of this assertion is that $|\lambda_i| < c$ also implies the following remark:

Remark. *Let A be an $n \times n$ Hermitian matrix and λ_i its eigenvalues. Then if the eigenvalues are bounded as $|\lambda_i| < c$, it is a necessary condition that*

$$|A_{ii}| < c\,, \quad i = 1, 2, \cdots, n\,. \tag{3.3}$$

This has already been pointed out, through a different argument, in [23], where unitarity bounds for larger matrices were being considered.

3.2 Necessary conditions for unitarity in a NHDM

When dealing with some NHDM model with many parameters, some general bounds may be extracted by looking at scattering matrices and using the conditions of eq. (3.3). In a general NHDM, we have that[3]

$$\begin{aligned} |\lambda_{ii,ii}| &< \frac{4\pi}{3}\,, \\ |\lambda_{ii,jj}| &< 4\pi\,, \\ |\lambda_{ii,jj} + 2\lambda_{ij,ji}| &< 4\pi\,. \end{aligned} \tag{3.4}$$

In particular, for any 3HDM we have the necessary (but not sufficient) unitarity constraints

$$\begin{aligned} |r_1|,\, |r_2|,\, |r_3| &< \frac{4\pi}{3}\,, \\ |r_4|,\, |r_5|,\, |r_6| &< 4\pi\,, \\ |r_4 + 2r_7|,\, |r_5 + 2r_8|,\, |r_6 + 2r_9| &< 4\pi\,, \end{aligned} \tag{3.5}$$

confirming the particular result of eq. (3.4) of [30], obtained for the case of the $\mathbb{Z}_2{\times}\mathbb{Z}_2{\times}\mathbb{Z}_2^{(CP)}$.

[3] We note the important fact that in some cases perturbative unitarity bounds supersede the usual perturbativity bound $|\lambda| < 4\pi$, as evidenced by $|\lambda_{ii,ii}| < \frac{4\pi}{3}$.

3.3 An improved procedure

Although it may not seem at first hand an improvement to the standard method, the technique with the leading principal minors yields five main advantages:

- Determinants are polynomial in nature and therefore more numerically stable, as root problems may occur in diagonalizations;
- Both determinants and diagonalization are tipically $\mathcal{O}(n^3)$, although the former is much faster;
- The use of eq. (3.3) enables a timely choice of the random matrices. We only consider random matrices which check $|A_{ii}| < c$;
- As it is much faster, it enables a much more thorough and (thus) reliable scan of the parameter space;
- Analytical inequalities are trivial to compute, regardless of the size of the scattering matrix.

Thus, we present an example of the use of this technique with the following procedure:

1. Sample a very large number of random Hermitian matrices by making them check eq. (3.3);
2. Loop through the random Hermitian matrices calculating the determinants $D_2(A + cI)$ and $D_2(cI - A)$;
3. Check positivity of the determinants;
4. Trim the remaining Hermitian matrices;
5. Go to step 2, but now compute $D_3(A + cI)$ and $D_3(cI - A)$ until we reach the full n-by-n determinants.

When finished, the remaining matrices are valid scattering matrices through unitarity.

We tested the comparison between the methods with minors and eigenvalues with a python code, which we include in the paper as a supplementary material file. In this test, we ran unitarity through 400000 symmetric matrices with size 5×5. We concluded that our method runs about four times faster in this example.[4]

The use of the remaining principal minors is not as advantageous as with the case of the diagonal elements. In fact, a matrix of size N has $2^N - 1$ principal minors. As our algorithm uses only $2N - 1$ principal minors, it scales linearly to larger matrices. Thus, regarding the number of operations, our algorithm is $\mathcal{O}(N)$.

The procedure proposed here is interesting even for simple 3×3 matrices. We will illustrate this point in section 4, using some 3×3 matrices which show up in our discussion of the scattering matrices for all symmetry-constrained 3HDM models, to be performed later, in section 5.

[4]The generalization from symmetric to Hermitian matrices is trivial to perform.

4 Conditions for larger matrices

In the symmetry-constrained 3HDM cases to be presented in section 5 below, we will find many matrices of large dimension. Even in the case of 3×3 matrices, we can use the formula for solutions of cubic equations or, else, we can utilize the new procedure described in section 3. In this section, we provide a few examples of the latter. Though simple, they illustrate well how powerful the procedure in section 3 is.

4.1 The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

In this case we will find,[5]

$$M_2^{++} \supset A = 2 \begin{pmatrix} r_1 & c_3 & c_5 \\ c_3^* & r_2 & c_{17} \\ c_5^* & c_{17}^* & r_3 \end{pmatrix}, \tag{4.1}$$

and the conditions are

$$\begin{aligned} &D_1(A + 8\pi I) > 0 \Rightarrow r_1 > -4\pi\,, \\ &D_2(A + 8\pi I) > 0 \Rightarrow (r_1 + 4\pi)\,(r_2 + 4\pi) - |c_3|^2 > 0\,, \\ &D_3(A + 8\pi I) > 0 \Rightarrow 2\Re\,[c_3 c_{17} c_5^*] + (r_1 + 4\pi)\,(r_2 + 4\pi)\,(r_3 + 4\pi) \\ &\qquad\qquad - (r_3 + 4\pi)\,|c_3|^2 - (r_1 + 4\pi)\,|c_{17}|^2 - (r_2 + 4\pi)\,|c_5|^2 > 0 \\ &\qquad\qquad \Leftrightarrow \det\,(A + 8\pi I) > 0\,, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &D_1(8\pi I - A) > 0 \Rightarrow r_1 < 4\pi\,, \\ &D_2(8\pi I - A) > 0 \Rightarrow (r_1 - 4\pi)\,(r_2 - 4\pi) - |c_3|^2 > 0\,, \\ &D_3(8\pi I - A) > 0 \Rightarrow \det\,(8\pi I - A) > 0\,. \end{aligned} \tag{4.3}$$

With these six conditions we have necessary and sufficient conditions for unitarity. We may also add, in consequence of eq. (3.3), that $|r_2| < 4\pi$ and $|r_3| < 4\pi$, althought it does not yield any new information.

The next matrix is

$$M_0^{+} \supset A = 2 \begin{pmatrix} r_1 & r_7 & r_8 \\ r_7 & r_2 & r_9 \\ r_8 & r_9 & r_3 \end{pmatrix}, \tag{4.4}$$

[5] The notation for the subscripts and superscripts in the M matrices is defined in section 5.

and the conditions are

$$
\begin{aligned}
D_1(A+8\pi I) > 0 &\Rightarrow r_1 > -4\pi\,, \\
D_2(A+8\pi I) > 0 &\Rightarrow (r_1+4\pi)\,(r_2+4\pi) - r_7^2 > 0\,, \\
D_3(A+8\pi I) > 0 &\Rightarrow 2r_7r_8r_9 + (r_1+4\pi)\,(r_2+4\pi)\,(r_3+4\pi) \\
&\qquad - (r_3+4\pi)\,r_7^2 - (r_1+4\pi)\,r_9^2 - (r_2+4\pi)\,r_8^2 > 0 \\
&\qquad \Leftrightarrow \det(A+8\pi I) > 0\,,
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
D_1(8\pi I - A) > 0 &\Rightarrow r_1 < 4\pi\,, \\
D_2(8\pi I - A) > 0 &\Rightarrow (r_1-4\pi)\,(r_2-4\pi) - r_7^2 > 0\,, \\
D_3(8\pi I - A) > 0 &\Rightarrow \det(8\pi I - A) > 0\,.
\end{aligned}
\tag{4.6}
$$

The next matrix is

$$
M_0^0 \supset A = 2\begin{pmatrix} 3r_1 & 2r_4+r_7 & 2r_5+r_8 \\ 2r_4+r_7 & 3r_2 & 2r_6+r_9 \\ 2r_5+r_8 & 2r_6+r_9 & 3r_3 \end{pmatrix}\,, \tag{4.7}
$$

and the conditions are

$$
\begin{aligned}
D_1(A+8\pi I) > 0 &\Rightarrow 3r_1 > -4\pi\,, \\
D_2(A+8\pi I) > 0 &\Rightarrow (3r_1+4\pi)\,(3r_2+4\pi) - (2r_4+r_7)^2 > 0\,, \\
D_3(A+8\pi I) > 0 &\Rightarrow \det(A+8\pi I) > 0\,,
\end{aligned}
\tag{4.8}
$$

and

$$
\begin{aligned}
D_1(8\pi I - A) > 0 &\Rightarrow 3r_1 < 4\pi\,, \\
D_2(8\pi I - A) > 0 &\Rightarrow (3r_1-4\pi)\,(3r_2-4\pi) - (2r_4+r_7)^2 > 0\,, \\
D_3(8\pi I - A) > 0 &\Rightarrow \det(8\pi I - A) > 0\,.
\end{aligned}
\tag{4.9}
$$

We note that this matrix yields a stronger bound on r_1, r_2, r_3 than the previous ones. We have $|r_i| < 4\pi/3$ for $i = 1, 2, 3$.

4.2 The S_3 symmetry

In the case of this symmetry, we will find

$$
M_0^+ \supset A = 2\begin{pmatrix} r_4 & c_{12} & c_{12}^* \\ c_{12}^* & r_5 & c_{11} \\ c_{12} & c_{11}^* & r_5 \end{pmatrix}\,, \tag{4.10}
$$

and the conditions are

$$\begin{aligned} &D_1(A+8\pi I)>0 \Rightarrow r_4>-4\pi\,,\\ &D_2(A+8\pi I)>0 \Rightarrow\, (r_4+4\pi)\,(r_5+4\pi)-|c_{12}|^2>0\,,\\ &D_3(A+8\pi I)>0 \Rightarrow 2\Re\left[c_{11}c_{12}^2\right]+(r_1+4\pi)\,(r_5+4\pi)^2\\ &\qquad\qquad\qquad\qquad -(r_4+4\pi)\,|c_{11}|^2-2\,(r_5+4\pi)\,|c_{12}|^2>0\\ &\qquad\qquad\qquad\qquad \Leftrightarrow \det\,(A+8\pi I)>0\,, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &D_1(8\pi I-A)>0 \Rightarrow r_4<4\pi\,,\\ &D_2(8\pi I-A)>0 \Rightarrow\, (r_4-4\pi)\,(r_5-4\pi)-|c_{11}|^2>0\,,\\ &D_3(8\pi I-A)>0 \Rightarrow\, \det\,(8\pi I-A)>0\,. \end{aligned} \tag{4.12}$$

The next matrix is

$$M_0^0 \supset A = 2\begin{pmatrix} r_4+2r_7 & 3c_{12} & 3c_{12}^* \\ 3c_{12}^* & r_5+2r_8 & 3c_{11} \\ 3c_{12} & 3c_{11}^* & r_5+2r_8 \end{pmatrix}\,, \tag{4.13}$$

and the conditions are

$$\begin{aligned} &D_1(A+8\pi I)>0 \Rightarrow (r_4+2r_7)+4\pi>0\,,\\ &D_2(A+8\pi I)>0 \Rightarrow\, (r_4+2r_7+4\pi)\,(r_5+2r_8+4\pi)-9\,|c_{12}|^2>0\,,\\ &D_3(A+8\pi I)>0 \Rightarrow\, \det\,(A+8\pi I)>0\,, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} &D_1(8\pi I-A)>0 \Rightarrow 4\pi-(r_4+2r_7)>0\,,\\ &D_2(8\pi I-A)>0 \Rightarrow\, (r_4+2r_7-4\pi)\,(r_5+2r_8-4\pi)-9\,|c_{12}|^2>0\,,\\ &D_3(8\pi I-A)>0 \Rightarrow\, \det\,(8\pi I-A)>0\,. \end{aligned} \tag{4.15}$$

5 Unitarity bounds for all symmetry-constrained 3HDM

In NHDM, there are neutral scalars and charge $\pm$ scalars (in units of the positron charge). Thus, in $2 \to 2$ scattering, the initial (and final) charges can be 0, $+$ (same scattering matrices as $-$), or $++$ (same matrices as $--$). Following the method provided in [26] for tree-level unitarity bounds, we present the eigenvalues for the matrices M_2^{++}, M_2^{+}, M_0^{+}, M_2^0, M_0^0, where $M_{2\mathcal{Y}}^{Q}$ are scattering matrices with hypercharge $2\mathcal{Y}$ and[6] electric charge Q. We will state that a matrix M is "equal" ($=$) to its block-diagonal form by only making use of permutations. To identify the relevant permutations, we use the algorithm presented in appendix C. For M_0^0 we will also use orthogonal matrices of the type

$$\mathcal{O} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}, \tag{5.1}$$

to further reduce the size of the matrices. For this operation we will use the symbol of "similar" ($\sim$). These operations will simplify the presentation, while at the same time preserving the final results. It is then without loss of generality that we use them.

We will denote the eigenvalues as $\Lambda_i^{Q,2\mathcal{Y}}$ for the ith eigenvalue of charge Q and hypercharge $\mathcal{Y}$.[7] These correspond to the eigenvalues of the corresponding matrices $M_{2\mathcal{Y}}^{Q}$. The unitarity bounds provided by using $|\Lambda| < 8\pi$ for symmetry-constrained 3HDMs are given in the following subsections.

5.1 The $\mathbb{Z}_2^{(\mathrm{CP})}$ symmetry

By imposing $G = \mathbb{Z}_2^{(\mathrm{CP})}$ we get the most general 3HDM but now with real coefficients. This is the smallest symmetry possible. In general, we must contend with 9×9 irreducible scattering matrices and, thus, its unitarity bounds should be obtained numerically. As mentioned in section 3, for these cases we advocate a faster procedure based on principle minors.

5.2 The $\mathbb{Z}_2$ symmetry

By imposing $G = \mathbb{Z}_2$ with representation $\mathrm{diag}(1, 1, -1)$ we get the quartic potential

$$\begin{aligned} V_{\mathbb{Z}_2} = &\sum_{i=1}^{3} r_i|\phi_i|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + 2r_5(\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + 2r_6(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) \\ &+ 2r_7|\phi_1^\dagger\phi_2|^2 + 2r_8|\phi_1^\dagger\phi_3|^2 + 2r_9|\phi_2^\dagger\phi_3|^2 + \Big[2c_1(\phi_1^\dagger\phi_1)(\phi_1^\dagger\phi_2) + c_3(\phi_1^\dagger\phi_2)^2 \\ &+ c_5(\phi_1^\dagger\phi_3)^2 + 2c_7(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_2) + 2c_{11}(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3) \\ &+ 2c_{13}(\phi_1^\dagger\phi_2)(\phi_3^\dagger\phi_3) + 2c_{14}(\phi_1^\dagger\phi_3)(\phi_3^\dagger\phi_2) + c_{17}(\phi_2^\dagger\phi_3)^2 + h.c.\Big], \end{aligned} \tag{5.2}$$

with the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}M_2^{++} = \mathrm{diag}\left\{\begin{pmatrix} r_1 & \sqrt{2}c_1 & c_3 & c_5 \\ \sqrt{2}c_1^* & r_4+r_7 & \sqrt{2}c_7 & \sqrt{2}c_{11} \\ c_3^* & \sqrt{2}c_7^* & r_2 & c_{17} \\ c_5^* & \sqrt{2}c_{11}^* & c_{17}^* & r_3 \end{pmatrix}, \begin{pmatrix} r_5+r_8 & c_{13}+c_{14} \\ c_{13}^*+c_{14}^* & r_6+r_9 \end{pmatrix}\right\} \tag{5.3}$$

[6] In our notation, $Q = T_3 + \mathcal{Y}$, where T_3 is the third component of weak isospin.

[7] Since when $Q = {++}$ only $2\mathcal{Y} = 2$ exists, we suppress the explicit reference to the hypercharge in the corresponding eigenvalues: $\Lambda_i^{++,2} \to \Lambda_i^{++}$.

JHEP08(2022)273

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned}
\Lambda^{++}_{1-4} &= \text{Eigenvalues of first matrix}\,,\\
\Lambda^{++}_{5,6} &= \pm\sqrt{4\,|c_{13}+c_{14}|^2+(r_5-r_6+r_8-r_9)^{\,2}}+r_5+r_6+r_8+r_9\,.
\end{aligned} \tag{5.4}$$

The matrix M_2^+. From M_2^+ we get

$$\frac{1}{2}\,M_2^+ = \text{diag}\left\{\begin{pmatrix} r_1 & c_1 & c_1 & c_3 & c_5\\ c_1^* & r_4 & r_7 & c_7 & c_{11}\\ c_1^* & r_7 & r_4 & c_7 & c_{11}\\ c_3^* & c_7^* & c_7^* & r_2 & c_{17}\\ c_5^* & c_{11}^* & c_{11}^* & c_{17}^* & r_3 \end{pmatrix}, \begin{pmatrix} r_5 & c_{13} & r_8 & c_{14}\\ c_{13}^* & r_6 & c_{14}^* & r_9\\ r_8 & c_{14} & r_5 & c_{13}\\ c_{14}^* & r_9 & c_{13}^* & r_6 \end{pmatrix}\right\}, \tag{5.5}$$

with eigenvalues of M_2^+:

$$\begin{aligned}
\Lambda^{+,2}_{1-5} &= \text{Eigenvalues of first matrix}\,,\\
\Lambda^{+,2}_{6,7} &= \Lambda^{++}_{5,6}\,,\\
\Lambda^{+,2}_{8,9} &= \pm\sqrt{4\,|c_{13}-c_{14}|^2+(r_5-r_6-r_8+r_9)^{\,2}}+r_5+r_6-r_8-r_9\,.
\end{aligned} \tag{5.6}$$

The matrix M_0^+. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \text{diag}\left\{\begin{pmatrix} r_1 & c_1^* & c_1 & r_7 & r_8\\ c_1 & r_4 & c_3 & c_7 & c_{14}\\ c_1^* & c_3^* & r_4 & c_7^* & c_{14}^*\\ r_7 & c_7^* & c_7 & r_2 & r_9\\ r_8 & c_{14}^* & c_{14} & r_9 & r_3 \end{pmatrix}, \begin{pmatrix} r_5 & c_{13} & c_5 & c_{11}\\ c_{13}^* & r_6 & c_{11} & c_{17}\\ c_5^* & c_{11}^* & r_5 & c_{13}^*\\ c_{11}^* & c_{17}^* & c_{13} & r_6 \end{pmatrix}\right\}, \tag{5.7}$$

with eigenvalues of M_0^+:

$$\begin{aligned}
\Lambda^{+,0}_{1-5} &= \text{Eigenvalues of first matrix}\,,\\
\Lambda^{+,0}_{6-9} &= \text{Eigenvalues of second matrix}\,.
\end{aligned} \tag{5.8}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.9}$$

with eigenvalues of M_2^0:

$$\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.10}$$

The matrix M_0^0. From M_0^0 we get

$$\frac{1}{2}\,M_0^0 \sim \text{diag}\left\{\frac{1}{2}\,M_0^+, A, B\right\}, \tag{5.11}$$

with

$$A = \begin{pmatrix} 3r_1 & 3c_1^* & 3c_1 & 2r_4+r_7 & 2r_5+r_8 \\ 3c_1 & r_4+2r_7 & 3c_3 & 3c_7 & 2c_{13}+c_{14} \\ 3c_1^* & 3c_3^* & r_4+2r_7 & 3c_7^* & 2c_{13}^*+c_{14}^* \\ 2r_4+r_7 & 3c_7^* & 3c_7 & 3r_2 & 2r_6+r_9 \\ 2r_5+r_8 & 2c_{13}^*+c_{14}^* & 2c_{13}+c_{14} & 2r_6+r_9 & 3r_3 \end{pmatrix}, \tag{5.12}$$

$$B = \begin{pmatrix} r_5+2r_8 & c_{13}+2c_{14} & 3c_5 & 3c_{11} \\ c_{13}^*+2c_{14}^* & r_6+2r_9 & 3c_{11} & 3c_{17} \\ 3c_5^* & 3c_{11}^* & r_5+2r_8 & c_{13}^*+2c_{14}^* \\ 3c_{11}^* & 3c_{17}^* & c_{13}+2c_{14} & r_6+2r_9 \end{pmatrix}, \tag{5.13}$$

with eigenvalues of M_0^0:

$$\begin{aligned} \Lambda^{0,0}_{1-5} &= \text{Eigenvalues of second matrix}\,, \\ \Lambda^{0,0}_{6-9} &= \text{Eigenvalues of third matrix}\,. \end{aligned} \tag{5.14}$$

5.3 The $\mathbb{Z}_2 \times \mathbb{Z}_2^{(\mathrm{CP})}$ symmetry

By imposing $G = \mathbb{Z}_2 \times \mathbb{Z}_2^{(\mathrm{CP})}$ we get the quartic potential

$$\begin{aligned} V_{\mathbb{Z}_2\times\mathbb{Z}_2^{(\mathrm{CP})}} =& \sum_{i=1}^{3} r_i|\phi_i|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + 2r_5(\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + 2r_6(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) \\ &+ 2r_7|\phi_1^\dagger\phi_2|^2 + 2r_8|\phi_1^\dagger\phi_3|^2 + 2r_9|\phi_2^\dagger\phi_3|^2 + \Big[2r_{10}(\phi_1^\dagger\phi_1)(\phi_1^\dagger\phi_2) + r_{11}(\phi_1^\dagger\phi_2)^2 \\ &+ r_{12}(\phi_1^\dagger\phi_3)^2 + 2r_{13}(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_2) + 2r_{14}(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3) \\ &+ 2r_{15}(\phi_1^\dagger\phi_2)(\phi_3^\dagger\phi_3) + 2r_{16}(\phi_1^\dagger\phi_3)(\phi_3^\dagger\phi_2) + r_{17}(\phi_2^\dagger\phi_3)^2 + h.c.\Big]\,, \end{aligned} \tag{5.15}$$

with the following scattering matrices. This case is obtained from eq. (5.2) by making all coefficients real. Here, and in similar cases below, we stress the fact that all parameters are real by changing the c_k in the notation of [28], into r_j with $j \geq 10$. Specifically, in this case, we do $(c_1, c_3, c_5, c_7, c_{11}, c_{13}, c_{14}, c_{17}) \to (r_{10}, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17})$.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}\, M_2^{++} = \mathrm{diag}\left\{ \begin{pmatrix} r_1 & \sqrt{2}r_{10} & r_{11} & r_{12} \\ \sqrt{2}r_{10} & r_4+r_7 & \sqrt{2}r_{13} & \sqrt{2}r_{14} \\ r_{11} & \sqrt{2}r_{13} & r_2 & r_{17} \\ r_{12} & \sqrt{2}r_{14} & r_{17} & r_3 \end{pmatrix}, \begin{pmatrix} r_5+r_8 & r_{15}+r_{16} \\ r_{15}+r_{16} & r_6+r_9 \end{pmatrix} \right\}, \tag{5.16}$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned} \Lambda^{++}_{1-4} &= \text{Eigenvalues of first matrix}\,, \\ \Lambda^{++}_{5,6} &= \pm\sqrt{4\,(r_{15}+r_{16})^2 + (r_5-r_6+r_8-r_9)^{\,2}} + r_5+r_6+r_8+r_9\,. \end{aligned} \tag{5.17}$$

JHEP08(2022)273

The matrix M_2^+. From M_2^+ we get

$$\frac{1}{2}\,M_2^+ = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_{10} & r_{10} & r_{11} & r_{12} \\ r_{10} & r_4 & r_7 & r_{13} & r_{14} \\ r_{10} & r_7 & r_4 & r_{13} & r_{14} \\ r_{11} & r_{13} & r_{13} & r_2 & r_{17} \\ r_{12} & r_{14} & r_{14} & r_{17} & r_3 \end{pmatrix}, \begin{pmatrix} r_5 & r_{15} & r_8 & r_{16} \\ r_{15} & r_6 & r_{16} & r_9 \\ r_8 & r_{16} & r_5 & r_{15} \\ r_{16} & r_9 & r_{15} & r_6 \end{pmatrix}\right\}, \tag{5.18}$$

with eigenvalues of M_2^+:

$$\begin{aligned} \Lambda^{+,2}_{1-5} &= \text{Eigenvalues of first matrix}\,, \\ \Lambda^{+,2}_{6,7} &= \Lambda^{++}_{5,6}\,, \\ \Lambda^{+,2}_{8,9} &= \pm\sqrt{4\,(r_{15}-r_{16})^2 + (r_5 - r_6 - r_8 + r_9)^{\,2}} + r_5 + r_6 - r_8 - r_9\,. \end{aligned} \tag{5.19}$$

The matrix M_0^+. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_{10} & r_{10} & r_7 & r_8 \\ r_{10} & r_4 & r_{11} & r_{13} & r_{16} \\ r_{10} & r_{11} & r_4 & r_{13} & r_{16} \\ r_7 & r_{13} & r_{13} & r_2 & r_9 \\ r_8 & r_{16} & r_{16} & r_9 & r_3 \end{pmatrix}, \begin{pmatrix} r_5 & r_{15} & r_{12} & r_{14} \\ r_{15} & r_6 & r_{14} & r_{17} \\ r_{12} & r_{14} & r_5 & r_{15} \\ r_{14} & r_{17} & r_{15} & r_6 \end{pmatrix}\right\}, \tag{5.20}$$

with eigenvalues of M_0^+:

$$\begin{aligned} \Lambda^{+,0}_{1-5} &= \text{Eigenvalues of first matrix}\,, \\ \Lambda^{+,0}_{6,7} &= \pm\sqrt{4\,(r_{14}+r_{15})^{\,2} + (r_5 - r_6 + r_{12} - r_{17})^{\,2}} + r_5 + r_6 + r_{12} + r_{17}\,, \\ \Lambda^{+,0}_{8,9} &= \pm\sqrt{4\,(r_{14}-r_{15})^{\,2} + (r_5 - r_6 - r_{12} + r_{17})^{\,2}} + r_5 + r_6 - r_{12} - r_{17}\,. \end{aligned} \tag{5.21}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.22}$$

with eigenvalues of M_2^0:

$$\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.23}$$

The matrix M_0^0. From M_0^0 we get

$$\frac{1}{2}\,M_0^0 \sim \mathrm{diag}\left\{\frac{1}{2}\,M_0^+, A, B\right\}, \tag{5.24}$$

with

$$A = \begin{pmatrix} 3r_1 & 3r_{10} & 3r_{10} & 2r_4+r_7 & 2r_5+r_8 \\ 3r_{10} & r_4+2r_7 & 3r_{11} & 3r_{13} & 2r_{15}+r_{16} \\ 3r_{10} & 3r_{11} & r_4+2r_7 & 3r_{13} & 2r_{15}+r_{16} \\ 2r_4+r_7 & 3r_{13} & 3r_{13} & 3r_2 & 2r_6+r_9 \\ 2r_5+r_8 & 2r_{15}+r_{16} & 2r_{15}+r_{16} & 2r_6+r_9 & 3r_3 \end{pmatrix}, \tag{5.25}$$

$$B = \begin{pmatrix} r_5+2r_8 & r_{15}+2r_{16} & 3r_{12} & 3r_{14} \\ r_{15}+2r_{16} & r_6+2r_9 & 3r_{14} & 3r_{17} \\ 3r_{12} & 3r_{14} & r_5+2r_8 & r_{15}+2r_{16} \\ 3r_{14} & 3r_{17} & r_{15}+2r_{16} & r_6+2r_9 \end{pmatrix}, \tag{5.26}$$

with eigenvalues of M_0^0:

$$\begin{aligned}
\Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,, \\
\Lambda^{0,0}_{10-14} &= \text{Eigenvalues of first matrix}\,, \\
\Lambda^{0,0}_{15,16} &= \pm\sqrt{4\left(-3r_{14}+r_{15}+2r_{16}\right)^2+\left(r_5-r_6+2r_8-2r_9-3r_{12}+3r_{17}\right)^2} \\
&\quad +r_5+r_6+2r_8+2r_9-3\left(r_{12}+r_{17}\right)\,, \\
\Lambda^{0,0}_{17,18} &= \pm\sqrt{4\left(3r_{14}+r_{15}+2r_{16}\right)^2+\left(r_5-r_6+2r_8-2r_9+3r_{12}-3r_{17}\right)^2} \\
&\quad +r_5+r_6+2r_8+2r_9+3\left(r_{12}+r_{17}\right)\,.
\end{aligned} \tag{5.27}$$

5.4 The $\mathbb{Z}_4$ symmetry

By imposing $G = \mathbb{Z}_4$ with representation $\text{diag}(i, -i, 1)$ we get the quartic potential

$$\begin{aligned}
V_{\mathbb{Z}_4} = &\sum_{i=1}^{3} r_i|\phi_i|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + 2r_5(\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + 2r_6(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) \\
&+ 2r_7|\phi_1^\dagger\phi_2|^2 + 2r_8|\phi_1^\dagger\phi_3|^2 + 2r_9|\phi_2^\dagger\phi_3|^2 + r_{10}\left[(\phi_1^\dagger\phi_2)^2 + h.c.\right] \\
&+ 2r_{11}\left[(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3) + h.c.\right]\,,
\end{aligned} \tag{5.28}$$

which can be easily achieved by setting from $\mathbb{Z}_2$ the constraints $\{c_1, c_5, c_7, c_{13}, c_{14}, c_{17}\} \to 0$.

Thus, we get the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}M_2^{++} = \text{diag}\left\{\begin{pmatrix} r_1 & r_{10} \\ r_{10} & r_2 \end{pmatrix}, \begin{pmatrix} r_4+r_7 & \sqrt{2}r_{11} \\ \sqrt{2}r_{11} & r_3 \end{pmatrix}, (r_5+r_8), (r_6+r_9)\right\}, \tag{5.29}$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned}
\Lambda^{++}_{1,2} &= \pm\sqrt{4r_{10}^2+\left(r_1-r_2\right)^2}+r_1+r_2\,, \\
\Lambda^{++}_{3,4} &= \pm\sqrt{8r_{11}^2+\left(-r_3+r_4+r_7\right)^2}+r_3+r_4+r_7\,, \\
\Lambda^{++}_5 &= 2(r_5+r_8)\,, \\
\Lambda^{++}_6 &= 2(r_6+r_9)\,.
\end{aligned} \tag{5.30}$$

The matrix M_2^+. From M_2^+ we get

$$\frac{1}{2}\,M_2^+ = \text{diag}\left\{\begin{pmatrix} r_4 & r_7 & r_{11} \\ r_7 & r_4 & r_{11} \\ r_{11} & r_{11} & r_3 \end{pmatrix}, \begin{pmatrix} r_1 & r_{10} \\ r_{10} & r_2 \end{pmatrix}, \begin{pmatrix} r_5 & r_8 \\ r_8 & r_5 \end{pmatrix}, \begin{pmatrix} r_6 & r_9 \\ r_9 & r_6 \end{pmatrix}\right\}, \tag{5.31}$$

with eigenvalues of M_2^+:

$$\begin{aligned} \Lambda^{+,2}_{1,2} &= \Lambda^{++}_{3,4}\,, \\ \Lambda^{+,2}_{3} &= 2(r_4 - r_7)\,, \\ \Lambda^{+,2}_{4,5} &= \Lambda^{++}_{1,2}\,, \\ \Lambda^{+,2}_{6,7} &= 2(r_5 \pm r_8) \\ \Lambda^{+,2}_{8,9} &= 2(r_6 \pm r_9)\,. \end{aligned} \tag{5.32}$$

The matrix M_0^+. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \text{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_8 \\ r_7 & r_2 & r_9 \\ r_8 & r_9 & r_3 \end{pmatrix}, \begin{pmatrix} r_4 & r_{10} \\ r_{10} & r_4 \end{pmatrix}, \begin{pmatrix} r_5 & r_{11} \\ r_{11} & r_6 \end{pmatrix}, \begin{pmatrix} r_6 & r_{11} \\ r_{11} & r_5 \end{pmatrix}\right\}, \tag{5.33}$$

with eigenvalues of M_0^+:

$$\begin{aligned} \Lambda^{+,0}_{1-3} &= \text{Roots of:} \\ &\quad x^3 - 2(r_1 + r_2 + r_3)x^2 + 4(-r_7^2 - r_8^2 - r_9^2 + r_1 r_2 + r_1 r_3 + r_2 r_3)x \\ &\quad + 8(r_3 r_7^2 - 2 r_8 r_9 r_7 + r_2 r_8^2 + r_1 r_9^2 - r_1 r_2 r_3) = 0\,, \\ \Lambda^{+,0}_{4,5} &= 2\left(r_4 \pm r_{10}\right)\,, \\ \Lambda^{+,0}_{6-9} &= \pm\sqrt{4 r_{11}^2 + (r_5 - r_6)^2} + r_5 + r_6\,. \end{aligned} \tag{5.34}$$

The matrix M_2^0. As shown in complete generality in appendix B,

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.35}$$

and, thus, the eigenvalues of M_2^0 and M_2^{++} coincide:

$$\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.36}$$

The matrix M_0^0. From M_0^0 we get

$$\begin{aligned} \frac{1}{2}\,M_0^0 \sim \text{diag}\Bigg\{ &\frac{1}{2} M_0^+, \begin{pmatrix} 3r_1 & 2r_4 + r_7 & 2r_5 + r_8 \\ 2r_4 + r_7 & 3r_2 & 2r_6 + r_9 \\ 2r_5 + r_8 & 2r_6 + r_9 & 3r_3 \end{pmatrix}, \begin{pmatrix} r_4 + 2r_7 & 3r_{10} \\ r_{10} & r_4 + 2r_7 \end{pmatrix}, \\ &\begin{pmatrix} r_5 + 2r_8 & 3r_{11} \\ 3r_{11} & r_6 + 2r_9 \end{pmatrix}, \begin{pmatrix} r_6 + 2r_9 & 3r_{11} \\ 3r_{11} & r_5 + 2r_8 \end{pmatrix} \Bigg\}, \end{aligned} \tag{5.37}$$

with eigenvalues of M_0^0:

$$\begin{aligned}
\Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,,\\
\Lambda^{0,0}_{10-12} &= \text{Roots of:}\\
&\quad x^3 + 2\left(-3r_1 - 3r_2 - 3r_3\right)x^2 + 4\Big(-4r_4^2 - 4r_7r_4 - 4r_5^2 - 4r_6^2 - r_7^2 - r_8^2 - r_9^2\\
&\quad + 9r_1r_2 + 9r_1r_3 + 9r_2r_3 - 4r_5r_8 - 4r_6r_9\Big)x + 8\Big(12r_3r_4^2 + 12r_2r_5^2 + 12r_1r_6^2\\
&\quad + 3r_3r_7^2 + 3r_2r_8^2 + 3r_1r_9^2 - 27r_1r_2r_3 - 16r_4r_5r_6 + 12r_3r_4r_7 - 8r_5r_6r_7\\
&\quad + 12r_2r_5r_8 - 8r_4r_6r_8 - 4r_6r_7r_8 - 8r_4r_5r_9 + 12r_1r_6r_9 - 4r_5r_7r_9 - 4r_4r_8r_9\\
&\quad - 2r_7r_8r_9\Big) = 0\,,\\
\Lambda^{0,0}_{13,14} &= 2\left(r_4 + 2r_7 \pm 3r_{10}\right)\,,\\
\Lambda^{0,0}_{15-18} &= \pm\sqrt{36r_{11}^2 + \left(r_5 - r_6 + 2r_8 - 2r_9\right)^2} + r_5 + r_6 + 2r_8 + 2r_9\,.
\end{aligned} \tag{5.38}$$

5.5 The $\mathbb{Z}_3$ symmetry

By imposing $G = \mathbb{Z}_3$ with representation $\text{diag}(e^{\frac{2\pi i}{3}}, e^{\frac{-2\pi i}{3}}, 1)$ we get the quartic potential

$$\begin{aligned}
V_{\mathbb{Z}_3} =& \sum_{i=1}^{3} r_i|\phi_i|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + 2r_5(\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + 2r_6(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)\\
&+ 2r_7|\phi_1^\dagger\phi_2|^2 + 2r_8|\phi_1^\dagger\phi_3|^2 + 2r_9|\phi_2^\dagger\phi_3|^2 + \Big[2c_4(\phi_1^\dagger\phi_2)(\phi_1^\dagger\phi_3)\\
&+ 2c_{11}(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3) + 2c_{12}(\phi_1^\dagger\phi_2)(\phi_3^\dagger\phi_2) + h.c.\Big]\,,
\end{aligned} \tag{5.39}$$

with the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}\,M_2^{++} = \text{diag}\left\{\begin{pmatrix} r_1 & \sqrt{2}c_4 \\ \sqrt{2}c_4^* & r_6 + r_9\end{pmatrix}, \begin{pmatrix} r_4 + r_7 & \sqrt{2}c_{11} \\ \sqrt{2}c_{11}^* & r_3\end{pmatrix}, \begin{pmatrix} r_5 + r_8 & \sqrt{2}c_{12} \\ \sqrt{2}c_{12}^* & r_2\end{pmatrix}\right\}\,, \tag{5.40}$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned}
\Lambda^{++}_{1,2} &= \pm\sqrt{8|c_4|^2 + \left(-r_1 + r_6 + r_9\right)^2} + r_1 + r_6 + r_9\,,\\
\Lambda^{++}_{3,4} &= \pm\sqrt{8|c_{11}|^2 + \left(-r_3 + r_4 + r_7\right)^2} + r_3 + r_4 + r_7\,,\\
\Lambda^{++}_{5,6} &= \pm\sqrt{8|c_{12}|^2 + \left(-r_2 + r_5 + r_8\right)^2} + r_2 + r_5 + r_8\,.
\end{aligned} \tag{5.41}$$

The matrix M_2^{+}. From M_2^{+} we get

$$\frac{1}{2}\,M_2^{+} = \text{diag}\left\{\begin{pmatrix} r_1 & c_4 & c_4 \\ c_4^* & r_6 & r_9 \\ c_4^* & r_9 & r_6\end{pmatrix}, \begin{pmatrix} r_4 & r_7 & c_{11} \\ r_7 & r_4 & c_{11} \\ c_{11}^* & c_{11}^* & r_3\end{pmatrix}, \begin{pmatrix} r_5 & c_{12} & r_8 \\ c_{12}^* & r_2 & c_{12}^* \\ r_8 & c_{12} & r_5\end{pmatrix}\right\}\,, \tag{5.42}$$

with eigenvalues of M_2^+:

$$\begin{aligned}
\Lambda_1^{+,2} &= 2(r_6 - r_9)\,,\\
\Lambda_{2,3}^{+,2} &= \Lambda_{1,2}^{++}\,,\\
\Lambda_4^{+,2} &= 2(r_4 - r_7)\,,\\
\Lambda_{5,6}^{+,2} &= \Lambda_{3,4}^{++}\\
\Lambda_7^{+,2} &= 2(r_5 - r_8)\\
\Lambda_{8,9}^{+,2} &= \Lambda_{5,6}^{++}\,.
\end{aligned} \tag{5.43}$$

The matrix M_0^+. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_8\\ r_7 & r_2 & r_9\\ r_8 & r_9 & r_3\end{pmatrix},\begin{pmatrix} r_4 & c_{12} & c_4\\ c_{12}^* & r_6 & c_{11}\\ c_4^* & c_{11}^* & r_5\end{pmatrix},\begin{pmatrix} r_5 & c_4 & c_{11}\\ c_4^* & r_4 & c_{12}^*\\ c_{11}^* & c_{12} & r_6\end{pmatrix}\right\}, \tag{5.44}$$

with eigenvalues of M_0^+:

$$\begin{aligned}
\Lambda_{1-3}^{+,0} = {} & \text{Roots of:}\\
& x^3 - 2(r_1+r_2+r_3)x^2 + 4(-r_7^2 - r_8^2 - r_9^2 + r_1r_2 + r_1r_3 + r_2r_3)x\\
& + 8\left(r_3r_7^2 - 2r_8r_9r_7 + r_2r_8^2 + r_1r_9^2 - r_1r_2r_3\right) = 0\,,\\
\Lambda_{4-6}^{+,0} = {} & \text{Roots of:}\\
& x^3 - 2(r_4+r_5+r_6)x^2 + 4(-|c_4|^2 - |c_{11}|^2 - |c_{12}|^2 + r_4r_5 + r_4r_6 + r_5r_6)x\\
& + 8\left(r_6|c_4|^2 + r_4|c_{11}|^2 + r_5|c_{12}|^2 - 2\Re(c_4c_{11}^*c_{12}^*) - r_4r_5r_6\right) = 0\,,\\
\Lambda_{7-9}^{+,0} = {} & \Lambda_{4-6}^{+,0}\,.
\end{aligned} \tag{5.45}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.46}$$

with eigenvalues of M_2^0:

$$\Lambda_{1-6}^{0,2} = \Lambda_{1-6}^{++}\,. \tag{5.47}$$

The matrix M_0^0. From M_0^0 we get

$$\begin{aligned}
\frac{1}{2}\,M_0^0 \sim \mathrm{diag}\Bigg\{&\frac{1}{2}M_0^+,\begin{pmatrix} 3r_1 & 2r_4+r_7 & 2r_5+r_8\\ 2r_4+r_7 & 3r_2 & 2r_6+r_9\\ 2r_5+r_8 & 2r_6+r_9 & 3r_3\end{pmatrix},\begin{pmatrix} r_4+2r_7 & 3c_{12} & 3c_4\\ 3c_{12}^* & r_6+2r_9 & 3c_{11}\\ 3c_4^* & 3c_{11}^* & r_5+2r_8\end{pmatrix},\\
&\begin{pmatrix} r_5+2r_8 & 3c_4 & 3c_{11}\\ 3c_4^* & r_4+2r_7 & 3c_{12}^*\\ 3c_{11}^* & 3c_{12} & r_6+2r_9\end{pmatrix}\Bigg\}\,,
\end{aligned} \tag{5.48}$$

with eigenvalues of M_0^0:

$$
\begin{aligned}
\Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,,\\
\Lambda^{0,0}_{10-12} &= \text{Roots of:}\\
&\quad x^3 + 2\left(-3r_1 - 3r_2 - 3r_3\right)x^2 + 4\Big(-4r_4^2 - 4r_7r_4 - 4r_5^2 - 4r_6^2 - r_7^2 - r_8^2 - r_9^2\\
&\quad + 9r_1r_2 + 9r_1r_3 + 9r_2r_3 - 4r_5r_8 - 4r_6r_9\Big)x + 8\Big(12r_3r_4^2 + 12r_2r_5^2 + 12r_1r_6^2\\
&\quad + 3r_3r_7^2 + 3r_2r_8^2 + 3r_1r_9^2 - 27r_1r_2r_3 - 16r_4r_5r_6 + 12r_3r_4r_7 - 8r_5r_6r_7\\
&\quad + 12r_2r_5r_8 - 8r_4r_6r_8 - 4r_6r_7r_8 - 8r_4r_5r_9 + 12r_1r_6r_9 - 4r_5r_7r_9 - 4r_4r_8r_9\\
&\quad - 2r_7r_8r_9\Big) = 0\,,\\
\Lambda^{0,0}_{13-15} &= \text{Roots of:}\\
&\quad x^3 + 2\left(-r_4 - r_5 - r_6 - 2r_7 - 2r_8 - 2r_9\right)x^2 + 4\Big(-9|c_4|^2 - 9|c_{11}|^2 - 9|c_{12}|^2\\
&\quad + r_4r_5 + r_4r_6 + r_5r_6 + 2r_5r_7 + 2r_6r_7 + 2r_4r_8 + 2r_6r_8 + 4r_7r_8 + 2r_4r_9\\
&\quad + 2r_5r_9 + 4r_7r_9 + 4r_8r_9\Big)x + 8\Big(9r_6|c_4|^2 + 18r_9|c_4|^2 + 9r_4|c_{11}|^2 + 9r_5|c_{12}|^2\\
&\quad + 18r_7|c_{11}|^2 + 18r_8|c_{12}|^2 - 54\Re(c_4c_{11}^*c_{12}^*) - r_4r_5r_6 - 2r_5r_6r_7 - 2r_4r_6r_8\\
&\quad - 4r_6r_7r_8 - 2r_4r_5r_9 - 4r_5r_7r_9 - 4r_4r_8r_9 - 8r_7r_8r_9\Big) = 0\,,\\
\Lambda^{0,0}_{16-18} &= \Lambda^{0,0}_{13-15}\,.
\end{aligned}
\tag{5.49}
$$

5.6 The $\mathbb{Z}_3 \rtimes \mathbb{Z}_2^{(\mathrm{CP})}$ symmetry

By imposing $G = \mathbb{Z}_3 \rtimes \mathbb{Z}_2^{(\mathrm{CP})}$ we get the quartic potential

$$
\begin{aligned}
V_{\mathbb{Z}_3 \rtimes \mathbb{Z}_2^{(\mathrm{CP})}} = & \sum_{i=1}^{3} r_i|\phi_i|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + 2r_5(\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + 2r_6(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)\\
& + 2r_7|\phi_1^\dagger\phi_2|^2 + 2r_8|\phi_1^\dagger\phi_3|^2 + 2r_9|\phi_2^\dagger\phi_3|^2 + 2r_{10}\left[(\phi_1^\dagger\phi_2)(\phi_1^\dagger\phi_3) + h.c.\right]\\
& + 2r_{11}\left[(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3) + h.c.\right] + 2r_{12}\left[(\phi_1^\dagger\phi_2)(\phi_3^\dagger\phi_2) + h.c.\right]\,,
\end{aligned}
\tag{5.50}
$$

which can be easily achieved by setting from $\mathbb{Z}_3$ the constraints $\{c_4, c_{11}, c_{12}\} \in \mathbb{R}$. Thus, we get the following eigenvalues.

The matrix M_2^{++}. From M_2^{++} we get

$$
\frac{1}{2}\,M_2^{++} = \mathrm{diag}\left\{ \begin{pmatrix} r_1 & \sqrt{2}r_{10} \\ \sqrt{2}r_{10} & r_6 + r_9 \end{pmatrix}, \begin{pmatrix} r_4 + r_7 & \sqrt{2}r_{11} \\ \sqrt{2}r_{11} & r_3 \end{pmatrix}, \begin{pmatrix} r_5 + r_8 & \sqrt{2}r_{12} \\ \sqrt{2}r_{12} & r_2 \end{pmatrix} \right\}, \tag{5.51}
$$

and thus we get the eigenvalues of M_2^{++}:

$$
\begin{aligned}
\Lambda^{++}_{1,2} &= \pm\sqrt{8r_{10}^2 + \left(-r_1 + r_6 + r_9\right)^2} + r_1 + r_6 + r_9\,,\\
\Lambda^{++}_{3,4} &= \pm\sqrt{8r_{11}^2 + \left(-r_3 + r_4 + r_7\right)^2} + r_3 + r_4 + r_7\,,\\
\Lambda^{++}_{5,6} &= \pm\sqrt{8r_{12}^2 + \left(-r_2 + r_5 + r_8\right)^2} + r_2 + r_5 + r_8\,.
\end{aligned}
\tag{5.52}
$$

JHEP08(2022)273

The matrix $\boldsymbol{M_2^+}$. From M_2^+ we get

$$\frac{1}{2}\,M_2^+ = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_{10} & r_{10} \\ r_{10} & r_6 & r_9 \\ r_{10} & r_9 & r_6 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 & r_{11} \\ r_7 & r_4 & r_{11} \\ r_{11} & r_{11} & r_3 \end{pmatrix}, \begin{pmatrix} r_5 & r_{12} & r_8 \\ r_{12} & r_2 & r_{12} \\ r_8 & r_{12} & r_5 \end{pmatrix}\right\}, \tag{5.53}$$

with eigenvalues of M_2^+:

$$\begin{aligned} \Lambda_1^{+,2} &= 2(r_6 - r_9)\,, \\ \Lambda_{2,3}^{+,2} &= \Lambda_{1,2}^{++}\,, \\ \Lambda_4^{+,2} &= 2(r_4 - r_7)\,, \\ \Lambda_{5,6}^{+,2} &= \Lambda_{3,4}^{++} \\ \Lambda_7^{+,2} &= 2(r_5 - r_8) \\ \Lambda_{8,9}^{+,2} &= \Lambda_{5,6}^{++}\,. \end{aligned} \tag{5.54}$$

The matrix $\boldsymbol{M_0^+}$. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_8 \\ r_7 & r_2 & r_9 \\ r_8 & r_9 & r_3 \end{pmatrix}, \begin{pmatrix} r_4 & r_{12} & r_{10} \\ r_{12} & r_6 & r_{11} \\ r_{10} & r_{11} & r_5 \end{pmatrix}, \begin{pmatrix} r_5 & r_{10} & r_{11} \\ r_{10} & r_4 & r_{12} \\ r_{11} & r_{12} & r_6 \end{pmatrix}\right\}, \tag{5.55}$$

with eigenvalues of M_0^+:

$$\begin{aligned} \Lambda_{1-3}^{+,0} =\ & \text{Roots of:} \\ & x^3 - 2(r_1+r_2+r_3)x^2 + 4(-r_7^2 - r_8^2 - r_9^2 + r_1r_2 + r_1r_3 + r_2r_3)x \\ & + 8\left(r_3r_7^2 - 2r_8r_9r_7 + r_2r_8^2 + r_1r_9^2 - r_1r_2r_3\right) = 0\,, \\ \Lambda_{4-6}^{+,0} =\ & \text{Roots of:} \\ & x^3 - 2(r_4+r_5+r_6)x^2 + 4(-r_{10}^2 - r_{11}^2 - r_{12}^2 + r_4r_5 + r_4r_6 + r_5r_6)x \\ & + 8\left(r_6r_{10}^2 + r_4r_{11}^2 + r_5r_{12}^2 - 2r_{10}r_{11}r_{12} - r_4r_5r_6\right) = 0\,, \\ \Lambda_{7-9}^{+,0} =\ & \Lambda_{4-6}^{+,0}\,. \end{aligned} \tag{5.56}$$

The matrix $\boldsymbol{M_2^0}$. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.57}$$

with eigenvalues of M_2^0:

$$\Lambda_{1-6}^{0,2} = \Lambda_{1-6}^{++}\,. \tag{5.58}$$

The matrix $\boldsymbol{M_0^0}$. From M_0^0 we get

$$\begin{aligned} \frac{1}{2}\,M_0^0 \sim \mathrm{diag}\Bigg\{ & \frac{1}{2}M_0^+, \begin{pmatrix} 3r_1 & 2r_4+r_7 & 2r_5+r_8 \\ 2r_4+r_7 & 3r_2 & 2r_6+r_9 \\ 2r_5+r_8 & 2r_6+r_9 & 3r_3 \end{pmatrix}, \begin{pmatrix} r_4+2r_7 & 3r_{12} & 3r_{10} \\ 3r_{12} & r_6+2r_9 & 3r_{11} \\ 3r_{10} & 3r_{11} & r_5+2r_8 \end{pmatrix}, \\ & \begin{pmatrix} r_5+2r_8 & 3r_{10} & 3r_{11} \\ 3r_{10} & r_4+2r_7 & 3r_{12} \\ 3r_{11} & 3r_{12} & r_6+2r_9 \end{pmatrix}\Bigg\}, \end{aligned} \tag{5.59}$$

with eigenvalues of M_0^0:

$$
\begin{aligned}
\Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,,\\
\Lambda^{0,0}_{10-12} &= \text{Roots of:}\\
&\quad x^3+2\left(-3r_1-3r_2-3r_3\right)x^2+4\Big(-4r_4^2-4r_7r_4-4r_5^2-4r_6^2-r_7^2-r_8^2-r_9^2\\
&\quad +9r_1r_2+9r_1r_3+9r_2r_3-4r_5r_8-4r_6r_9\Big)x+8\Big(12r_3r_4^2+12r_2r_5^2+12r_1r_6^2\\
&\quad +3r_3r_7^2+3r_2r_8^2+3r_1r_9^2-27r_1r_2r_3-16r_4r_5r_6+12r_3r_4r_7-8r_5r_6r_7\\
&\quad +12r_2r_5r_8-8r_4r_6r_8-4r_6r_7r_8-8r_4r_5r_9+12r_1r_6r_9-4r_5r_7r_9-4r_4r_8r_9\\
&\quad -2r_7r_8r_9\Big)=0\,,\\
\Lambda^{0,0}_{13-15} &= \text{Roots of:}\\
&\quad x^3+2\left(-r_4-r_5-r_6-2r_7-2r_8-2r_9\right)x^2+4\Big(-9r_{10}^2-9r_{11}^2-9r_{12}^2\\
&\quad +r_4r_5+r_4r_6+r_5r_6+2r_5r_7+2r_6r_7+2r_4r_8+2r_6r_8+4r_7r_8+2r_4r_9\\
&\quad +2r_5r_9+4r_7r_9+4r_8r_9\Big)x+8\Big(9r_6r_{10}^2+18r_9r_{10}^2+9r_4r_{11}^2+9r_5r_{12}^2\\
&\quad +18r_7r_{11}^2+18r_8r_{12}^2-54r_{10}r_{11}r_{12}-r_4r_5r_6-2r_5r_6r_7-2r_4r_6r_8\\
&\quad -4r_6r_7r_8-2r_4r_5r_9-4r_5r_7r_9-4r_4r_8r_9-8r_7r_8r_9\Big)=0\,,\\
\Lambda^{0,0}_{16-18} &= \Lambda^{0,0}_{13-15}\,.
\end{aligned}
\tag{5.60}
$$

5.7 The $\mathrm{U}(1)_2$ symmetry

By imposing $G=\mathrm{U}(1)_2$ with representation $\mathrm{diag}(1,1,e^{i\alpha})$, with $\alpha\neq\{0,\pi\}$, we get the quartic potential

$$
\begin{aligned}
V_{\mathrm{U}(1)_2} =& \sum_{i=1}^{3} r_i|\phi_i|^4+2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)+2r_5(\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3)+2r_6(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)\\
&+2r_7|\phi_1^\dagger\phi_2|^2+2r_8|\phi_1^\dagger\phi_3|^2+2r_9|\phi_2^\dagger\phi_3|^2+\Big[2c_1(\phi_1^\dagger\phi_1)(\phi_1^\dagger\phi_2)+c_3(\phi_1^\dagger\phi_2)^2\\
&+2c_7(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_2)+2c_{13}(\phi_1^\dagger\phi_2)(\phi_3^\dagger\phi_3)+2c_{14}(\phi_1^\dagger\phi_3)(\phi_3^\dagger\phi_2)+h.c.\Big]\,,
\end{aligned}
\tag{5.61}
$$

which can be easily achieved by setting from $\mathbb{Z}_2$ the constraints $\{c_5,c_{17}\}\to 0$.

Thus, we get the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$
\frac{1}{2}\,M_2^{++}=\mathrm{diag}\left\{\begin{pmatrix} r_1 & \sqrt{2}c_1 & c_3\\ \sqrt{2}c_1^* & r_4+r_7 & \sqrt{2}c_7\\ c_3^* & \sqrt{2}c_7^* & r_2\end{pmatrix},\begin{pmatrix} r_5+r_8 & c_{13}+c_{14}\\ c_{13}^*+c_{14}^* & r_6+r_9\end{pmatrix},r_3\right\}\,,
\tag{5.62}
$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned}
\Lambda_{1-3}^{++} &= \text{Roots of:} \\
&\quad x^3 + 2x^2\,(-r_1 - r_2 - r_4 - r_7) + 4x\Big(-2|c_1|^2 - |c_3|^2 - 2|c_7|^2 + r_1r_2 + r_1r_4 \\
&\quad + r_2r_4 + r_1r_7 + r_2r_7\Big) + 8\Big(2r_1|c_7|^2 + 2r_2|c_1|^2 + (r_4+r_7)|c_3|^2 - 4\Re(c_1c_3^*c_7) \\
&\quad - r_1r_2r_4 - r_1r_2r_7\Big) = 0\,, \\
\Lambda_{4,5}^{++} &= \pm\sqrt{4|c_{13}+c_{14}|^2 + (r_5 - r_6 + r_8 - r_9)^{\,2}} + r_5 + r_6 + r_8 + r_9\,, \\
\Lambda_6^{++} &= 2r_3\,.
\end{aligned} \tag{5.63}$$

The matrix M_2^+. From M_2^+ we get

$$\frac{1}{2}\,M_2^+ = \text{diag}\left\{\begin{pmatrix} r_1 & c_1 & c_1 & c_3 \\ c_1^* & r_4 & r_7 & c_7 \\ c_1^* & r_7 & r_4 & c_7 \\ c_3^* & c_7^* & c_7^* & r_2 \end{pmatrix}, \begin{pmatrix} r_5 & c_{13} & r_8 & c_{14} \\ c_{13}^* & r_6 & c_{14}^* & r_9 \\ r_8 & c_{14} & r_5 & c_{13} \\ c_{14}^* & r_9 & c_{13}^* & r_6 \end{pmatrix}, r_3\right\}, \tag{5.64}$$

with eigenvalues of M_2^+:

$$\begin{aligned}
\Lambda_{1-3}^{+,2} &= \Lambda_{1-3}^{++}\,, \\
\Lambda_4^{+,2} &= 2(r_4 - r_7)\,, \\
\Lambda_{5,6}^{+,2} &= \Lambda_{4,5}^{++}\,, \\
\Lambda_{7,8}^{+,2} &= \pm\sqrt{4|c_{13}-c_{14}|^2 + (r_5 - r_6 - r_8 + r_9)^{\,2}} + r_5 + r_6 - r_8 - r_9 \\
\Lambda_9^{+,2} &= \Lambda_6^{++}\,.
\end{aligned} \tag{5.65}$$

The matrix M_0^+. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \text{diag}\left\{\begin{pmatrix} r_1 & c_1^* & c_1 & r_7 & r_8 \\ c_1 & r_4 & c_3 & c_7 & c_{14} \\ c_1^* & c_3^* & r_4 & c_7^* & c_{14}^* \\ r_7 & c_7^* & c_7 & r_2 & r_9 \\ r_8 & c_{14}^* & c_{14} & r_9 & r_3 \end{pmatrix}, \begin{pmatrix} r_5 & c_{13} \\ c_{13}^* & r_6 \end{pmatrix}, \begin{pmatrix} r_5 & c_{13}^* \\ c_{13} & r_6 \end{pmatrix}\right\}, \tag{5.66}$$

with eigenvalues of M_0^+:

$$\begin{aligned}
\Lambda_{1-5}^{+,0} &= \text{Eigenvalues of first matrix}\,, \\
\Lambda_{6,7}^{+,0} &= \pm\sqrt{4|c_{13}|^2 + (r_5 - r_6)^{\,2}} + r_5 + r_6\,, \\
\Lambda_{8,9}^{+,0} &= \Lambda_{6,7}^{+,0}\,.
\end{aligned} \tag{5.67}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.68}$$

with eigenvalues of M_2^0:

$$\Lambda_{1-6}^{0,2} = \Lambda_{1-6}^{++}\,. \tag{5.69}$$

The matrix M_0^0. From M_0^0 we get

$$\frac{1}{2}\,M_0^0 \sim \mathrm{diag}\left\{\frac{1}{2}M_0^+,\begin{pmatrix} 3r_1 & 3c_1^* & 3c_1 & 2r_4+r_7 & 2r_5+r_8 \\ 3c_1 & r_4+2r_7 & 3c_3 & 3c_7 & 2c_{13}+c_{14} \\ 3c_1^* & 3c_3^* & r_4+2r_7 & 3c_7^* & 2c_{13}^*+c_{14}^* \\ 2r_4+r_7 & 3c_7^* & 3c_7 & 3r_2 & 2r_6+r_9 \\ 2r_5+r_8 & 2c_{13}^*+c_{14}^* & 2c_{13}+c_{14} & 2r_6+r_9 & 3r_3 \end{pmatrix},\right.$$
$$\left.\begin{pmatrix} r_5+2r_8 & c_{13}+2c_{14} \\ c_{13}^*+2c_{14}^* & r_6+2r_9 \end{pmatrix},\begin{pmatrix} r_5+2r_8 & c_{13}^*+2c_{14}^* \\ c_{13}+2c_{14} & r_6+2r_9 \end{pmatrix}\right\}. \tag{5.70}$$

with eigenvalues of M_0^0:

$$\begin{aligned}
\Lambda_{1-9}^{0,0} &= \Lambda_{1-9}^{+,0}\,,\\
\Lambda_{10-14}^{0,0} &= \text{Eigenvalues of second matrix}\,,\\
\Lambda_{15,16}^{0,0} &= \pm\sqrt{4|c_{13}+2c_{14}|^2+(r_5-r_6+2r_8-2r_9)^{\,2}}+r_5+r_6+2r_8+2r_9\,,\\
\Lambda_{17-18}^{0,0} &= \Lambda_{15-16}^{0,0}\,.
\end{aligned} \tag{5.71}$$

5.8 The $\mathrm{U(1)_1}$ symmetry

By imposing $G = \mathrm{U}(1)_1$ with representation $\mathrm{diag}(e^{i\alpha}, e^{-i\alpha}, 1)$, with $\alpha \neq \{0, \pi/2, 2\pi/3, \pi\}$, we get the quartic potential

$$\begin{aligned}
V_{\mathrm{U}(1)_1} =& \sum_{i=1}^{3} r_i|\phi_i|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + 2r_5(\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + 2r_6(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)\\
&+ 2r_7|\phi_1^\dagger\phi_2|^2 + 2r_8|\phi_1^\dagger\phi_3|^2 + 2r_9|\phi_2^\dagger\phi_3|^2 + 2r_{11}\left[(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3) + h.c.\right]\,,
\end{aligned} \tag{5.72}$$

which can be easily achieved by setting from $\mathbb{Z}_4$ the constraint $r_{10} \to 0$.

Thus, we get the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}\,M_2^{++} = \mathrm{diag}\left\{\begin{pmatrix} r_4+r_7 & \sqrt{2}r_{11} \\ \sqrt{2}r_{11} & r_3 \end{pmatrix}, r_1, r_2, (r_5+r_8), (r_6+r_9)\right\}\,, \tag{5.73}$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned}
\Lambda_{1,2}^{++} &= \pm\sqrt{8r_{11}^2+(-r_3+r_4+r_7)^{\,2}}+r_3+r_4+r_7\,,\\
\Lambda_3^{++} &= 2r_1\,,\\
\Lambda_4^{++} &= 2r_2\,,\\
\Lambda_5^{++} &= 2(r_5+r_8)\,,\\
\Lambda_6^{++} &= 2(r_6+r_9)\,.
\end{aligned} \tag{5.74}$$

JHEP08(2022)273

The matrix M_2^+. From M_2^+ we get

$$\frac{1}{2}\,M_2^+ = \text{diag}\left\{\begin{pmatrix} r_4 & r_7 & r_{11} \\ r_7 & r_4 & r_{11} \\ r_{11} & r_{11} & r_3 \end{pmatrix}, \begin{pmatrix} r_5 & r_8 \\ r_8 & r_5 \end{pmatrix}, \begin{pmatrix} r_6 & r_9 \\ r_9 & r_6 \end{pmatrix}, r_1, r_2 \right\}, \tag{5.75}$$

with eigenvalues of M_2^+:

$$\begin{aligned} \Lambda^{+,2}_{1,2} &= \Lambda^{++}_{1,2}\,, \\ \Lambda^{+,2}_{3} &= 2\left(r_4 - r_7\right)\,, \\ \Lambda^{+,2}_{4,5} &= 2(r_5 \pm r_8)\,, \\ \Lambda^{+,2}_{6,7} &= 2(r_6 \pm r_9)\,, \\ \Lambda^{+,2}_{8,9} &= \Lambda^{++}_{3,4}\,. \end{aligned} \tag{5.76}$$

The matrix M_0^+. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \text{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_8 \\ r_7 & r_2 & r_9 \\ r_8 & r_9 & r_3 \end{pmatrix}, \begin{pmatrix} r_5 & r_{11} \\ r_{11} & r_6 \end{pmatrix}, \begin{pmatrix} r_6 & r_{11} \\ r_{11} & r_5 \end{pmatrix}, r_4, r_4 \right\}, \tag{5.77}$$

with eigenvalues of M_0^+:

$$\begin{aligned} \Lambda^{+,0}_{1-3} &= \text{Roots of:} \\ &\qquad x^3 - 2(r_1+r_2+r_3)x^2 + 4(-r_7^2 - r_8^2 - r_9^2 + r_1 r_2 + r_1 r_3 + r_2 r_3)x \\ &\qquad + 8(r_3 r_7^2 - 2 r_8 r_9 r_7 + r_2 r_8^2 + r_1 r_9^2 - r_1 r_2 r_3) = 0\,, \\ \Lambda^{+,0}_{4,5} &= \pm\sqrt{4 r_{11}^2 + (r_5 - r_6)^{\,2}} + r_5 + r_6\,, \\ \Lambda^{+,0}_{6,7} &= \Lambda^{+,0}_{4,5}\,, \\ \Lambda^{+,0}_{8,9} &= 2 r_4\,. \end{aligned} \tag{5.78}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.79}$$

with eigenvalues of M_2^0:

$$\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.80}$$

The matrix M_0^0. From M_0^0 we get

$$\begin{aligned} \frac{1}{2}\,M_0^0 \sim \text{diag}&\left\{\frac{1}{2}M_0^+, \begin{pmatrix} 3r_1 & 2r_4 + r_7 & 2r_5 + r_8 \\ 2r_4 + r_7 & 3r_2 & 2r_6 + r_9 \\ 2r_5 + r_8 & 2r_6 + r_9 & 3r_3 \end{pmatrix}, \begin{pmatrix} r_5 + 2r_8 & 3r_{11} \\ 3r_{11} & r_6 + 2r_9 \end{pmatrix}, \right. \\ &\left. \begin{pmatrix} r_6 + 2r_9 & 3r_{11} \\ 3r_{11} & r_5 + 2r_8 \end{pmatrix}, (r_4 + 2r_7), (r_4 + 2r_7) \right\}. \end{aligned} \tag{5.81}$$

with eigenvalues of M_0^0:

$$\begin{aligned}
\Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,,\\
\Lambda^{0,0}_{10-12} &= \text{Roots of:}\\
&\quad x^3 + 2\left(-3r_1 - 3r_2 - 3r_3\right)x^2 + 4\Big(-4r_4^2 - 4r_7r_4 - 4r_5^2 - 4r_6^2 - r_7^2 - r_8^2 - r_9^2\\
&\quad + 9r_1r_2 + 9r_1r_3 + 9r_2r_3 - 4r_5r_8 - 4r_6r_9\Big)x + 8\Big(12r_3r_4^2 + 12r_2r_5^2 + 12r_1r_6^2\\
&\quad + 3r_3r_7^2 + 3r_2r_8^2 + 3r_1r_9^2 - 27r_1r_2r_3 - 16r_4r_5r_6 + 12r_3r_4r_7 - 8r_5r_6r_7\\
&\quad + 12r_2r_5r_8 - 8r_4r_6r_8 - 4r_6r_7r_8 - 8r_4r_5r_9 + 12r_1r_6r_9 - 4r_5r_7r_9 - 4r_4r_8r_9\\
&\quad - 2r_7r_8r_9\Big) = 0\,,\\
\Lambda^{0,0}_{13,14} &= \pm\sqrt{36r_{11}^2 + \left(r_5 - r_6 + 2r_8 - 2r_9\right)^2} + r_5 + r_6 + 2\left(r_8 + r_9\right)\,,\\
\Lambda^{0,0}_{15,16} &= \Lambda^{0,0}_{13,14}\,,\\
\Lambda^{0,0}_{17,18} &= 2(r_4 + 2r_7)\,.
\end{aligned} \tag{5.82}$$

5.9 The $\mathrm{U}(1) \times \mathbb{Z}_2$ symmetry

By imposing $G = \mathrm{U}(1) \times \mathbb{Z}_2$ with representation $\mathrm{diag}(1, -1, e^{i\alpha})$, with $\alpha \neq k\pi/2, k \in \mathbb{Z}$, we get the quartic potential[8]

$$\begin{aligned}
V_{\mathrm{U}(1)\times\mathbb{Z}_2} = &\sum_{i=1}^{3} r_i|\phi_i|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + 2r_5(\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + 2r_6(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)\\
&+ 2r_7|\phi_1^\dagger\phi_2|^2 + 2r_8|\phi_1^\dagger\phi_3|^2 + 2r_9|\phi_2^\dagger\phi_3|^2 + r_{10}\left[(\phi_1^\dagger\phi_2)^2 + h.c.\right]\,,
\end{aligned} \tag{5.83}$$

which can be easily achieved by setting from $\mathbb{Z}_4$ the constraint $r_{11} \to 0$.

Thus, we get the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}M_2^{++} = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_{10}\\ r_{10} & r_2\end{pmatrix}, (r_4 + r_7), (r_5 + r_8), (r_6 + r_9), r_3\right\}\,, \tag{5.84}$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned}
\Lambda^{++}_{1,2} &= \pm\sqrt{4r_{10}^2 + \left(r_1 - r_2\right)^2} + r_1 + r_2\,,\\
\Lambda^{++}_{3} &= 2(r_4 + r_7)\,,\\
\Lambda^{++}_{4} &= 2(r_5 + r_8)\,,\\
\Lambda^{++}_{5} &= 2(r_6 + r_9)\,,\\
\Lambda^{++}_{6} &= 2r_3\,.
\end{aligned} \tag{5.85}$$

[8] In [28], the authors state that $\alpha \neq \{0, \pi\}$ but if $\alpha \neq k\pi/2, k \in \mathbb{Z}$ we also get a generator for $\mathbb{Z}_4$.

JHEP08(2022)273

The matrix M_2^+. From M_2^+ we get

$$\frac{1}{2}\,M_2^+ = \text{diag}\left\{\begin{pmatrix} r_1 & r_{10} \\ r_{10} & r_2 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix}, \begin{pmatrix} r_5 & r_8 \\ r_8 & r_5 \end{pmatrix}, \begin{pmatrix} r_6 & r_9 \\ r_9 & r_6 \end{pmatrix}, r_3\right\}, \tag{5.86}$$

with eigenvalues of M_2^+:

$$\begin{aligned} \Lambda^{+,2}_{1,2} &= \Lambda^{++}_{1,2}\,, \\ \Lambda^{+,2}_{3,4} &= 2(r_4 \pm r_7)\,, \\ \Lambda^{+,2}_{5,6} &= 2(r_5 \pm r_8)\,, \\ \Lambda^{+,2}_{7,8} &= 2(r_6 \pm r_9)\,, \\ \Lambda^{+,2}_{9} &= \Lambda^{++}_{6}\,. \end{aligned} \tag{5.87}$$

The matrix M_0^+. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \text{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_8 \\ r_7 & r_2 & r_9 \\ r_8 & r_9 & r_3 \end{pmatrix}, \begin{pmatrix} r_4 & r_{10} \\ r_{10} & r_4 \end{pmatrix}, r_5, r_5, r_6, r_6\right\}, \tag{5.88}$$

with eigenvalues of M_0^+:

$$\begin{aligned} \Lambda^{+,0}_{1-3} &= \text{Roots of:} \\ &\quad x^3 - 2(r_1+r_2+r_3)x^2 + 4(-r_7^2 - r_8^2 - r_9^2 + r_1r_2 + r_1r_3 + r_2r_3)x \\ &\quad + 8(r_3r_7^2 - 2r_8r_9r_7 + r_2r_8^2 + r_1r_9^2 - r_1r_2r_3) = 0\,, \\ \Lambda^{+,0}_{4,5} &= 2\,(r_4 \pm r_{10})\,, \\ \Lambda^{+,0}_{6,7} &= 2r_5\,, \\ \Lambda^{+,0}_{8,9} &= 2r_6\,. \end{aligned} \tag{5.89}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.90}$$

with eigenvalues of M_2^0:

$$\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.91}$$

The matrix M_0^0. From M_0^0 we get

$$\begin{aligned} \frac{1}{2}\,M_0^0 \sim \text{diag}\Bigg\{&\frac{1}{2}M_0^+, \begin{pmatrix} 3r_1 & 2r_4+r_7 & 2r_5+r_8 \\ 2r_4+r_7 & 3r_2 & 2r_6+r_9 \\ 2r_5+r_8 & 2r_6+r_9 & 3r_3 \end{pmatrix}, \begin{pmatrix} r_4+2r_7 & 3r_{10} \\ 3r_{10} & r_4+2r_7 \end{pmatrix}, \\ &(r_5+2r_8), (r_5+2r_8), (r_6+2r_9), (r_6+2r_9)\Bigg\}\,. \end{aligned} \tag{5.92}$$

with eigenvalues of M_0^0:

$$
\begin{aligned}
\Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,, \\
\Lambda^{0,0}_{10-12} &= \text{Roots of:} \\
& x^3 + 2\left(-3r_1 - 3r_2 - 3r_3\right)x^2 + 4\Big(-4r_4^2 - 4r_7r_4 - 4r_5^2 - 4r_6^2 - r_7^2 - r_8^2 - r_9^2 \\
& + 9r_1r_2 + 9r_1r_3 + 9r_2r_3 - 4r_5r_8 - 4r_6r_9\Big)x + 8\Big(12r_3r_4^2 + 12r_2r_5^2 + 12r_1r_6^2 \\
& + 3r_3r_7^2 + 3r_2r_8^2 + 3r_1r_9^2 - 27r_1r_2r_3 - 16r_4r_5r_6 + 12r_3r_4r_7 - 8r_5r_6r_7 \\
& + 12r_2r_5r_8 - 8r_4r_6r_8 - 4r_6r_7r_8 - 8r_4r_5r_9 + 12r_1r_6r_9 - 4r_5r_7r_9 - 4r_4r_8r_9 \\
& - 2r_7r_8r_9\Big) = 0\,, \\
\Lambda^{0,0}_{13,14} &= 2\left(\pm 3r_{10} + r_4 + 2r_7\right)\,, \\
\Lambda^{0,0}_{15,16} &= 2(r_5 + 2r_8)\,, \\
\Lambda^{0,0}_{17,18} &= 2(r_6 + 2r_9)\,.
\end{aligned} \tag{5.93}
$$

5.10 The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

By imposing $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ we get the quartic potential of the model originally proposed by Weinberg [31],

$$
\begin{aligned}
V_{\mathbb{Z}_2\times\mathbb{Z}_2} = \sum_{i=1}^{3} r_i|\phi_i|^4 &+ 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + 2r_5(\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + 2r_6(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) \\
&+ 2r_7|\phi_1^\dagger\phi_2|^2 + 2r_8|\phi_1^\dagger\phi_3|^2 + 2r_9|\phi_2^\dagger\phi_3|^2 + \Big[c_3(\phi_1^\dagger\phi_2)^2 + c_5(\phi_1^\dagger\phi_3)^2 \\
&+ c_{17}(\phi_2^\dagger\phi_3)^2 + h.c.\Big]\,,
\end{aligned} \tag{5.94}
$$

which can be easily achieved by setting from the $\mathbb{Z}_2$ symmetric 3HDM potential the constraints $\{c_1, c_7, c_{11}, c_{13}, c_{14}\} \to 0$.

Thus, we get the following scattering matrices which reproduce in the limit of real coefficients the conditions (91)–(100) of ref. [27].

The matrix M_2^{++}. From M_2^{++} we get

$$
\frac{1}{2}\,M_2^{++} = \text{diag}\left\{\begin{pmatrix} r_1 & c_3 & c_5 \\ c_3^* & r_2 & c_{17} \\ c_5^* & c_{17}^* & r_3 \end{pmatrix}, (r_4 + r_7), (r_5 + r_8), (r_6 + r_9)\right\}\,, \tag{5.95}
$$

and thus we get the eigenvalues of M_2^{++}:

$$
\begin{aligned}
\Lambda^{++}_{1-3} &= \text{Roots of:} \\
& x^3 + 2\left(-r_1 - r_2 - r_3\right)x^2 + 4\left(-|c_3|^2 - |c_5|^2 - |c_{17}|^2 + r_1r_2 + r_1r_3 + r_2r_3\right)x \\
& + 8\left(r_3|c_3|^2 + r_2|c_5|^2 + r_1|c_{17}|^2 - 2\text{Re}(c_3c_5^*c_{17}) - r_1r_2r_3\right) = 0\,, \\
\Lambda^{++}_4 &= 2(r_4 + r_7)\,, \\
\Lambda^{++}_5 &= 2(r_5 + r_8)\,, \\
\Lambda^{++}_6 &= 2(r_6 + r_9)\,.
\end{aligned} \tag{5.96}
$$

The matrix M_2^+. From M_2^+ we get

$$\frac{1}{2}\,M_2^+ = \mathrm{diag}\left\{\begin{pmatrix} r_1 & c_3 & c_5 \\ c_3^* & r_2 & c_{17} \\ c_5^* & c_{17}^* & r_3 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix}, \begin{pmatrix} r_5 & r_8 \\ r_8 & r_5 \end{pmatrix}, \begin{pmatrix} r_6 & r_9 \\ r_9 & r_6 \end{pmatrix}\right\}, \tag{5.97}$$

with eigenvalues of M_2^+:

$$\begin{aligned} \Lambda_{1-3}^{+,2} &= \Lambda_{1-3}^{++}\,, \\ \Lambda_{4,5}^{+,2} &= 2(r_4 \pm r_7)\,, \\ \Lambda_{6,7}^{+,2} &= 2(r_5 \pm r_8)\,, \\ \Lambda_{8,9}^{+,2} &= 2(r_6 \pm r_9)\,. \end{aligned} \tag{5.98}$$

The matrix M_0^+. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_8 \\ r_7 & r_2 & r_9 \\ r_8 & r_9 & r_3 \end{pmatrix}, \begin{pmatrix} r_4 & c_3 \\ c_3^* & r_4 \end{pmatrix}, \begin{pmatrix} r_5 & c_5 \\ c_5^* & r_5 \end{pmatrix}, \begin{pmatrix} r_6 & c_{17} \\ c_{17}^* & r_6 \end{pmatrix}\right\}, \tag{5.99}$$

with eigenvalues of M_0^+:

$$\begin{aligned} \Lambda_{1-3}^{+,0} = {} & \text{Roots of:} \\ & x^3 - 2(r_1+r_2+r_3)x^2 + 4(-r_7^2 - r_8^2 - r_9^2 + r_1 r_2 + r_1 r_3 + r_2 r_3)x \\ & + 8\left(r_3 r_7^2 - 2 r_8 r_9 r_7 + r_2 r_8^2 + r_1 r_9^2 - r_1 r_2 r_3\right) = 0\,, \\ \Lambda_{4,5}^{+,0} = {} & 2\left(r_4 \pm \sqrt{c_3}\sqrt{c_3^*}\right)\,, \\ \Lambda_{6,7}^{+,0} = {} & 2\left(r_5 \pm \sqrt{c_5}\sqrt{c_5^*}\right)\,, \\ \Lambda_{8,9}^{+,0} = {} & 2\left(r_6 \pm \sqrt{c_{17}}\sqrt{c_{17}^*}\right)\,. \end{aligned} \tag{5.100}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.101}$$

with eigenvalues of M_2^0:

$$\Lambda_{1-6}^{0,2} = \Lambda_{1-6}^{++}\,. \tag{5.102}$$

The matrix M_0^0. From M_0^0 we get

$$\begin{aligned} \frac{1}{2}\,M_0^0 \sim \mathrm{diag}\Bigg\{ & \frac{1}{2}M_0^+, \begin{pmatrix} 3r_1 & 2r_4+r_7 & 2r_5+r_8 \\ 2r_4+r_7 & 3r_2 & 2r_6+r_9 \\ 2r_5+r_8 & 2r_6+r_9 & 3r_3 \end{pmatrix}, \begin{pmatrix} r_4+2r_7 & 3c_3 \\ 3c_3^* & r_4+2r_7 \end{pmatrix}, \\ & \begin{pmatrix} r_5+2r_8 & 3c_5 \\ 3c_5^* & r_5+2r_8 \end{pmatrix}, \begin{pmatrix} r_6+2r_9 & 3c_{17} \\ 3c_{17}^* & r_6+2r9 \end{pmatrix}\Bigg\}\,. \end{aligned} \tag{5.103}$$

with eigenvalues of M_0^0:

$$\Lambda^{0,0}_{1-9} = \Lambda^{+,0}_{1-9}\,,$$
$$\Lambda^{0,0}_{10-12} = \text{Roots of:}$$
$$x^3 + 2\left(-3r_1 - 3r_2 - 3r_3\right)x^2 + 4\Big(-4r_4^2 - 4r_7r_4 - 4r_5^2 - 4r_6^2 - r_7^2 - r_8^2 - r_9^2 + 9r_1r_2 + 9r_1r_3 + 9r_2r_3 - 4r_5r_8 - 4r_6r_9\Big)x + 8\Big(12r_3r_4^2 + 12r_2r_5^2 + 12r_1r_6^2 + 3r_3r_7^2 + 3r_2r_8^2 + 3r_1r_9^2 - 27r_1r_2r_3 - 16r_4r_5r_6 + 12r_3r_4r_7 - 8r_5r_6r_7 + 12r_2r_5r_8 - 8r_4r_6r_8 - 4r_6r_7r_8 - 8r_4r_5r_9 + 12r_1r_6r_9 - 4r_5r_7r_9 - 4r_4r_8r_9 - 2r_7r_8r_9\Big) = 0\,,$$
$$\Lambda^{0,0}_{13,14} = 2\left(\pm 3\sqrt{c_3}\sqrt{c_3^*} + r_4 + 2r_7\right)\,,$$
$$\Lambda^{0,0}_{15,16} = 2\left(\pm 3\sqrt{c_5}\sqrt{c_5^*} + r_5 + 2r_8\right)\,,$$
$$\Lambda^{0,0}_{17,18} = 2\left(-3\sqrt{c_{17}}\sqrt{c_{17}^*} + r_6 + 2r_9\right)\,. \tag{5.104}$$

5.11 The $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{(\mathrm{CP})}$ symmetry

By imposing $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{(\mathrm{CP})}$, the so called Branco model [32], we get the quartic potential

$$V_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2^{(\mathrm{CP})}} = \sum_{i=1}^{3} r_i|\phi_i|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + 2r_5(\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + 2r_6(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) + 2r_7|\phi_1^\dagger\phi_2|^2 + 2r_8|\phi_1^\dagger\phi_3|^2 + 2r_9|\phi_2^\dagger\phi_3|^2 + r_{10}\left[(\phi_1^\dagger\phi_2)^2 + h.c.\right] + r_{11}\left[(\phi_1^\dagger\phi_3)^2 + h.c.\right] + r_{12}\left[(\phi_2^\dagger\phi_3)^2 + h.c.\right]\,, \tag{5.105}$$

which can be easily achieved by setting from $\mathbb{Z}_2 \times \mathbb{Z}_2$ the constraints $\{c_3, c_5, c_{17}\} \in \mathbb{R}$.

Thus, we get the following scattering matrices which reproduce the conditions (91)–(100) of refs. [27, 30].

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}\,M_2^{++} = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_{10} & r_{11} \\ r_{10} & r_2 & r_{12} \\ r_{11} & r_{12} & r_3 \end{pmatrix}, (r_4+r_7), (r_5+r_8), (r_6+r_9)\right\}\,, \tag{5.106}$$

and thus we get the eigenvalues of M_2^{++}:

$$\Lambda^{++}_{1-3} = \text{Roots of:}$$
$$x^3 + 2\left(-r_1 - r_2 - r_3\right)x^2 + 4\left(-r_{10}^2 - r_{11}^2 - r_{12}^2 + r_1r_2 + r_1r_3 + r_2r_3\right)x + 8\left(r_3r_{10}^2 + r_2r_{11}^2 + r_1r_{12}^2 - 2r_{10}r_{11}r_{12} - r_1r_2r_3\right) = 0\,,$$
$$\Lambda^{++}_4 = 2(r_4 + r_7)\,,$$
$$\Lambda^{++}_5 = 2(r_5 + r_8)\,,$$
$$\Lambda^{++}_6 = 2(r_6 + r_9)\,. \tag{5.107}$$

JHEP08(2022)273

The matrix $\boldsymbol{M_2^+}$. From M_2^+ we get

$$\frac{1}{2}\,M_2^+ = \text{diag}\left\{\begin{pmatrix} r_1 & r_{10} & r_{11} \\ r_{10} & r_2 & r_{12} \\ r_{11} & r_{12} & r_3 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix}, \begin{pmatrix} r_5 & r_8 \\ r_8 & r_5 \end{pmatrix}, \begin{pmatrix} r_6 & r_9 \\ r_9 & r_6 \end{pmatrix}\right\}, \tag{5.108}$$

with eigenvalues of M_2^+:

$$\begin{aligned} \Lambda^{+,2}_{1-3} &= \Lambda^{++}_{1-3}\,, \\ \Lambda^{+,2}_{4,5} &= 2(r_4 \pm r_7)\,, \\ \Lambda^{+,2}_{6,7} &= 2(r_5 \pm r_8)\,, \\ \Lambda^{+,2}_{8,9} &= 2(r_6 \pm r_9)\,. \end{aligned} \tag{5.109}$$

The matrix $\boldsymbol{M_0^+}$. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \text{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_8 \\ r_7 & r_2 & r_9 \\ r_8 & r_9 & r_3 \end{pmatrix}, \begin{pmatrix} r_4 & r_{10} \\ r_{10} & r_4 \end{pmatrix}, \begin{pmatrix} r_5 & r_{11} \\ r_{11} & r_5 \end{pmatrix}, \begin{pmatrix} r_6 & r_{12} \\ r_{12} & r_6 \end{pmatrix}\right\}, \tag{5.110}$$

with eigenvalues of M_0^+:

$$\begin{aligned} \Lambda^{+,0}_{1-3} &= \text{Roots of:} \\ &\qquad x^3 - 2(r_1+r_2+r_3)x^2 + 4\left(-r_7^2 - r_8^2 - r_9^2 + r_1 r_2 + r_1 r_3 + r_2 r_3\right)x \\ &\qquad + 8\left(r_3 r_7^2 - 2 r_8 r_9 r_7 + r_2 r_8^2 + r_1 r_9^2 - r_1 r_2 r_3\right) = 0\,, \\ \Lambda^{+,0}_{4,5} &= 2\left(r_4 \pm r_{10}\right)\,, \\ \Lambda^{+,0}_{6,7} &= 2\left(r_5 \pm r_{11}\right)\,, \\ \Lambda^{+,0}_{8,9} &= 2\left(r_6 \pm r_{12}\right)\,. \end{aligned} \tag{5.111}$$

The matrix $\boldsymbol{M_2^0}$. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.112}$$

with eigenvalues of M_2^0:

$$\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.113}$$

The matrix $\boldsymbol{M_0^0}$. From M_0^0 we get

$$\begin{aligned} \frac{1}{2}\,M_0^0 \sim \text{diag}\Bigg\{ &\frac{1}{2}M_0^+, \begin{pmatrix} 3r_1 & 2r_4+r_7 & 2r_5+r_8 \\ 2r_4+r_7 & 3r_2 & 2r_6+r_9 \\ 2r_5+r_8 & 2r_6+r_9 & 3r_3 \end{pmatrix}, \begin{pmatrix} r_4+2r_7 & 3r_{10} \\ 3r_{10} & r_4+2r_7 \end{pmatrix}, \\ &\begin{pmatrix} r_5+2r_8 & 3r_{11} \\ 3r_{11} & r_5+2r_8 \end{pmatrix}, \begin{pmatrix} r_6+2r_9 & 3r_{12} \\ 3r_{12} & r_6+2r9 \end{pmatrix}\Bigg\}\,. \end{aligned} \tag{5.114}$$

with eigenvalues of M_0^0:

$$
\begin{aligned}
\Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,,\\
\Lambda^{0,0}_{10-12} &= \text{Roots of:}\\
&\quad x^3+2\left(-3r_1-3r_2-3r_3\right)x^2+4\Big(-4r_4^2-4r_7r_4-4r_5^2-4r_6^2-r_7^2-r_8^2-r_9^2\\
&\quad +9r_1r_2+9r_1r_3+9r_2r_3-4r_5r_8-4r_6r_9\Big)x+8\Big(12r_3r_4^2+12r_2r_5^2+12r_1r_6^2\\
&\quad +3r_3r_7^2+3r_2r_8^2+3r_1r_9^2-27r_1r_2r_3-16r_4r_5r_6+12r_3r_4r_7-8r_5r_6r_7\\
&\quad +12r_2r_5r_8-8r_4r_6r_8-4r_6r_7r_8-8r_4r_5r_9+12r_1r_6r_9-4r_5r_7r_9-4r_4r_8r_9\\
&\quad -2r_7r_8r_9\Big)=0\,,\\
\Lambda^{0,0}_{13,14} &= 2\left(r_4+2r_7\pm 3r_{10}\right)\,,\\
\Lambda^{0,0}_{15,16} &= 2\left(r_5+2r_8\pm 3r_{11}\right)\,,\\
\Lambda^{0,0}_{17,18} &= 2\left(r_6+2r_9\pm 3r_{12}\right)\,. \qquad (5.115)
\end{aligned}
$$

5.12 The U(1) × U(1) symmetry

By imposing $G = \text{U}(1)\times\text{U}(1)$ with representation $\text{diag}(1,e^{i\alpha},e^{i\beta})$, with $\alpha \neq \{0,\pi\}$ and $\beta \neq \{0,\pi,\pm\alpha\}$, we get the quartic potential

$$
\begin{aligned}
V_{\text{U}(1)\times\text{U}(1)} =& \sum_{i=1}^{3} r_i|\phi_i|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + 2r_5(\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + 2r_6(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)\\
&+ 2r_7|\phi_1^\dagger\phi_2|^2 + 2r_8|\phi_1^\dagger\phi_3|^2 + 2r_9|\phi_2^\dagger\phi_3|^2\,, \qquad (5.116)
\end{aligned}
$$

which can be easily achieved by setting from $\text{U}(1)_1$ the constraint $c_{11}\to 0$.

The matrix M_2^{++}. From M_2^{++} we get

$$
\frac{1}{2}\,M_2^{++} = \text{diag}\left\{r_1, r_2, r_3, (r_4+r_7), (r_5+r_8), (r_6+r_9)\right\}\,, \qquad (5.117)
$$

and thus we get the eigenvalues of M_2^{++}:

$$
\begin{aligned}
\Lambda_1^{++} &= 2r_1\,,\\
\Lambda_2^{++} &= 2r_2\,,\\
\Lambda_3^{++} &= 2r_3\,,\\
\Lambda_4^{++} &= 2(r_4+r_7)\,,\\
\Lambda_5^{++} &= 2(r_5+r_8)\,,\\
\Lambda_6^{++} &= 2(r_6+r_9)\,. \qquad (5.118)
\end{aligned}
$$

The matrix M_2^{+}. From M_2^{+} we get

$$
\frac{1}{2}\,M_2^{+} = \text{diag}\left\{\begin{pmatrix} r_4 & r_7\\ r_7 & r_4\end{pmatrix}, \begin{pmatrix} r_5 & r_8\\ r_8 & r_5\end{pmatrix}, \begin{pmatrix} r_6 & r_9\\ r_9 & r_6\end{pmatrix}, r_1, r_2, r_3\right\}\,, \qquad (5.119)
$$

with eigenvalues of M_2^+:

$$
\begin{aligned}
\Lambda_{1,2}^{+,2} &= 2(r_4 \pm r_7)\,, \\
\Lambda_{3,4}^{+,2} &= 2(r_5 \pm r_8)\,, \\
\Lambda_{5,6}^{+,2} &= 2(r_6 \pm r_9)\,, \\
\Lambda_{7}^{+,2} &= \Lambda_1^{++}\,, \\
\Lambda_{8}^{+,2} &= \Lambda_2^{++}\,, \\
\Lambda_{9}^{+,2} &= \Lambda_3^{++}\,.
\end{aligned} \tag{5.120}
$$

The matrix M_0^+. From M_0^+ we get

$$
\frac{1}{2}\,M_0^+ = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_8 \\ r_7 & r_2 & r_9 \\ r_8 & r_9 & r_3 \end{pmatrix}, r_4, r_4, r_5, r_5, r_6, r_6\right\}, \tag{5.121}
$$

with eigenvalues of M_0^+:

$$
\begin{aligned}
\Lambda_{1-3}^{+,0} = \text{Roots of:}& \\
& x^3 - 2(r_1+r_2+r_3)x^2 + 4\left(-r_7^2 - r_8^2 - r_9^2 + r_1r_2 + r_1r_3 + r_2r_3\right)x \\
& + 8\left(r_3r_7^2 - 2r_8r_9r_7 + r_2r_8^2 + r_1r_9^2 - r_1r_2r_3\right) = 0\,, \\
\Lambda_{4,5}^{+,0} = 2r_4\,,& \\
\Lambda_{6,7}^{+,0} = 2r_5\,,& \\
\Lambda_{8,9}^{+,0} = 2r_6\,.&
\end{aligned} \tag{5.122}
$$

The matrix M_2^0. From M_2^0 we get

$$
\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.123}
$$

with eigenvalues of M_2^0:

$$
\Lambda_{1-6}^{0,2} = \Lambda_{1-6}^{++}\,. \tag{5.124}
$$

The matrix M_0^0. From M_0^0 we get

$$
\begin{aligned}
\frac{1}{2}\,M_0^0 \sim \mathrm{diag}\Bigg\{&\frac{1}{2}M_0^+, \begin{pmatrix} 3r_1 & 2r_4+r_7 & 2r_5+r_8 \\ 2r_4+r_7 & 3r_2 & 2r_6+r_9 \\ 2r_5+r_8 & 2r_6+r_9 & 3r_3 \end{pmatrix}, (r_4+2r_7), (r_4+2r_7), \\
&(r_5+2r_8), (r_5+2r_8), (r_6+2r_9), (r_6+2r_9)\Bigg\}\,.
\end{aligned} \tag{5.125}
$$

with eigenvalues of M_0^0:

$$
\begin{aligned}
\Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,, \\
\Lambda^{0,0}_{10-12} &= \text{Roots of:} \\
&\quad x^3 + 2\left(-3r_1 - 3r_2 - 3r_3\right)x^2 + 4\Big(-4r_4^2 - 4r_7r_4 - 4r_5^2 - 4r_6^2 - r_7^2 - r_8^2 - r_9^2 \\
&\quad + 9r_1r_2 + 9r_1r_3 + 9r_2r_3 - 4r_5r_8 - 4r_6r_9\Big)x + 8\Big(12r_3r_4^2 + 12r_2r_5^2 + 12r_1r_6^2 \\
&\quad + 3r_3r_7^2 + 3r_2r_8^2 + 3r_1r_9^2 - 27r_1r_2r_3 - 16r_4r_5r_6 + 12r_3r_4r_7 - 8r_5r_6r_7 \\
&\quad + 12r_2r_5r_8 - 8r_4r_6r_8 - 4r_6r_7r_8 - 8r_4r_5r_9 + 12r_1r_6r_9 - 4r_5r_7r_9 - 4r_4r_8r_9 \\
&\quad - 2r_7r_8r_9\Big) = 0\,, \\
\Lambda^{0,0}_{13,14} &= 2(r_4 + 2r_7)\,, \\
\Lambda^{0,0}_{15,16} &= 2(r_5 + 2r_8)\,, \\
\Lambda^{0,0}_{17,18} &= 2(r_6 + 2r_9)\,.
\end{aligned} \tag{5.126}
$$

5.13 The U(2) symmetry

By imposing $G = \mathrm{U}(2)$ we get the quartic potential

$$
\begin{aligned}
V_{\mathrm{U}(2)} =& r_1\left[(\phi_1^\dagger\phi_1) + (\phi_2^\dagger\phi_2)\right]^2 + r_3|\phi_3|^4 + 2r_5(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) \\
&+ 2r_7\left[|\phi_1^\dagger\phi_2|^2 - (\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)\right] + 2r_8\left[|\phi_1^\dagger\phi_3|^2 + |\phi_2^\dagger\phi_3|^2\right]\,,
\end{aligned} \tag{5.127}
$$

with the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$
\frac{1}{2}\,M_2^{++} = \operatorname{diag}\left\{r_1, r_1, r_1, r_3, (r_5 + r_8), (r_5 + r_8)\right\}\,, \tag{5.128}
$$

and thus we get the eigenvalues of M_2^{++}:

$$
\begin{aligned}
\Lambda^{++}_{1,2,3} &= 2r_1\,, \\
\Lambda^{++}_{4} &= 2r_3\,, \\
\Lambda^{++}_{5,6} &= 2(r_5 + r_8)\,.
\end{aligned} \tag{5.129}
$$

The matrix M_2^{+}. From M_2^{+} we get

$$
\frac{1}{2}\,M_2^{+} = \operatorname{diag}\left\{\begin{pmatrix} r_1 - r_7 & r_7 \\ r_7 & r_1 - r_7 \end{pmatrix}, \begin{pmatrix} r_5 & r_8 \\ r_8 & r_5 \end{pmatrix}, \begin{pmatrix} r_5 & r_8 \\ r_8 & r_5 \end{pmatrix}, r_1, r_1, r_3\right\}\,, \tag{5.130}
$$

with eigenvalues of M_2^{+}:

$$
\begin{aligned}
\Lambda^{+,2}_{1} &= 2(r_1 - 2r_7)\,, \\
\Lambda^{+,2}_{2-5} &= 2(r_5 \pm r_8)\,, \\
\Lambda^{+,2}_{6-8} &= 2r_1\,, \\
\Lambda^{+,2}_{9} &= 2r_3\,.
\end{aligned} \tag{5.131}
$$

JHEP08(2022)273

The matrix M_0^+. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \text{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_8 \\ r_7 & r_1 & r_8 \\ r_8 & r_8 & r_3 \end{pmatrix}, (r_1-r_7), (r_1-r_7), r_5, r_5, r_5, r_5\right\}\,, \tag{5.132}$$

with eigenvalues of M_0^+:

$$\begin{aligned}
&\Lambda^{+,0}_{1,2} = \pm\sqrt{r_1^2 - 2\,(r_3-r_7)\,r_1 + r_3^2 + r_7^2 + 8r_8^2 - 2r_3r_7} + r_1 + r_3 + r_7\,,\\
&\Lambda^{+,0}_{3-5} = 2(r_1-r_7)\,,\\
&\Lambda^{+,0}_{6-9} = 2r_5\,.
\end{aligned} \tag{5.133}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.134}$$

with eigenvalues of M_2^0:

$$\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.135}$$

The matrix M_0^0. From M_0^0 we get

$$\begin{aligned}
\frac{1}{2}\,M_0^0 \sim \text{diag}&\left\{\frac{1}{2}M_0^+, \begin{pmatrix} 3r_1 & 2r_1-r_7 & 2r_5+r_8 \\ 2r_1-r_7 & 3r_1 & 2r_5+r_8 \\ 2r_5+r_8 & 2r_5+r_8 & 3r_3 \end{pmatrix}, (r_1+r_7), (r_1+r_7),\right.\\
&\left.\quad (r_5+2r_8), (r_5+2r_8), (r_5+2r_8), (r_5+2r_8)\right\}.
\end{aligned} \tag{5.136}$$

with eigenvalues of M_0^0:

$$\begin{aligned}
&\Lambda^{0,0}_{1-9} = \Lambda^{+,0}_{1-9}\,,\\
&\Lambda^{0,0}_{10,11} = \pm\sqrt{(-5r_1-3r_3+r_7)^2 + 4\left(8r_5^2 + 8r_8r_5 + 2r_8^2 - 15r_1r_3 + 3r_3r_7\right)} + 5r_1 + 3r_3 - r_7,\\
&\Lambda^{0,0}_{12-14} = 2(r_1+r_7)\,,\\
&\Lambda^{0,0}_{15-18} = 2(r_5+2r_8)\,.
\end{aligned} \tag{5.137}$$

5.14 The O(2) symmetry

By imposing $G = \text{O}(2)$ we get the quartic potential

$$\begin{aligned}
V_{\text{O}(2)} =& r_1\left[(\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2\right] + r_3|\phi_3|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)\\
&+ 2r_5(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) + 2r_7|\phi_1^\dagger\phi_2|^2\\
&+ 2r_8\left[|\phi_1^\dagger\phi_3|^2 + |\phi_2^\dagger\phi_3|^2\right] + 2r_{10}\left[(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3) + h.c.\right]\,,
\end{aligned} \tag{5.138}$$

with the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}\,M_2^{++} = \text{diag}\left\{\begin{pmatrix} r_4+r_7 & \sqrt{2}r_{10} \\ \sqrt{2}r_{10} & r_3 \end{pmatrix}, (r_5+r_8), (r_5+r_8), r_1, r_1\right\}\,, \tag{5.139}$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned} \Lambda_{1,2}^{++} &= \pm\sqrt{8r_{10}^2 + (-r_3+r_4+r_7)^{\,2}} + r_3 + r_4 + r_7\,, \\ \Lambda_{3,4}^{++} &= 2(r_5+r_8)\,, \\ \Lambda_{5,6}^{++} &= 2r_1\,. \end{aligned} \tag{5.140}$$

The matrix M_2^{+}. From M_2^{+} we get

$$\frac{1}{2}\,M_2^{+} = \text{diag}\left\{\begin{pmatrix} r_4 & r_7 & r_{10} \\ r_7 & r_4 & r_{10} \\ r_{10} & r_{10} & r_3 \end{pmatrix}, \begin{pmatrix} r_5 & r_8 \\ r_8 & r_5 \end{pmatrix}, \begin{pmatrix} r_5 & r_8 \\ r_8 & r_5 \end{pmatrix}, r_1, r_1\right\}\,, \tag{5.141}$$

with eigenvalues of M_2^{+}:

$$\begin{aligned} \Lambda_{1,2}^{+,2} &= \pm\sqrt{8r_{10}^2 + (-r_3+r_4+r_7)^{\,2}} + r_3 + r_4 + r_7\,, \\ \Lambda_{3}^{+,2} &= 2(r_4-r_7)\,, \\ \Lambda_{4-7}^{+,2} &= 2(r_5 \pm r_8)\,, \\ \Lambda_{8,9}^{+,2} &= 2r_1\,. \end{aligned} \tag{5.142}$$

The matrix M_0^{+}. From M_0^{+} we get

$$\frac{1}{2}\,M_0^{+} = \text{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_8 \\ r_7 & r_1 & r_8 \\ r_8 & r_8 & r_3 \end{pmatrix}, \begin{pmatrix} r_5 & r_{10} \\ r_{10} & r_5 \end{pmatrix}, \begin{pmatrix} r_5 & r_{10} \\ r_{10} & r_5 \end{pmatrix}, r_4, r_4\right\}\,, \tag{5.143}$$

with eigenvalues of M_0^{+}:

$$\begin{aligned} \Lambda_{1,2}^{+,0} &= \pm\sqrt{(r_1-r_3+r_7)^{\,2} + 8r_8^2} + r_1 + r_3 + r_7\,, \\ \Lambda_{3}^{+,0} &= 2(r_1-r_7)\,, \\ \Lambda_{4-7}^{+,0} &= 2\left(r_5 \pm r_{10}\right)\,, \\ \Lambda_{8,9}^{+,0} &= 2r_4\,. \end{aligned} \tag{5.144}$$

The matrix M_2^{0}. From M_2^{0} we get

$$\frac{1}{2}\,M_2^{0} = \frac{1}{2}\,M_2^{++}\,, \tag{5.145}$$

with eigenvalues of M_2^{0}:

$$\Lambda_{1-6}^{0,2} = \Lambda_{1-6}^{++}\,. \tag{5.146}$$

The matrix M_0^0. From M_0^0 we get

$$\frac{1}{2}\,M_0^0 \sim \mathrm{diag}\left\{\frac{1}{2}M_0^+,\begin{pmatrix}3r_1 & 2r_4+r_7 & 2r_5+r_8\\ r_4+r_7 & 3r_1 & 2r_5+r_8\\ 2r_5+r_8 & 2r_5+r_8 & 3r_3\end{pmatrix},\begin{pmatrix}r_5+2r_8 & 3r_{10}\\ 3r_{10} & r_5+2r_8\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}r_5+2r_8 & 3r_{10}\\ 3r_{10} & r_5+2r_8\end{pmatrix},(r_4+2r_7),(r_4+2r_7)\right\}. \tag{5.147}$$

with eigenvalues of M_0^0:

$$\begin{aligned}\Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,,\\ \Lambda^{0,0}_{10,11} &= \pm\sqrt{8\,(2r_5+r_8)^{\,2}+(3r_1-3r_3+2r_4+r_7)^{\,2}}\\ &\quad +3r_1+3r_3+2r_4+r_7\,,\\ \Lambda^{0,0}_{12} &= 6r_1-2\,(2r_4+r_7)\,,\\ \Lambda^{0,0}_{13-16} &= 2\,(r_5+2r_8\pm 3r_{10})\,,\\ \Lambda^{0,0}_{17,18} &= 2(r_4+2r_7)\,.\end{aligned} \tag{5.148}$$

5.15 The D_4 symmetry

By imposing $G = D_4$ we get the quartic potential

$$\begin{aligned}V_{D_4} =& r_1\left[(\phi_1^\dagger\phi_1)^2+(\phi_2^\dagger\phi_2)^2\right]+r_3|\phi_3|^4+2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)\\ &+2r_5(\phi_1^\dagger\phi_1+\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)+2r_7|\phi_1^\dagger\phi_2|^2\\ &+2r_8\left[|\phi_1^\dagger\phi_3|^2+|\phi_2^\dagger\phi_3|^2\right]+r_{10}\left[(\phi_1^\dagger\phi_2)^2+h.c.\right]\\ &+2r_{11}\left[(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3)+h.c.\right]\,,\end{aligned} \tag{5.149}$$

which can be easily achieved by setting from $\mathbb{Z}_4$ the constraints $r_2 \to r_1, r_6 \to r_5, r_9 \to r_8$.

Thus, we get the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}\,M_2^{++} = \mathrm{diag}\left\{\begin{pmatrix}r_4+r_7 & \sqrt{2}r_{11}\\ \sqrt{2}r_{11} & r_3\end{pmatrix},\begin{pmatrix}r_1 & r_{10}\\ r_{10} & r_1\end{pmatrix},(r_5+r_8),(r_5+r_8)\right\}, \tag{5.150}$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned}\Lambda^{++}_{1,2} &= \pm\sqrt{8r_{11}^2+(-r_3+r_4+r_7)^{\,2}}+r_3+r_4+r_7\,,\\ \Lambda^{++}_{3,4} &= 2\,(r_1\pm r_{10})\,,\\ \Lambda^{++}_{5,6} &= 2(r_5+r_8)\,.\end{aligned} \tag{5.151}$$

The matrix M_2^+. From M_2^+ we get

$$\frac{1}{2}\,M_2^{+} = \mathrm{diag}\left\{\begin{pmatrix}r_4 & r_7 & r_{11}\\ r_7 & r_4 & r_{11}\\ r_{11} & r_{11} & r_3\end{pmatrix},\begin{pmatrix}r_1 & r_{10}\\ r_{10} & r_1\end{pmatrix},\begin{pmatrix}r_5 & r_8\\ r_8 & r_5\end{pmatrix},\begin{pmatrix}r_5 & r_8\\ r_8 & r_5\end{pmatrix}\right\}, \tag{5.152}$$

JHEP08(2022)273

with eigenvalues of M_2^+:

$$
\begin{aligned}
\Lambda^{+,2}_{1,2} &= \Lambda^{++}_{1,2}\,, \\
\Lambda^{+,2}_{3} &= 2(r_4 - r_7)\,, \\
\Lambda^{+,2}_{4,5} &= \Lambda^{++}_{3,4}\,, \\
\Lambda^{+,2}_{6-9} &= 2(r_5 \pm r_8)\,.
\end{aligned} \tag{5.153}
$$

The matrix $\boldsymbol{M_0^+}$. From M_0^+ we get

$$
\frac{1}{2}\,M_0^+ = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_8 \\ r_7 & r_1 & r_8 \\ r_8 & r_8 & r_3 \end{pmatrix}, \begin{pmatrix} r_4 & r_{10} \\ r_{10} & r_4 \end{pmatrix}, \begin{pmatrix} r_5 & r_{11} \\ r_{11} & r_5 \end{pmatrix}, \begin{pmatrix} r_5 & r_{11} \\ r_{11} & r_5 \end{pmatrix}\right\}\,, \tag{5.154}
$$

with eigenvalues of M_0^+:

$$
\begin{aligned}
\Lambda^{+,0}_{1,2} &= \pm\sqrt{8r_8^2 + (r_1 - r_3 + r_7)^{\,2}} + r_1 + r_3 + r_7\,, \\
\Lambda^{+,0}_{3} &= 2(r_1 - r_7)\,, \\
\Lambda^{+,0}_{4,5} &= 2\left(r_4 \pm r_{10}\right)\,, \\
\Lambda^{+,0}_{6-9} &= 2\left(r_5 \pm r_{11}\right)\,.
\end{aligned} \tag{5.155}
$$

The matrix $\boldsymbol{M_2^0}$. From M_2^0 we get

$$
\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.156}
$$

with eigenvalues of M_2^0:

$$
\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.157}
$$

The matrix $\boldsymbol{M_0^0}$. From M_0^0 we get

$$
\begin{aligned}
\frac{1}{2}\,M_0^0 \sim \mathrm{diag}\Bigg\{&\frac{1}{2}M_0^+, \begin{pmatrix} 3r_1 & 2r_4 + r_7 & 2r_5 + r_8 \\ 2r_4 + r_7 & 3r_1 & 2r_5 + r_8 \\ 2r_5 + r_8 & 2r_5 + r_8 & 3r_3 \end{pmatrix}, \begin{pmatrix} r_4 + 2r_7 & 3r_{10} \\ 3r_{10} & r_4 + 2r_7 \end{pmatrix}, \\
&\begin{pmatrix} r_5 + 2r_8 & 3r_{11} \\ 3r_{11} & r_5 + 2r_8 \end{pmatrix}, \begin{pmatrix} r_5 + 2r_8 & 3r_{11} \\ 3r_{11} & r_5 + 2r_8 \end{pmatrix}\Bigg\}\,,
\end{aligned} \tag{5.158}
$$

with eigenvalues of M_0^0:

$$
\begin{aligned}
\Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,, \\
\Lambda^{0,0}_{10,11} &= \pm\sqrt{(3r_1 - 3r_3 + 2r_4 + r_7)^{\,2} + 8\,(2r_5 + r_8)^{\,2}} + 3r_1 + 3r_3 + 2r_4 + r_7,\,, \\
\Lambda^{0,0}_{12} &= 6r_1 - 2\left(2r_4 + r_7\right)\,, \\
\Lambda^{0,0}_{13,14} &= 2\left(\pm 3r_{10} + r_4 + 2r_7\right)\,, \\
\Lambda^{0,0}_{15-18} &= 2\left(\pm 3r_{11} + r_5 + 2r_8\right)\,.
\end{aligned} \tag{5.159}
$$

5.16 The S_3 symmetry

By imposing $G = S_3$ we get the quartic potential

$$\begin{aligned} V_{S_3} =& r_1 \left[(\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2\right] + r_3|\phi_3|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) \\ &+ 2r_5(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) + 2r_7|\phi_1^\dagger\phi_2|^2 + 2r_8\left[|\phi_1^\dagger\phi_3|^2 + |\phi_2^\dagger\phi_3|^2\right] \\ &+ \left[2c_{11}(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3) + 2c_{12}\left[(\phi_1^\dagger\phi_2)(\phi_3^\dagger\phi_2) + (\phi_2^\dagger\phi_1)(\phi_3^\dagger\phi_1)\right] + h.c.\right], \end{aligned} \tag{5.160}$$

which can be easily achieved by setting from $\mathbb{Z}_3$ the constraints $r_2 \to r_1, r_6 \to r_5, r_9 \to r_8, c_4 \to c_{12}^*$.

Thus, we get the following scattering matrices which reproduce in the limit of real coefficients the conditions (37a)–(37l) of ref. [25].

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}M_2^{++} = \text{diag}\left\{\begin{pmatrix} r_1 & \sqrt{2}c_{12} \\ \sqrt{2}c_{12}^* & r_5+r_8\end{pmatrix}, \begin{pmatrix} r_5+r_8 & \sqrt{2}c_{12} \\ \sqrt{2}c_{12}^* & r_1\end{pmatrix}, \begin{pmatrix} r_4+r_7 & \sqrt{2}c_{11} \\ \sqrt{2}c_{11}^* & r_3\end{pmatrix}\right\}, \tag{5.161}$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned} \Lambda_{1,2}^{++} &= \pm\sqrt{8|c_{12}|^2 + (-r_1+r_5+r_8)^2} + r_1 + r_5 + r_8, \\ \Lambda_{3,4}^{++} &= \Lambda_{1,2}^{++}, \\ \Lambda_{5,6}^{++} &= \pm\sqrt{8|c_{11}|^2 + (-r_3+r_4+r_7)^2} + r_3 + r_4 + r_7. \end{aligned} \tag{5.162}$$

The matrix M_2^{+}. From M_2^{+} we get

$$\frac{1}{2}M_2^{+} = \text{diag}\left\{\begin{pmatrix} r_1 & c_{12}^* & c_{12}^* \\ c_{12} & r_5 & r_8 \\ c_{12} & r_8 & r_5\end{pmatrix}, \begin{pmatrix} r_5 & c_{12} & r_8 \\ c_{12}^* & r_1 & c_{12}^* \\ r_8 & c_{12} & r_5\end{pmatrix}, \begin{pmatrix} r_4 & r_7 & c_{11} \\ r_7 & r_4 & c_{11} \\ c_{11}^* & c_{11}^* & r_3\end{pmatrix}\right\}, \tag{5.163}$$

with eigenvalues of M_2^{+}:

$$\begin{aligned} \Lambda_1^{+,2} &= 2(r_5 - r_8), \\ \Lambda_{2,3}^{+,2} &= \Lambda_{1,2}^{++}, \\ \Lambda_{4-6}^{+,2} &= \Lambda_{1-3}^{+,2}, \\ \Lambda_7^{+,2} &= 2(r_4 - r_7) \\ \Lambda_{8,9}^{+,2} &= \Lambda_{5,6}^{++}. \end{aligned} \tag{5.164}$$

The matrix M_0^{+}. From M_0^{+} we get

$$\frac{1}{2}M_0^{+} = \text{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_8 \\ r_7 & r_1 & r_8 \\ r_8 & r_8 & r_3\end{pmatrix}, \begin{pmatrix} r_4 & c_{12} & c_{12}^* \\ c_{12}^* & r_5 & c_{11} \\ c_{12} & c_{11}^* & r_5\end{pmatrix}, \begin{pmatrix} r_5 & c_{12}^* & c_{11} \\ c_{12} & r_4 & c_{12}^* \\ c_{11}^* & c_{12} & r_5\end{pmatrix}\right\}, \tag{5.165}$$

with eigenvalues of M_0^+:

$$\begin{aligned}
\Lambda_1^{+,0} &= 2\left(r_1 - r_7\right)\,, \\
\Lambda_{2,3}^{+,0} &= \pm\sqrt{\left(r_1 - r_3 + r_7\right)^2 + 8r_8^2} + r_1 + r_3 + r_7\,, \\
\Lambda_{4-6}^{+,0} &= \text{Roots of:} \\
&\quad x^3 - 2(r_4 + 2r_5)x^2 + 4(-2|c_{12}|^2 - |c_{11}|^2 + 2r_4r_5 + r_5^2)x \\
&\quad + 8\left(2r_5|c_{12}|^2 + r_4|c_{11}|^2 - 2\Re(c_{11}c_{12}^2) - r_4r_5^2\right) = 0\,, \\
\Lambda_{7-9}^{+,0} &= \Lambda_{4-6}^{+,0}\,.
\end{aligned} \tag{5.166}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.167}$$

with eigenvalues of M_2^0:

$$\Lambda_{1-6}^{0,2} = \Lambda_{1-6}^{++}\,. \tag{5.168}$$

The matrix M_0^0. From M_0^0 we get

$$\begin{aligned}
\frac{1}{2}\,M_0^0 \sim \text{diag}\Bigg\{ &\frac{1}{2}M_0^+, \begin{pmatrix} 3r_1 & 2r_4 + r_7 & 2r_5 + r_8 \\ 2r_4 + r_7 & 3r_1 & 2r_5 + r_8 \\ 2r_5 + r_8 & 2r_5 + r_8 & 3r_3 \end{pmatrix}, \begin{pmatrix} r_4 + 2r_7 & 3c_{12} & 3c_{12}^* \\ 3c_{12}^* & r_5 + 2r_8 & 3c_{11} \\ 3c_{12} & 3c_{11}^* & r_5 + 2r_8 \end{pmatrix}, \\
&\begin{pmatrix} r_5 + 2r_8 & 3c_{12}^* & 3c_{11} \\ 3c_{12} & r_4 + 2r_7 & 3c_{12}^* \\ 3c_{11}^* & 3c_{12} & r_5 + 2r_8 \end{pmatrix} \Bigg\}\,,
\end{aligned} \tag{5.169}$$

with eigenvalues of M_0^0:

$$\begin{aligned}
\Lambda_{1-9}^{0,0} &= \Lambda_{1-9}^{+,0}\,, \\
\Lambda_{10}^{0,0} &= 6r_1 - 2\left(2r_4 + r_7\right)\,, \\
\Lambda_{11,12}^{0,0} &= \pm\sqrt{\left(3r_1 - 3r_3 + 2r_4 + r_7\right)^2 + 8\left(2r_5 + r_8\right)^2} + 3r_1 + 3r_3 + 2r_4 + r_7\,, \\
\Lambda_{13-15}^{0,0} &= \text{Roots of:} \\
&\quad x^3 + 2\left(-r_4 - 2r_5 - 2r_7 - 4r_8\right)x^2 + 4\left(-18|c_{12}|^2 - 9|c_{11}|^2\right. \\
&\quad \left. + r_5^2 + 2r_4r_5 + 4r_7r_5 + 4r_8r_5 + 4r_8^2 + 4r_4r_8 + 8r_7r_8\right)x \\
&\quad + 8\left[\left(r_4 + 2r_7\right)\left(9|c_{11}|^2 - \left(r_5 + 2r_8\right)^2\right)\right. \\
&\quad \left. - 54\Re(c_{11}c_{12}^2) + \left(r_5 + 2r_8\right)|c_{12}|^2\right] = 0\,, \\
\Lambda_{16-18}^{0,0} &= \Lambda_{13-15}^{0,0}\,.
\end{aligned} \tag{5.170}$$

5.17 The $S_3 \times \mathbb{Z}_2^{(\mathrm{CP})}$ symmetry

By imposing $G = S_3 \times \mathbb{Z}_2^{(\mathrm{CP})}$ we get the quartic potential

$$\begin{aligned} V_{S_3\times\mathbb{Z}_2^{(\mathrm{CP})}} =& r_1 \left[(\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2\right] + r_3|\phi_3|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) \\ &+ 2r_5(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) + 2r_7|\phi_1^\dagger\phi_2|^2 + 2r_8\left[|\phi_1^\dagger\phi_3|^2 + |\phi_2^\dagger\phi_3|^2\right] \\ &+ 2r_{10}\left[(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3) + h.c.\right] + 2r_{11}\left[(\phi_1^\dagger\phi_2)(\phi_3^\dagger\phi_2) + (\phi_2^\dagger\phi_1)(\phi_3^\dagger\phi_1) + h.c.\right] , \end{aligned} \tag{5.171}$$

which can be easily achieved by setting from S_3 the constraints $\{c_{11}, c_{12}^*\} \in \mathbb{R}$.

Thus, we get the following scattering matrices which reproduce the conditions (37a)–(37l) of ref. [25].

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}M_2^{++} = \mathrm{diag}\left\{\begin{pmatrix} r_1 & \sqrt{2}r_{11} \\ \sqrt{2}r_{11} & r_5+r_8\end{pmatrix}, \begin{pmatrix} r_5+r_8 & \sqrt{2}r_{11} \\ \sqrt{2}r_{11} & r_1\end{pmatrix}, \begin{pmatrix} r_4+r_7 & \sqrt{2}r_{10} \\ \sqrt{2}r_{10} & r_3\end{pmatrix}\right\} , \tag{5.172}$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned} \Lambda_{1,2}^{++} &= \pm\sqrt{8r_{11}^2 + (-r_1+r_5+r_8)^2} + r_1 + r_5 + r_8 , \\ \Lambda_{3,4}^{++} &= \Lambda_{1,2}^{++} , \\ \Lambda_{5,6}^{++} &= \pm\sqrt{8r_{10}^2 + (-r_3+r_4+r_7)^2} + r_3 + r_4 + r_7 . \end{aligned} \tag{5.173}$$

The matrix M_2^{+}. From M_2^{+} we get

$$\frac{1}{2}M_2^{+} = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_{11} & r_{11} \\ r_{11} & r_5 & r_8 \\ r_{11} & r_8 & r_5\end{pmatrix}, \begin{pmatrix} r_5 & r_{11} & r_8 \\ r_{11} & r_1 & r_{11} \\ r_8 & r_{11} & r_5\end{pmatrix}, \begin{pmatrix} r_4 & r_7 & r_{10} \\ r_7 & r_4 & r_{10} \\ r_{10} & r_{10} & r_3\end{pmatrix}\right\} , \tag{5.174}$$

with eigenvalues of M_2^{+}:

$$\begin{aligned} \Lambda_1^{+,2} &= 2(r_5 - r_8) , \\ \Lambda_{2,3}^{+,2} &= \Lambda_{1,2}^{++} , \\ \Lambda_{4-6}^{+,2} &= \Lambda_{1-3}^{+,2} , \\ \Lambda_7^{+,2} &= 2(r_4 - r_7) \\ \Lambda_{8,9}^{+,2} &= \Lambda_{5,6}^{++} . \end{aligned} \tag{5.175}$$

The matrix M_0^{+}. From M_0^{+} we get

$$\frac{1}{2}M_0^{+} = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_8 \\ r_7 & r_1 & r_8 \\ r_8 & r_8 & r_3\end{pmatrix}, \begin{pmatrix} r_4 & r_{11} & r_{11} \\ r_{11} & r_5 & r_{10} \\ r_{11} & r_{10} & r_5\end{pmatrix}, \begin{pmatrix} r_5 & r_{11} & r_{10} \\ r_{11} & r_4 & r_{11} \\ r_{10} & r_{11} & r_5\end{pmatrix}\right\} , \tag{5.176}$$

with eigenvalues of M_0^+:

$$\begin{aligned}
\Lambda_1^{+,0} &= 2\,(r_1 - r_7)\ , \\
\Lambda_{2,3}^{+,0} &= \pm\sqrt{(r_1 - r_3 + r_7)^{\,2} + 8r_8^2} + r_1 + r_3 + r_7\,, \\
\Lambda_{4-6}^{+,0} &= \pm\sqrt{(-r_4 + r_5 + r_{10})^{\,2} + 8r_{11}^2} + r_4 + r_5 + r_{10}\,, \\
\Lambda_{7-9}^{+,0} &= \Lambda_{4-6}^{+,0}\,.
\end{aligned} \tag{5.177}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.178}$$

with eigenvalues of M_2^0:

$$\Lambda_{1-6}^{0,2} = \Lambda_{1-6}^{++}\,. \tag{5.179}$$

The matrix M_0^0. From M_0^0 we get

$$\begin{aligned}
\frac{1}{2}\,M_0^0 \sim \mathrm{diag}\Bigg\{&\frac{1}{2}M_0^+,\begin{pmatrix} 3r_1 & 2r_4+r_7 & 2r_5+r_8 \\ 2r_4+r_7 & 3r_1 & 2r_5+r_8 \\ 2r_5+r_8 & 2r_5+r_8 & 3r_3 \end{pmatrix},\begin{pmatrix} r_4+2r_7 & 3r_{11} & 3r_{11} \\ 3r_{11} & r_5+2r_8 & 3r_{10} \\ 3r_{11} & 3r_{10} & r_5+2r_8 \end{pmatrix}, \\
&\begin{pmatrix} r_5+2r_8 & 3r_{11} & 3r_{10} \\ 3r_{11} & r_4+2r_7 & 3r_{11} \\ 3r_{10} & 3r_{11} & r_5+2r_8 \end{pmatrix}\Bigg\}\,,
\end{aligned} \tag{5.180}$$

with eigenvalues of M_0^0:

$$\begin{aligned}
\Lambda_{1-9}^{0,0} &= \Lambda_{1-9}^{+,0}\,, \\
\Lambda_{10}^{0,0} &= 6r_1 - 2\,(2r_4 + r_7)\ , \\
\Lambda_{11,12}^{0,0} &= \pm\sqrt{(3r_1 - 3r_3 + 2r_4 + r_7)^{\,2} + 8\,(2r_5 + r_8)^{\,2}} + 3r_1 + 3r_3 + 2r_4 + r_7\,, \\
\Lambda_{13,14}^{0,0} &= \pm\sqrt{(-r_4 + r_5 - 2r_7 + 2r_8 + 3r_{10})^{\,2} + 72r_{11}^2} + r_4 + r_5 + 2r_7 + 2r_8 + 3r_{10}\,, \\
\Lambda_{15}^{0,0} &= 2\,(r_5 + 2r_8 - 3r_{10})\ , \\
\Lambda_{16-18}^{0,0} &= \Lambda_{13-15}^{0,0}\,.
\end{aligned} \tag{5.181}$$

5.18 The $CP4$ symmetry

By imposing $G = CP4$ we get the quartic potential

$$\begin{aligned}
V_{CP4} =& r_1(\phi_1^\dagger\phi_1)^2 + r_2\left[(\phi_2^\dagger\phi_2)^2 + (\phi_3^\dagger\phi_3)^2\right] + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3) \\
&+ 2r_6(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) + 2r_7\left[|\phi_1^\dagger\phi_2|^2 + |\phi_1^\dagger\phi_3|^2\right] + 2r_9|\phi_2^\dagger\phi_3|^2 \\
&+ 2r_{10}\left[(\phi_2^\dagger\phi_1)(\phi_3^\dagger\phi_1) + h.c.\right] + r_{11}\left[(\phi_1^\dagger\phi_2)^2 - (\phi_1^\dagger\phi_3)^2 + h.c.\right] \\
&+ \left[c_{17}(\phi_2^\dagger\phi_3)^2 + 2c_{16}(\phi_2^\dagger\phi_3)(\phi_2^\dagger\phi_2 - \phi_3^\dagger\phi_3) + h.c.\right].
\end{aligned} \tag{5.182}$$

Thus, we get the following scattering matrices which reproduce the conditions (4.24)–(4.32) of ref. [26].

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}\,M_2^{++} = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_{11} & \sqrt{2}r_{10} & -r_{11} \\ r_{11} & r_2 & \sqrt{2}c_{16} & c_{17} \\ \sqrt{2}r_{10} & \sqrt{2}c_{16}^* & r_6+r_9 & -\sqrt{2}c_{16} \\ -r_{11} & c_{17}^* & -\sqrt{2}c_{16}^* & r_2 \end{pmatrix}, (r_4+r_7), (r_4+r_7)\right\}, \tag{5.183}$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned}
\Lambda_{1-4}^{++} = \text{Roots of:}& \\
& x^4 + 2\left(-r_1 - 2r_2 - r_6 - r_9\right)x^3 + 4\Big[-4\left|c_{16}\right|^2 - \left|c_{17}\right|^2 + r_1\left(2r_2 + r_6 + r_9\right) \\
& + r_2\left(r_2 + 2\left(r_6 + r_9\right)\right) - 2\left(r_{10}^2 + r_{11}^2\right)\Big]x^2 + 8\Big[4\left(r_1 + r_2\right)\left|c_{16}\right|^2 \\
& + \left(r_1 + r_6 + r_9\right)\left|c_{17}\right|^2 + 4\Re\left(c_{16}^2 c_{17}^*\right) + 2r_{11}^2\left(\Re\left(c_{17}\right) + r_2 + r_6 + r_9\right) \\
& - 8r_{10}r_{11}\Re\left(c_{16}\right) - r_2\left(-4r_{10}^2 + r_2\left(r_6 + r_9\right) + r_1\left(r_2 + 2\left(r_6 + r_9\right)\right)\right)\Big]x \\
& + 16\Big[4r_{11}^2\left|c_{16}\right|^2 - 4r_1r_2\left|c_{16}\right|^2 + 2r_{10}^2\left|c_{17}\right|^2 - r_1r_6\left|c_{17}\right|^2 - r_1r_9\left|c_{17}\right|^2 \\
& - 2r_{11}^2\left(c_{16}^*\right)^2 - 2c_{16}^2r_1c_{17}^* - 2c_{17}r_1\left(c_{16}^*\right)^2 + 4c_{16}r_{10}r_{11}c_{17}^* \\
& + 4c_{17}r_{10}r_{11}c_{16}^* - 2r_6r_{11}^2\Re\left(c_{17}\right) - 2r_9r_{11}^2\Re\left(c_{17}\right) + 8r_2r_{10}r_{11}\Re\left(c_{16}\right) \\
& - 2c_{16}^2r_{11}^2 - 2r_2^2r_{10}^2 - 2r_2r_6r_{11}^2 - 2r_2r_9r_{11}^2 + r_1r_2^2r_6 + r_1r_2^2r_9\Big] = 0\,, \\
\Lambda_{5,6}^{++} = 2(r_4 + r_7)\,.&
\end{aligned} \tag{5.184}$$

The matrix M_2^{+}. From M_2^{+} we get

$$\frac{1}{2}\,M_2^{+} \sim \mathrm{diag}\left\{\frac{1}{2}\begin{pmatrix} r_1 & r_{11} & 2r_{10} & -r_{11} \\ r_{11} & r_2 & 2c_{16} & c_{17} \\ 2r_{10} & 2c_{16}^* & 2(r_6+r_9) & -2c_{16} \\ -r_{11} & c_{17}^* & -2c_{16}^* & r_2 \end{pmatrix}, r_6 - r_9, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix}\right\}. \tag{5.185}$$

Here, the procedure we have mentioned thus far, which includes the algorithm in appendix C, yields a 5×5 matrix. Supplemented by a suitable rotation, it can be written in the 4×4 and 1×1 blocks given in eq. (5.185). The eigenvalues of M_2^{+} are:

$$\begin{aligned}
\Lambda_{1-4}^{+,2} &= \Lambda_{1-4}^{++}\,, \\
\Lambda_{5}^{+,2} &= 2(r_6 - r_9)\,, \\
\Lambda_{6,7}^{+,2} &= 2(r_4 \pm r_7)\,, \\
\Lambda_{8,9}^{+,2} &= \Lambda_{6,7}^{++}\,.
\end{aligned} \tag{5.186}$$

The matrix M_0^{+}. From M_0^{+} we get

$$\frac{1}{2}\,M_0^{+} = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_7 & 0 & 0 \\ r_7 & r_2 & r_9 & c_{16}^* & c_{16} \\ r_7 & r_9 & r_2 & -c_{16}^* & -c_{16} \\ 0 & c_{16} & -c_{16} & r_6 & c_{17} \\ 0 & c_{16}^* & -c_{16}^* & c_{17}^* & r_6 \end{pmatrix}, \begin{pmatrix} r_4 & r_{11} & r_{10} & 0 \\ r_{11} & r_4 & 0 & r_{10} \\ r_{10} & 0 & r_4 & -r_{11} \\ 0 & r_{10} & -r_{11} & r_4 \end{pmatrix}\right\}, \tag{5.187}$$

As in eq. (5.185), each matrix can be further block diagonalized in an easy way, leading to one 3×3 block, one 2×2 block, and two identical 2×2 blocks. The eigenvalues of M_0^+ are:

$$\begin{aligned}
\Lambda^{+,0}_{1-3} = &\text{ Roots of:} \\
& x^3 - 2(-r_2 - 2r_6 + r_9)x^2 + 4(-4\,|c_{16}|^{\,2} - |c_{17}|^{\,2} + r_6^2 + 2r_2r_6 - 2r_6r_9)x \\
& + 8\left(4r_6\,|c_{16}|^{\,2} + (r_2 - r_9)\left(|c_{17}|^{\,2} - r_6^2\right) - 4\Re\left(c_{16}^2 c_{17}^*\right)\right) = 0\,, \\
\Lambda^{+,0}_{4,5} = &\pm\sqrt{8r_7^2 + (-r_1 + r_2 + r_9)^{\,2}} + r_1 + r_2 + r_9\,, \\
\Lambda^{+,0}_{6-7} = &\Lambda^{+,0}_{8-9} = 2\left(r_4 \pm \sqrt{r_{10}^2 + r_{11}^2}\right)\,.
\end{aligned} \tag{5.188}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.189}$$

with eigenvalues of M_2^0:

$$\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.190}$$

The matrix M_0^0. From M_0^0 we get the eigenvalues of M_0^0:

$$\begin{aligned}
\Lambda^{0,0}_{1-9} = &\Lambda^{+,0}_{1-9}\,, \\
\Lambda^{0,0}_{10-12} = &\text{ Roots of:} \\
& x^3 + 2\,(-3r_2 - 3r_9)\,x^2 + 4\left(-36\,|c_{16}|^{\,2} - 9\,|c_{17}|^{\,2} - 3r_6^2\right. \\
& \left.+6r_2r_6 + 12r_2r_9 - 6r_6r_9\right)x + 8\left[36\,(r_6 + 2r_9)\,|c_{16}|^{\,2}\right. \\
& - (3r_2 - 2r_6 - r_9)\left((r_6 + 2r_9)^{\,2} - 9\,|c_{17}|^{\,2}\right) \\
& \left.-108\Re\left(c_{16}^2 c_{17}^*\right)\right] = 0\,, \\
\Lambda^{0,0}_{13,14} = &\pm\sqrt{8\,(2r_4 + r_7)^{\,2} + (-3r_1 + 3r_2 + 2r_6 + r_9)^{\,2}} + 3r_1 + 3r_2 + 2r_6 + r_9\,, \\
\Lambda^{0,0}_{15-18} = &2\left(\pm3\sqrt{r_{10}^2 + r_{11}^2} + r_4 + 2r_7\right)\,,
\end{aligned} \tag{5.191}$$

where we have suppressed the form of the matrix due to its size.

5.19 The SU(3) symmetry

By imposing $G = \mathrm{SU}(3)$ we get the quartic potential

$$\begin{aligned}
V_{\mathrm{SU}(3)} = &r_1\left[(\phi_1^\dagger\phi_1) + (\phi_2^\dagger\phi_2) + (\phi_3^\dagger\phi_3)\right]^2 \\
& + 2r_7\left[|\phi_1^\dagger\phi_2|^2 + |\phi_1^\dagger\phi_3|^2 + |\phi_2^\dagger\phi_3|^2\right. \\
& \left.-(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) - (\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) - (\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)\right]\,,
\end{aligned} \tag{5.192}$$

with the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}\,M_2^{++} = \mathrm{diag}(r_1, r_1, r_1, r_1, r_1, r_1)\,, \tag{5.193}$$

and thus we get the eigenvalues of M_2^{++}:

$$\Lambda^{++}_{1-6} = 2r_1\,. \tag{5.194}$$

The matrix M_2^{+}. From M_2^{+} we get

$$\frac{1}{2}\,M_2^{+} = \mathrm{diag}\left\{\begin{pmatrix} r_1 - r_7 & r_7 \\ r_7 & r_1 - r_7\end{pmatrix}, \begin{pmatrix} r_1 - r_7 & r_7 \\ r_7 & r_1 - r_7\end{pmatrix}, \begin{pmatrix} r_1 - r_7 & r_7 \\ r_7 & r_1 - r_7\end{pmatrix}, r_1, r_1, r_1\right\}, \tag{5.195}$$

with eigenvalues of M_2^{+}:

$$\begin{aligned}
\Lambda^{+,2}_{1} &= 2r_1\,,\\
\Lambda^{+,2}_{2} &= 2\left(r_1 - 2r_7\right)\,,\\
\Lambda^{+,2}_{3,4} &= \Lambda^{+,2}_{1,2}\,,\\
\Lambda^{+,2}_{5,6} &= \Lambda^{+,2}_{1,2}\\
\Lambda^{+,2}_{7-9} &= \Lambda^{+,2}_{1}\,.
\end{aligned} \tag{5.196}$$

The matrix M_0^{+}. From M_0^{+} we get

$$\frac{1}{2}\,M_0^{+} = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_7 \\ r_7 & r_1 & r_7 \\ r_7 & r_7 & r_1\end{pmatrix}, r_1 - r_7, r_1 - r_7, r_1 - r_7, r_1 - r_7, r_1 - r_7, r_1 - r_7\right\}, \tag{5.197}$$

with eigenvalues of M_0^{+}:

$$\begin{aligned}
\Lambda^{+,0}_{1} &= 2\left(r_1 + 2r_7\right)\,,\\
\Lambda^{+,0}_{2,3} &= 2(r_1 - r_7)\,,\\
\Lambda^{+,0}_{4-9} &= 2(r_1 - r_7)\,.
\end{aligned} \tag{5.198}$$

The matrix M_2^{0}. From M_2^{0} we get

$$\frac{1}{2}\,M_2^{0} = \frac{1}{2}\,M_2^{++}\,, \tag{5.199}$$

with eigenvalues of M_2^{0}:

$$\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.200}$$

The matrix M_0^0. From M_0^0 we get

$$\frac{1}{2}\,M_0^0 \sim \mathrm{diag}\left\{\frac{1}{2}M_0^+,\begin{pmatrix}3r_1 & 2r_1+r_7 & 2r_1-r_7\\ 2r_1-r_7 & 3r_1 & 2r_1-r_7\\ 2r_1-r_7 & 2r_1-r_7 & 3r_1\end{pmatrix},(r_1+r_7),\right.$$
$$\left.(r_1+r_7),(r_1+r_7),(r_1+r_7),(r_1+r_7),(r_1+r_7)\right\}, \tag{5.201}$$

with eigenvalues of M_0^0:

$$\begin{aligned}\Lambda_{1-9}^{0,0} &= \Lambda_{1-9}^{+,0}\,,\\ \Lambda_{10}^{0,0} &= 14r_1-4r_7\,,\\ \Lambda_{11,12}^{0,0} &= 2\left(r_1+r_7\right)\,,\\ \Lambda_{13-18}^{0,0} &= 2\left(r_1+r_7\right)\,.\end{aligned} \tag{5.202}$$

5.20 The A_4 symmetry

Imposing $G = A_4$ we get the quartic potential

$$\begin{aligned}V_{A_4} =& \frac{r_1+2r_4}{3}\left[(\phi_1^\dagger\phi_1)+(\phi_2^\dagger\phi_2)+(\phi_3^\dagger\phi_3)\right]^2+\frac{2(r_1-r_4)}{3}\left[(\phi_1^\dagger\phi_1)^2+(\phi_2^\dagger\phi_2)^2\right.\\ &\left.+(\phi_3^\dagger\phi_3)^2-(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)-(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)-(\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1)\right]\\ &+2r_7\left(|\phi_1^\dagger\phi_2|^2+|\phi_2^\dagger\phi_3|^2+|\phi_3^\dagger\phi_1|^2\right)\\ &+\left[c_3\left[(\phi_1^\dagger\phi_2)^2+(\phi_2^\dagger\phi_3)^2+(\phi_3^\dagger\phi_1)^2\right]+h.c.\right],\end{aligned} \tag{5.203}$$

which can be easily achieved by setting from $\mathbb{Z}_2\times\mathbb{Z}_2$ with the constraints $r_2 = r_3 = r_1$, $r_5 = r_6 = r_4$, $r_8 = r_9 = r_7$ and $c_5^* = c_{17} = c_3$. It can not be easily achieved from $\mathbb{Z}_3$ due to our choice of basis. Nevertheless, we work it out in appendix D.

Thus, we get the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}\,M_2^{++} = \mathrm{diag}\left\{\begin{pmatrix}r_1 & c_3 & c_3^*\\ c_3^* & r_1 & c_3\\ c_3 & c_3^* & r_1\end{pmatrix},(r_4+r_7),(r_4+r_7),(r_4+r_7)\right\}, \tag{5.204}$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned}\Lambda_1^{++} &= 2\left(2\mathrm{Re}(c_3)+r_1\right)\,,\\ \Lambda_{2,3}^{++} &= 2\left(\pm\sqrt{3}\,|\Im\left(c_3\right)|-\Re\left(c_3\right)+r_1\right)\,,\\ \Lambda_{4-6}^{++} &= 2(r_4+r_7)\,.\end{aligned} \tag{5.205}$$

The matrix M_2^+. From M_2^+ we get

$$\frac{1}{2}\,M_2^+ = \mathrm{diag}\left\{ \begin{pmatrix} r_1 & c_3 & c_3^* \\ c_3^* & r_1 & c_3 \\ c_3 & c_3^* & r_1 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix} \right\}, \tag{5.206}$$

with eigenvalues of M_2^+:

$$\begin{aligned} \Lambda^{+,2}_{1-3} &= \Lambda^{++}_{1-3}\,, \\ \Lambda^{+,2}_{4-9} &= 2(r_4 \pm r_7)\,. \end{aligned} \tag{5.207}$$

The matrix M_0^+. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \mathrm{diag}\left\{ \begin{pmatrix} r_1 & r_7 & r_7 \\ r_7 & r_1 & r_7 \\ r_7 & r_7 & r_1 \end{pmatrix}, \begin{pmatrix} r_4 & c_3 \\ c_3^* & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & c_3^* \\ c_3 & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & c_3 \\ c_3^* & r_4 \end{pmatrix} \right\}, \tag{5.208}$$

with eigenvalues of M_0^+:

$$\begin{aligned} \Lambda^{+,0}_{1} &= 2\left(r_1 + 2r_7\right)\,, \\ \Lambda^{+,0}_{2,3} &= 2\left(r_1 - r_7\right)\,, \\ \Lambda^{+,0}_{4-9} &= 2\left(r_4 \pm \sqrt{c_3}\sqrt{c_3^*}\right)\,. \end{aligned} \tag{5.209}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.210}$$

with eigenvalues of M_2^0:

$$\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.211}$$

The matrix M_0^0. From M_0^0 we get

$$\begin{aligned} \frac{1}{2}\,M_0^0 \sim \mathrm{diag}\Bigg\{ &\frac{1}{2}M_0^+, \begin{pmatrix} 3r_1 & 2r_4+r_7 & 2r_4+r_7 \\ 2r_4+r_7 & 3r_1 & 2r_4+r_7 \\ 2r_4+r_7 & 2r_4+r_7 & 3r_1 \end{pmatrix}, \begin{pmatrix} r_4+2r_7 & 3c_3 \\ 3c_3^* & r_4+2r_7 \end{pmatrix}, \\ &\begin{pmatrix} r_4+2r_7 & 3c_3^* \\ 3c_3 & r_4+2r_7 \end{pmatrix}, \begin{pmatrix} r_4+2r_7 & 3c_3 \\ 3c_3^* & r_4+2r_7 \end{pmatrix} \Bigg\}\,. \end{aligned} \tag{5.212}$$

with eigenvalues of M_0^0:

$$\begin{aligned} \Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,, \\ \Lambda^{0,0}_{10} &= 6r_1 + 8r_4 + 4r_7\,, \\ \Lambda^{0,0}_{11,12} &= 6r_1 - 2\left(2r_4 + r_7\right)\,, \\ \Lambda^{0,0}_{13-18} &= \pm 6\sqrt{c_3}\sqrt{c_3^*} + 2r_4 + 4r_7\,. \end{aligned} \tag{5.213}$$

5.21 The S_4 symmetry

Imposing $G = S_4$ we get the quartic potential

$$\begin{aligned} V_{S_4} =& \frac{r_1 + 2r_4}{3} \left[(\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2) + (\phi_3^\dagger \phi_3)\right]^2 + \frac{2(r_1 - r_4)}{3} \left[(\phi_1^\dagger \phi_1)^2 + (\phi_2^\dagger \phi_2)^2 \right. \\ & \left. +(\phi_3^\dagger \phi_3)^2 - (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) - (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3) - (\phi_3^\dagger \phi_3)(\phi_1^\dagger \phi_1)\right] \\ & + 2r_7 \left(|\phi_1^\dagger \phi_2|^2 + |\phi_2^\dagger \phi_3|^2 + |\phi_3^\dagger \phi_1|^2\right) \\ & + r_{10} \left((\phi_1^\dagger \phi_2)^2 + (\phi_2^\dagger \phi_3)^2 + (\phi_3^\dagger \phi_1)^2 + (\phi_2^\dagger \phi_1)^2 + (\phi_3^\dagger \phi_2)^2 + (\phi_1^\dagger \phi_3)^2\right) . \end{aligned} \tag{5.214}$$

which can be easily achieved by setting from A_4 with the constraint $c_3 \in \mathbb{R}$. It can not be easily achieved from D_4 or S_3 due to our choice of basis.

Thus, we get the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2} M_2^{++} = \operatorname{diag}\left\{ \begin{pmatrix} r_1 & r_{10} & r_{10} \\ r_{10} & r_1 & r_{10} \\ r_{10} & r_{10} & r_1 \end{pmatrix}, (r_4 + r_7), (r_4 + r_7), (r_4 + r_7) \right\} , \tag{5.215}$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned} \Lambda_1^{++} &= 2\left(r_1 + 2r_{10}\right) , \\ \Lambda_{2,3}^{++} &= 2\left(r_1 - r_{10}\right) , \\ \Lambda_{4-6}^{++} &= 2(r_4 + r_7) . \end{aligned} \tag{5.216}$$

The matrix M_2^{+}. From M_2^{+} we get

$$\frac{1}{2} M_2^{+} = \operatorname{diag}\left\{ \begin{pmatrix} r_1 & r_{10} & r_{10} \\ r_{10} & r_1 & r_{10} \\ r_{10} & r_{10} & r_1 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix} \right\} , \tag{5.217}$$

with eigenvalues of M_2^{+}:

$$\begin{aligned} \Lambda_{1-3}^{+,2} &= \Lambda_{1-3}^{++} , \\ \Lambda_{4-9}^{+,2} &= 2(r_4 \pm r_7) . \end{aligned} \tag{5.218}$$

The matrix M_0^{+}. From M_0^{+} we get

$$\frac{1}{2} M_0^{+} = \operatorname{diag}\left\{ \begin{pmatrix} r_1 & r_7 & r_7 \\ r_7 & r_1 & r_7 \\ r_7 & r_7 & r_1 \end{pmatrix}, \begin{pmatrix} r_4 & r_{10} \\ r_{10} & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & r_{10} \\ r_{10} & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & r_{10} \\ r_{10} & r_4 \end{pmatrix} \right\} , \tag{5.219}$$

with eigenvalues of M_0^{+}:

$$\begin{aligned} \Lambda_1^{+,0} &= 2\left(r_1 + 2r_7\right) , \\ \Lambda_{2,3}^{+,0} &= 2\left(r_1 - r_7\right) , \\ \Lambda_{4-9}^{+,0} &= 2\left(r_4 \pm r_{10}\right) . \end{aligned} \tag{5.220}$$

JHEP08(2022)273

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.221}$$

with eigenvalues of M_2^0:

$$\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.222}$$

The matrix M_0^0. From M_0^0 we get

$$\begin{aligned}\frac{1}{2}\,M_0^0 \sim \mathrm{diag}\Bigg\{&\frac{1}{2}M_0^+,\begin{pmatrix}3r_1 & 2r_4+r_7 & 2r_4+r_7\\ 2r_4+r_7 & 3r_1 & 2r_4+r_7\\ 2r_4+r_7 & 2r_4+r_7 & 3r_1\end{pmatrix},\begin{pmatrix}r_4+2r_7 & 3r_{10}\\ 3r_{10} & r_4+2r_7\end{pmatrix},\\ &\begin{pmatrix}r_4+2r_7 & 3r_{10}\\ 3r_{10} & r_4+2r_7\end{pmatrix},\begin{pmatrix}r_4+2r_7 & 3r_{10}\\ 3r_{10} & r_4+2r_7\end{pmatrix}\Bigg\}\,.\end{aligned} \tag{5.223}$$

with eigenvalues of M_0^0:

$$\begin{aligned}\Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,,\\ \Lambda^{0,0}_{10} &= 6r_1+8r_4+4r_7\,,\\ \Lambda^{0,0}_{11,12} &= 6r_1-2\left(2r_4+r_7\right)\,,\\ \Lambda^{0,0}_{13-18} &= 2\left(r_4+2r_7\pm 3r_{10}\right)\,.\end{aligned} \tag{5.224}$$

5.22 The SO(3) symmetry

Imposing $G=\mathrm{SO}(3)$ we get the quartic potential

$$\begin{aligned}V_{\mathrm{SO}(3)} =&\frac{r_1+2r_4}{3}\left[(\phi_1^\dagger\phi_1)+(\phi_2^\dagger\phi_2)+(\phi_3^\dagger\phi_3)\right]^2+\frac{2(r_1-r_4)}{3}\left[(\phi_1^\dagger\phi_1)^2+(\phi_2^\dagger\phi_2)^2\right.\\ &\left.+(\phi_3^\dagger\phi_3)^2-(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)-(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)-(\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1)\right]\\ &+2r_7\left(|\phi_1^\dagger\phi_2|^2+|\phi_2^\dagger\phi_3|^2+|\phi_3^\dagger\phi_1|^2\right)\\ &+(r_1-r_4-r_7)\left((\phi_1^\dagger\phi_2)^2+(\phi_2^\dagger\phi_3)^2+(\phi_3^\dagger\phi_1)^2+(\phi_2^\dagger\phi_1)^2+(\phi_3^\dagger\phi_2)^2+(\phi_1^\dagger\phi_3)^2\right)\,.\end{aligned} \tag{5.225}$$

which can be easily achieved by setting from S_4 with the constraint $r_{10}\to r_1-r_4-r_7$.

Thus, we get the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}\,M_2^{++} = \mathrm{diag}\left\{\begin{pmatrix}r_1 & r_1-r_4-r_7 & r_1-r_4-r_7\\ r_1-r_4-r_7 & r_1 & r_1-r_4-r_7\\ r_1-r_4-r_7 & r_1-r_4-r_7 & r_1\end{pmatrix},(r_4+r_7),(r_4+r_7),(r_4+r_7)\right\}\,, \tag{5.226}$$

and thus we get the eigenvalues of M_2^{++}:

$$\begin{aligned}\Lambda^{++}_1 &= 2\left(3r_1-2r_4-2r_7\right)\,,\\ \Lambda^{++}_{2,3} &= 2(r_4+r_7)\,,\\ \Lambda^{++}_{4-6} &= 2(r_4+r_7)\,.\end{aligned} \tag{5.227}$$

The matrix M_2^+. From M_2^+ we get

$$\frac{1}{2}\,M_2^+ = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_1-r_4-r_7 & r_1-r_4-r_7 \\ r_1-r_4-r_7 & r_1 & r_1-r_4-r_7 \\ r_1-r_4-r_7 & r_1-r_4-r_7 & r_1 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 \\ r_7 & r_4 \end{pmatrix}\right\}, \tag{5.228}$$

with eigenvalues of M_2^+:

$$\begin{aligned} \Lambda^{+,2}_{1-3} &= \Lambda^{++}_{1-3}\,, \\ \Lambda^{+,2}_{4-9} &= 2(r_4 \pm r_7)\,. \end{aligned} \tag{5.229}$$

The matrix M_0^+. From M_0^+ we get

$$\begin{aligned} \frac{1}{2}\,M_0^+ =& \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_7 \\ r_7 & r_1 & r_7 \\ r_7 & r_7 & r_1 \end{pmatrix}, \begin{pmatrix} r_4 & r_1-r_4-r_7 \\ r_1-r_4-r_7 & r_4 \end{pmatrix},\right. \\ & \left.\begin{pmatrix} r_4 & r_1-r_4-r_7 \\ r_1-r_4-r_7 & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & r_1-r_4-r_7 \\ r_1-r_4-r_7 & r_4 \end{pmatrix}\right\}, \end{aligned} \tag{5.230}$$

with eigenvalues of M_0^+:

$$\begin{aligned} \Lambda^{+,0}_{1} &= 2\left(r_1 + 2r_7\right)\,, \\ \Lambda^{+,0}_{2,3} &= 2\left(r_1 - r_7\right)\,, \\ \Lambda^{+,0}_{4-9} &= 2\left(r_4 \pm \left(r_1 - r_4 - r_7\right)\right)\,. \end{aligned} \tag{5.231}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.232}$$

with eigenvalues of M_2^0:

$$\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.233}$$

The matrix M_0^0. From M_0^0 we get

$$\begin{aligned} \frac{1}{2}\,M_0^0 \sim \mathrm{diag}&\left\{\frac{1}{2}M_0^+, \begin{pmatrix} 3r_1 & 2r_4+r_7 & 2r_4+r_7 \\ 2r_4+r_7 & 3r_1 & 2r_4+r_7 \\ 2r_4+r_7 & 2r_4+r_7 & 3r_1 \end{pmatrix}, \begin{pmatrix} r_4+2r_7 & 3\left(r_1-r_4-r_7\right) \\ 3\left(r_1-r_4-r_7\right) & r_4+2r_7 \end{pmatrix},\right. \\ &\left.\begin{pmatrix} r_4+2r_7 & 3\left(r_1-r_4-r_7\right) \\ 3\left(r_1-r_4-r_7\right) & r_4+2r_7 \end{pmatrix}, \begin{pmatrix} r_4+2r_7 & 3\left(r_1-r_4-r_7\right) \\ 3\left(r_1-r_4-r_7\right) & r_4+2r_7 \end{pmatrix}\right\}. \end{aligned} \tag{5.234}$$

with eigenvalues of M_0^0:

$$\begin{aligned} \Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,, \\ \Lambda^{0,0}_{10} &= 6r_1 + 8r_4 + 4r_7\,, \\ \Lambda^{0,0}_{11,12} &= 6r_1 - 2\left(2r_4 + r_7\right)\,, \\ \Lambda^{0,0}_{13-18} &= 2(r_4 + 2r_7) \pm 6(r_1 - r_4 - r_7)\,. \end{aligned} \tag{5.235}$$

JHEP08(2022)273

5.23 The $\Delta(54)$ symmetry

Imposing $G = \Delta(54)$ we get the quartic potential

$$
\begin{aligned}
V_{\Delta(54)} =& \frac{r_1+2r_4}{3}\left[(\phi_1^\dagger\phi_1)+(\phi_2^\dagger\phi_2)+(\phi_3^\dagger\phi_3)\right]^2 + \frac{2(r_1-r_4)}{3}\left[(\phi_1^\dagger\phi_1)^2+(\phi_2^\dagger\phi_2)^2\right. \\
&\left.+(\phi_3^\dagger\phi_3)^2-(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)-(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)-(\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1)\right] \\
&+2r_7\left(|\phi_1^\dagger\phi_2|^2+|\phi_2^\dagger\phi_3|^2+|\phi_3^\dagger\phi_1|^2\right) \\
&+\left[2c_{11}\left((\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3)+(\phi_2^\dagger\phi_1)(\phi_3^\dagger\phi_1)+(\phi_3^\dagger\phi_2)(\phi_1^\dagger\phi_2)\right)+h.c.\right]. \qquad (5.236)
\end{aligned}
$$

which can not be easily achieved from S_3 due to our choice of basis. The details are contained in appendix C

We get the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$
\frac{1}{2}M_2^{++} = \operatorname{diag}\left\{\begin{pmatrix} r_1 & \sqrt{2}c_{11}^* \\ \sqrt{2}c_{11} & r_4+r_7\end{pmatrix}, \begin{pmatrix} r_4+r_7 & \sqrt{2}c_{11} \\ \sqrt{2}c_{11}^* & r_1\end{pmatrix}, \begin{pmatrix} r_4+r_7 & \sqrt{2}c_{11} \\ \sqrt{2}c_{11}^* & r_1\end{pmatrix}\right\}, \qquad (5.237)
$$

and thus we get the eigenvalues of M_2^{++}:

$$
\Lambda_{1-6}^{++} = \pm\sqrt{8\,|c_{11}|^{\,2}+(-r_1+r_4+r_7)^{\,2}}+r_1+r_4+r_7\,. \qquad (5.238)
$$

The matrix M_2^{+}. From M_2^{+} we get

$$
\frac{1}{2}M_2^{+} = \operatorname{diag}\left\{\begin{pmatrix} r_1 & c_{11}^* & c_{11}^* \\ c_{11} & r_4 & r_7 \\ c_{11} & r_7 & r_4\end{pmatrix}, \begin{pmatrix} r_4 & r_7 & c_{11} \\ r_7 & r_4 & c_{11} \\ c_{11}^* & c_{11}^* & r_1\end{pmatrix}, \begin{pmatrix} r_4 & c_{11} & r_7 \\ c_{11}^* & r_1 & c_{11}^* \\ r_7 & c_{11} & r_4\end{pmatrix}\right\}, \qquad (5.239)
$$

with eigenvalues of M_2^{+}:

$$
\begin{aligned}
\Lambda_1^{+,2} &= 2\,(r_4-r_7)\;, \\
\Lambda_{2,3}^{+,2} &= \pm\sqrt{8\,|c_{11}|^{\,2}+(-r_1+r_4+r_7)^{\,2}}+r_1+r_4+r_7\,, \\
\Lambda_{4-6}^{+,2} &= \Lambda_{1-3}^{+,2}\,, \\
\Lambda_{7-9}^{+,2} &= \Lambda_{1-3}^{+,2}\,. \qquad (5.240)
\end{aligned}
$$

The matrix M_0^{+}. From M_0^{+} we get

$$
\frac{1}{2}M_0^{+} = \operatorname{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_7 \\ r_7 & r_1 & r_7 \\ r_7 & r_7 & r_1\end{pmatrix}, \begin{pmatrix} r_4 & c_{11} & c_{11}^* \\ c_{11}^* & r_4 & c_{11} \\ c_{11} & c_{11}^* & r_4\end{pmatrix}, \begin{pmatrix} r_4 & c_{11}^* & c_{11} \\ c_{11} & r_4 & c_{11}^* \\ c_{11}^* & c_{11} & r_4\end{pmatrix}\right\}, \qquad (5.241)
$$

with eigenvalues of M_0^{+}:

$$
\begin{aligned}
\Lambda_1^{+,0} &= 2\,(r_1+2r_7)\;, \\
\Lambda_{2,3}^{+,0} &= 2\,(r_1-r_7)\;, \\
\Lambda_4^{+,0} &= 2\,(2\Re\,(c_{11})+r_4)\;, \\
\Lambda_{5,6}^{+,0} &= 2\left(\pm\sqrt{3}\,|\Im\,(c_{11})|-\Re\,(c_{11})+r_4\right)\;, \\
\Lambda_{7-9}^{+,0} &= \Lambda_{4-6}^{+,0}\,. \qquad (5.242)
\end{aligned}
$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.243}$$

with eigenvalues of M_2^0:

$$\Lambda^{0,2}_{1-6} = \Lambda^{++}_{1-6}\,. \tag{5.244}$$

The matrix M_0^0. From M_0^0 we get

$$\begin{aligned}\frac{1}{2}\,M_0^0 \sim \mathrm{diag}\Bigg\{&\frac{1}{2}M_0^+,\begin{pmatrix}3r_1 & 2r_4+r_7 & 2r_4+r_7\\ 2r_4+r_7 & 3r_1 & 2r_4+r_7\\ 2r_4+r_7 & 2r_4+r_7 & 3r_1\end{pmatrix},\\ &\begin{pmatrix}r_4+2r_7 & 3c_{11} & 3c_{11}^*\\ 3c_{11}^* & r_4+2r_7 & 3c_{11}\\ 3c_{11} & 3c_{11}^* & r_4+2r_7\end{pmatrix},\begin{pmatrix}r_4+2r_7 & 3c_{11}^* & 3c_{11}\\ 3c_{11} & r_4+2r_7 & 3c_{11}^*\\ 3c_{11}^* & 3c_{11} & r_4+2r_7\end{pmatrix}\Bigg\}\,.\end{aligned} \tag{5.245}$$

with eigenvalues of M_0^0:

$$\begin{aligned}\Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,,\\ \Lambda^{0,0}_{10} &= 6r_1+8r_4+4r_7\,,\\ \Lambda^{0,0}_{11,12} &= 6r_1-2\left(2r_4+r_7\right)\,,\\ \Lambda^{0,0}_{13} &= 2\left(6\Re\left(c_{11}\right)+r_4+2r_7\right)\,,\\ \Lambda^{0,0}_{14,15} &= 2\left(\pm3\sqrt{3}\left|\Im\left(c_{11}\right)\right|-3\Re\left(c_{11}\right)+r_4+2r_7\right)\,,\\ \Lambda^{0,0}_{16-18} &= \Lambda^{0,0}_{13-15}\,.\end{aligned} \tag{5.246}$$

5.24 The $\Delta(54)\rtimes\mathbb{Z}_2^{(\mathrm{CP})}$ symmetry

Imposing $G=\Delta(54)\rtimes\mathbb{Z}_2^{(\mathrm{CP})}$ we get the quartic potential

$$\begin{aligned}V_{\Delta(54)\rtimes\mathbb{Z}_2^{(\mathrm{CP})}} =& \frac{r_1+2r_4}{3}\left[(\phi_1^\dagger\phi_1)+(\phi_2^\dagger\phi_2)+(\phi_3^\dagger\phi_3)\right]^2+\frac{2(r_1-r_4)}{3}\Big[(\phi_1^\dagger\phi_1)^2+(\phi_2^\dagger\phi_2)^2\\ &+(\phi_3^\dagger\phi_3)^2-(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)-(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)-(\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1)\Big]\\ &+2r_7\left(|\phi_1^\dagger\phi_2|^2+|\phi_2^\dagger\phi_3|^2+|\phi_3^\dagger\phi_1|^2\right)\\ &+2r_{10}\left[(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3)+(\phi_2^\dagger\phi_1)(\phi_3^\dagger\phi_1)+(\phi_3^\dagger\phi_2)(\phi_1^\dagger\phi_2)+h.c.\right]\,.\end{aligned} \tag{5.247}$$

which can be easily achieved by setting from $\Delta(54)$ the constraints $c_{11}\in\mathbb{R}$.

Thus we get the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}\,M_2^{++} = \mathrm{diag}\left\{\begin{pmatrix}r_1 & \sqrt{2}r_{10}\\ \sqrt{2}r_{10} & r_4+r_7\end{pmatrix},\begin{pmatrix}r_4+r_7 & \sqrt{2}r_{10}\\ \sqrt{2}r_{10} & r_1\end{pmatrix},\begin{pmatrix}r_4+r_7 & \sqrt{2}r_{10}\\ \sqrt{2}r_{10} & r_1\end{pmatrix}\right\}\,, \tag{5.248}$$

and thus we get the eigenvalues of M_2^{++}:

$$\Lambda^{++}_{1-6} = \pm\sqrt{8r_{10}^2+\left(-r_1+r_4+r_7\right)^2}+r_1+r_4+r_7\,. \tag{5.249}$$

The matrix M_2^+. From M_2^+ we get

$$\frac{1}{2}\,M_2^+ = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_{10} & r_{10} \\ r_{10} & r_4 & r_7 \\ r_{10} & r_7 & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & r_7 & r_{10} \\ r_7 & r_4 & r_{10} \\ r_{10} & r_{10} & r_1 \end{pmatrix}, \begin{pmatrix} r_4 & r_{10} & r_7 \\ r_{10} & r_1 & r_{10} \\ r_7 & r_{10} & r_4 \end{pmatrix}\right\}, \tag{5.250}$$

with eigenvalues of M_2^+:

$$\begin{aligned} \Lambda_1^{+,2} &= 2\,(r_4 - r_7)\;, \\ \Lambda_{2,3}^{+,2} &= \pm\sqrt{8r_{10}^2 + (-r_1 + r_4 + r_7)^{\,2}} + r_1 + r_4 + r_7\,, \\ \Lambda_{4-6}^{+,2} &= \Lambda_{1-3}^{+,2}\,, \\ \Lambda_{7-9}^{+,2} &= \Lambda_{1-3}^{+,2}\,. \end{aligned} \tag{5.251}$$

The matrix M_0^+. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \mathrm{diag}\left\{\begin{pmatrix} r_1 & r_7 & r_7 \\ r_7 & r_1 & r_7 \\ r_7 & r_7 & r_1 \end{pmatrix}, \begin{pmatrix} r_4 & r_{10} & r_{10} \\ r_{10} & r_4 & r_{10} \\ r_{10} & r_{10} & r_4 \end{pmatrix}, \begin{pmatrix} r_4 & r_{10} & r_{10} \\ r_{10} & r_4 & r_{10} \\ r_{10} & r_{10} & r_4 \end{pmatrix}\right\}, \tag{5.252}$$

with eigenvalues of M_0^+:

$$\begin{aligned} \Lambda_1^{+,0} &= 2\,(r_1 + 2r_7)\;, \\ \Lambda_{2,3}^{+,0} &= 2\,(r_1 - r_7)\;, \\ \Lambda_4^{+,0} &= 2\,(r_4 + 2r_{10})\;, \\ \Lambda_{5,6}^{+,0} &= 2\,(r_4 - r_{10})\;, \\ \Lambda_{7-9}^{+,0} &= \Lambda_{4-6}^{+,0}\,. \end{aligned} \tag{5.253}$$

The matrix M_2^0. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.254}$$

with eigenvalues of M_2^0:

$$\Lambda_{1-6}^{0,2} = \Lambda_{1-6}^{++}\,. \tag{5.255}$$

The matrix M_0^0. From M_0^0 we get

$$\begin{aligned} \frac{1}{2}\,M_0^0 \sim \mathrm{diag}\Bigg\{ & \frac{1}{2}M_0^+, \begin{pmatrix} 3r_1 & 2r_4 + r_7 & 2r_4 + r_7 \\ 2r_4 + r_7 & 3r_1 & 2r_4 + r_7 \\ 2r_4 + r_7 & 2r_4 + r_7 & 3r_1 \end{pmatrix}, \\ & \begin{pmatrix} r_4 + 2r_7 & 3r_{10} & 3r_{10} \\ 3r_{10} & r_4 + 2r_7 & 3r_{10} \\ 3r_{10} & 3r_{10} & r_4 + 2r_7 \end{pmatrix}, \begin{pmatrix} r_4 + 2r_7 & 3r_{10} & 3r_{10} \\ 3r_{10} & r_4 + 2r_7 & 3r_{10} \\ 3r_{10} & 3r_{10} & r_4 + 2r_7 \end{pmatrix}\Bigg\}\,. \end{aligned} \tag{5.256}$$

with eigenvalues of M_0^0:

$$\begin{aligned}
\Lambda_{1-9}^{0,0} &= \Lambda_{1-9}^{+,0}\,,\\
\Lambda_{10}^{0,0} &= 6r_1+8r_4+4r_7\,,\\
\Lambda_{11,12}^{0,0} &= 6r_1-2\left(2r_4+r_7\right)\,,\\
\Lambda_{13}^{0,0} &= 2\left(6r_{10}+r_4+2r_7\right)\,,\\
\Lambda_{14,15}^{0,0} &= 2\left(r_4+2r_7-3r_{10}\right)\,,\\
\Lambda_{16-18}^{0,0} &= \Lambda_{13-15}^{0,0}\,.
\end{aligned} \tag{5.257}$$

5.25 The $\Sigma(36)$ symmetry

Imposing $G=\Sigma(36)$ we get the quartic potential

$$\begin{aligned}
V_{\Sigma(36)} =& \frac{r_1+2r_4}{3}\left[(\phi_1^\dagger\phi_1)+(\phi_2^\dagger\phi_2)+(\phi_3^\dagger\phi_3)\right]^2+\frac{2(r_1-r_4)}{3}\left[(\phi_1^\dagger\phi_1)^2+(\phi_2^\dagger\phi_2)^2\right.\\
&\left.+(\phi_3^\dagger\phi_3)^2-(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)-(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)-(\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1)\right]\\
&+2r_7\left(|\phi_1^\dagger\phi_2|^2+|\phi_2^\dagger\phi_3|^2+|\phi_3^\dagger\phi_1|^2\right)\\
&+(r_1-r_4-r_7)\left((\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3)+(\phi_2^\dagger\phi_1)(\phi_3^\dagger\phi_1)+(\phi_3^\dagger\phi_2)(\phi_1^\dagger\phi_2)+h.c.\right)\,.
\end{aligned} \tag{5.258}$$

which can be easily achieved by setting from $\Delta(54)$ with the constraint $c_{11}\to r_1-r_4-r_7$.

Thus, we get the following scattering matrices.

The matrix M_2^{++}. From M_2^{++} we get

$$\frac{1}{2}M_2^{++}=\mathrm{diag}\left\{\begin{pmatrix} r_1 & \frac{r_1-r_4-r_7}{\sqrt{2}}\\ \frac{r_1-r_4-r_7}{\sqrt{2}} & r_4+r_7\end{pmatrix},\begin{pmatrix} r_4+r_7 & \frac{r_1-r_4-r_7}{\sqrt{2}}\\ \frac{r_1-r_4-r_7}{\sqrt{2}} & r_1\end{pmatrix},\begin{pmatrix} r_4+r_7 & \frac{r_1-r_4-r_7}{\sqrt{2}}\\ \frac{r_1-r_4-r_7}{\sqrt{2}} & r_1\end{pmatrix}\right\}, \tag{5.259}$$

and thus we get the eigenvalues of M_2^{++}:

$$\Lambda_{1-6}^{++}=\pm\sqrt{3}\,|-r_1+r_4+r_7|+(r_1+r_4+r_7)\,. \tag{5.260}$$

The matrix M_2^{+}. From M_2^{+} we get

$$\begin{aligned}
\frac{1}{2}M_2^{+} =\mathrm{diag}&\left\{\begin{pmatrix} r_1 & \frac{1}{2}(r_1-r_4-r_7) & \frac{1}{2}(r_1-r_4-r_7)\\ \frac{1}{2}(r_1-r_4-r_7) & r_4 & r_7\\ \frac{1}{2}(r_1-r_4-r_7) & r_7 & r_4\end{pmatrix},\right.\\
&\begin{pmatrix} r_4 & r_7 & \frac{1}{2}(r_1-r_4-r_7)\\ r_7 & r_4 & \frac{1}{2}(r_1-r_4-r_7)\\ \frac{1}{2}(r_1-r_4-r_7) & \frac{1}{2}(r_1-r_4-r_7) & r_1\end{pmatrix},\\
&\left.\begin{pmatrix} r_4 & \frac{1}{2}(r_1-r_4-r_7) & r_7\\ \frac{1}{2}(r_1-r_4-r_7) & r_1 & \frac{1}{2}(r_1-r_4-r_7)\\ r_7 & \frac{1}{2}(r_1-r_4-r_7) & r_4\end{pmatrix}\right\},
\end{aligned} \tag{5.261}$$

JHEP08(2022)273

with eigenvalues of M_2^+:

$$\begin{aligned}
\Lambda_1^{+,2} &= 2\,(r_4 - r_7)\ , \\
\Lambda_{2,3}^{+,2} &= \Lambda_{1,2}^{++}\ , \\
\Lambda_{4-6}^{+,2} &= \Lambda_{1-3}^{+,2}\,, \\
\Lambda_{7-9}^{+,2} &= \Lambda_{1-3}^{+,2}\,.
\end{aligned} \tag{5.262}$$

The matrix $\boldsymbol{M_0^+}$. From M_0^+ we get

$$\frac{1}{2}\,M_0^+ = \mathrm{diag}\left\{ \begin{pmatrix} r_1 & r_7 & r_7 \\ r_7 & r_1 & r_7 \\ r_7 & r_7 & r_1 \end{pmatrix}, \begin{pmatrix} r_4 & \frac{1}{2}(r_1-r_4-r_7) & \frac{1}{2}(r_1-r_4-r_7) \\ \frac{1}{2}(r_1-r_4-r_7) & r_4 & \frac{1}{2}(r_1-r_4-r_7) \\ \frac{1}{2}(r_1-r_4-r_7) & \frac{1}{2}(r_1-r_4-r_7) & r_4 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} r_4 & \frac{1}{2}(r_1-r_4-r_7) & \frac{1}{2}(r_1-r_4-r_7) \\ \frac{1}{2}(r_1-r_4-r_7) & r_4 & \frac{1}{2}(r_1-r_4-r_7) \\ \frac{1}{2}(r_1-r_4-r_7) & \frac{1}{2}(r_1-r_4-r_7) & r_4 \end{pmatrix} \right\}, \tag{5.263}$$

with eigenvalues of M_0^+:

$$\begin{aligned}
\Lambda_1^{+,0} &= 2\,(r_1 + 2r_7)\ , \\
\Lambda_{2,3}^{+,0} &= 2\,(r_1 - r_7)\ , \\
\Lambda_4^{+,0} &= \Lambda_2^{+,0}\,, \\
\Lambda_{5,6}^{+,0} &= -r_1 + 3r_4 + r_7\,, \\
\Lambda_{7-9}^{+,0} &= \Lambda_{4-6}^{+,0}\,.
\end{aligned} \tag{5.264}$$

The matrix $\boldsymbol{M_2^0}$. From M_2^0 we get

$$\frac{1}{2}\,M_2^0 = \frac{1}{2}\,M_2^{++}\,, \tag{5.265}$$

with eigenvalues of M_2^0:

$$\Lambda_{1-6}^{0,2} = \Lambda_{1-6}^{++}\,. \tag{5.266}$$

The matrix $\boldsymbol{M_0^0}$. From M_0^0 we get

$$\frac{1}{2}\,M_0^0 \sim \mathrm{diag}\left\{ \frac{1}{2}M_0^+, \begin{pmatrix} 3r_1 & 2r_4+r_7 & 2r_4+r_7 \\ 2r_4+r_7 & 3r_1 & 2r_4+r_7 \\ 2r_4+r_7 & 2r_4+r_7 & 3r_1 \end{pmatrix}, \right.$$
$$\frac{1}{2}\begin{pmatrix} 2(r_4+2r_7) & 3(r_1-r_4-r_7) & 3(r_1-r_4-r_7) \\ 3(r_1-r_4-r_7) & 2(r_4+2r_7) & 3(r_1-r_4-r_7) \\ 3(r_1-r_4-r_7) & 3(r_1-r_4-r_7) & 2(r_4+2r_7) \end{pmatrix},$$
$$\left. \frac{1}{2}\begin{pmatrix} 2(r_4+2r_7) & 3(r_1-r_4-r_7) & 3(r_1-r_4-r_7) \\ 3(r_1-r_4-r_7) & 2(r_4+2r_7) & 3(r_1-r_4-r_7) \\ 3(r_1-r_4-r_7) & 3(r_1-r_4-r_7) & 2(r_4+2r_7) \end{pmatrix} \right\}. \tag{5.267}$$

with eigenvalues of M_0^0:

$$\begin{aligned}
\Lambda^{0,0}_{1-9} &= \Lambda^{+,0}_{1-9}\,,\\
\Lambda^{0,0}_{10} &= 6r_1 + 8r_4 + 4r_7\,,\\
\Lambda^{0,0}_{11,12} &= 6r_1 - 2\left(2r_4 + r_7\right)\,,\\
\Lambda^{0,0}_{13} &= \Lambda^{0,0}_{11}\,,\\
\Lambda^{0,0}_{14,15} &= -3r_1 + 5r_4 + 7r_7\,,\\
\Lambda^{0,0}_{16-18} &= \Lambda^{0,0}_{13-15}\,.
\end{aligned} \tag{5.268}$$

6 Conclusions

Having found one elementary scalar particle, the most important issue is the determination of how many such scalars exist in nature. The possibility that there could be three Higgs doublets has several interesting features.

A 3HDM, in what we denote here by the symmetry-constrained $\mathbb{Z}_2 \times \mathbb{Z}_2$ version, was originally proposed by Weinberg [31], in order to have a model which simultaneously allows for CP violation and for the natural flavour conservation (NFC) mechanism [6, 7] designed to preclude flavour-changing neutral scalar exchanges. It is also the simplest NHDM where one can have the fifth type of fermion NFC couplings to scalars. Indeed, one can show that the usual NFC is only stable under the renormalization group if one single Higgs doublet has Yukawa couplings to the right-handed fermions of each electric charge [33]. This yields only five cases, dubbed in [34] types I, II, X, Y, and Z. The first four are possible in the 2HDM. The fifth, where each charged fermion sector (up-type quarks, down-type quarks, and charged leptons) couples to a different scalar, becomes possible in 3HDM (and for $N > 3$). 3HDM are also interesting because the list of all symmetry-constrained limits is known [9–12], while no such list exists (currently) for larger N.

Such models *must* obey the theoretical bounds from bounded from below potential, verification that the chosen solution of the stationarity equations is the global minimum, and perturbative unitarity. This article lists explicitly and exhaustively the perturbative unitarity conditions for all symmetry-constrained 3HDM.

We have explored the method advocated in [26] of classifying the scattering matrices by the charge Q and hypercharge $\mathcal{Y}$ of the initial/final states. If there is an additional substructure induced by the charges of the symmetry group, it is identified easily via the new algorithm we propose in appendix C, without the prior need to study the implications of each specific symmetry in detail. Appendix D will be useful for those wishing to relate the conditions in a large group with those in one of its subgroups, when the former and the latter are naturally written in different basis for the group generators.

An important part of this article is also the use of principal minors in order to obtain unitarity bounds *without* the need to perform matrix diagonalizations. This is explained in detail in section 3, with examples provided in section 4.

Together, these results will be necessary for anyone interested in the rich and varied landscape of properties and signals of 3HDM.

JHEP08(2022)273

An interesting avenue for further exploration concerns the relation between the unitarity bounds on the quartics couplings $z_{ij,kl}$ of (2.1), on the one hand, and physical scalar masses, on the other. If the vacuum expectation values (vev) of the scalar fields are non-vanishing, then, in general, the physical masses involve also the μ_{ij} couplings (to be precise, those μ_{ij} not fixed by the quartic couplings and vevs via the stationarity equations). Thus, in general, there is no direct relation. For example, the soflty broken $\mathbb{Z}_2$ 2HDM has a μ_{12} coupling which controls the decoupling limit for all masses heavier than the 125 GeV Higgs. Thus, in that case, one cannot in general find bounds on masses arising from unitarity bounds.[9] In contrast, in the $\mu_{12} = 0$, exact $\mathbb{Z}_2$-symmetric 2HDM, which does not have a decoupling limit [35], unitarity bounds do turn into bounds on scalar masses. The connection between symmetries, decoupling, and the impact on masses due to unitarity bounds could be fruitful, especially given the fact that a symmetry-constrained NHDM will exhibit decoupling if and only if the vacuum also satisfies the same symmetry [36].

An additional extension would be to consider the finite energy contributions. The results presented here are valid for energies large enough that the s-, t-, and u-channels can be ignored, but low enough that there is no significant RGE running of the quartic couplings. Indeed, as shown in [37], and used in a specific case, for example, in [38], a very precise calculation would encompass finite scattering energy contributions. This will involve cubic couplings and alter slightly the bounds on quartic couplings. These extensions lie beyond the scope of the present study.

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[9] However, as pointed out in [24], the situation is changed if the hWW coupling of the 125 GeV Higgs with the charged vector bosons has a fixed difference from the SM predictions. Such a difference, usually parametrized by $k_V - 1$, is constrained by experiment.

A Notations of the 3HDM

A.1 As in Ferreira and Silva

The notation in ref. [28] is

$$z_{ij,kl} = \begin{pmatrix} \begin{pmatrix} r_1 & c_1 & c_2 \\ c_1^* & r_4 & c_6 \\ c_2^* & c_6^* & r_5 \end{pmatrix} & \begin{pmatrix} c_1 & c_3 & c_4 \\ r_7 & c_7 & c_8 \\ c_9^* & c_{12} & c_{13} \end{pmatrix} & \begin{pmatrix} c_2 & c_4 & c_5 \\ c_9 & c_{10} & c_{11} \\ r_8 & c_{14} & c_{15} \end{pmatrix} \\ \begin{pmatrix} c_1^* & r_7 & c_9 \\ c_3^* & c_7^* & c_{12}^* \\ c_4^* & c_8^* & c_{13}^* \end{pmatrix} & \begin{pmatrix} r_4 & c_7 & c_{10} \\ c_7^* & r_2 & c_{16} \\ c_{10}^* & c_{16}^* & r_6 \end{pmatrix} & \begin{pmatrix} c_6 & c_8 & c_{11} \\ c_{12}^* & c_{16} & c_{17} \\ c_{14}^* & r_9 & c_{18} \end{pmatrix} \\ \begin{pmatrix} c_2^* & c_9^* & r_8 \\ c_4^* & c_{10}^* & c_{14}^* \\ c_5^* & c_{11}^* & c_{15}^* \end{pmatrix} & \begin{pmatrix} c_6^* & c_{12} & c_{14} \\ c_8^* & c_{16}^* & r_9 \\ c_{11}^* & c_{17}^* & c_{18}^* \end{pmatrix} & \begin{pmatrix} r_5 & c_{13} & c_{15} \\ c_{13}^* & r_6 & c_{18} \\ c_{15}^* & c_{18}^* & r_3 \end{pmatrix} \end{pmatrix} . \tag{A.1}$$

In general, the parameters c (r) are complex (real).

A.2 As in Varzielas and Ivanov

The (partial) notation in ref. [11] is:

$$z_{ij,kl} = \frac{1}{2} \begin{pmatrix} \begin{pmatrix} 2\lambda_1 & \times & \times \\ \times & \lambda_{12} & \times \\ \times & \times & \lambda_{13} \end{pmatrix} & \begin{pmatrix} \times & 2\bar{\lambda}_{12} & \lambda_6^* \\ \lambda'_{12} & \times & \times \\ \times & \lambda_7 & \times \end{pmatrix} & \begin{pmatrix} \times & \lambda_6^* & 2\bar{\lambda}_{31}^* \\ \times & \times & \lambda_5 \\ \lambda'_{13} & \bar{\lambda}'_8 & \times \end{pmatrix} \\ \begin{pmatrix} \times & \lambda'_{12} & \times \\ 2\bar{\lambda}_{12}^* & \times & \lambda_7^* \\ \lambda_6 & \times & \times \end{pmatrix} & \begin{pmatrix} \lambda_{12} & \times & \times \\ \times & 2\lambda_2 & \times \\ \times & \times & \lambda_{23} \end{pmatrix} & \begin{pmatrix} \times & \times & \lambda_5 \\ \lambda_7^* & \times & 2\bar{\lambda}_{23} \\ \bar{\lambda}'^*_8 & \lambda'_{23} & \times \end{pmatrix} \\ \begin{pmatrix} \times & \times & \lambda'_{13} \\ \lambda_6 & \times & \bar{\lambda}'^*_8 \\ 2\bar{\lambda}_{31} & \lambda_5^* & \times \end{pmatrix} & \begin{pmatrix} \times & \lambda_7 & \bar{\lambda}'_8 \\ \times & \times & \lambda'_{23} \\ \lambda_5^* & 2\bar{\lambda}_{23}^* & \times \end{pmatrix} & \begin{pmatrix} \lambda_{13} & \times & \times \\ \times & \lambda_{23} & \times \\ \times & \times & 2\lambda_3 \end{pmatrix} \end{pmatrix} , \tag{A.2}$$

where the entries denoted here by "$\times$" have not been named in ref. [11].

B Proof that $M_2^0 = M_2^{++}$

The proof is trivial and it follows from the definition of the matrices. Let

$$V_4 = \lambda_{ij,kl} (\Phi_i^\dagger \Phi_j)(\Phi_k^\dagger \Phi_l) , \tag{B.1}$$

with $\Phi_i^T = \begin{pmatrix} w_i^+ & n_i \end{pmatrix}$. Because

$$\begin{aligned} (M_2^{++})_{\alpha\beta} &= \frac{\partial^2 V_4}{\partial S_\alpha^{--} \partial S_\beta^{++}} , \\ (M_2^0)_{\alpha\beta} &= \frac{\partial^2 V_4}{\partial S_\alpha^{0*} \partial S_\beta^0} , \end{aligned} \tag{B.2}$$

JHEP08(2022)273

where $S^0_\alpha = \{n_1 n_1, n_1 n_2, n_2 n_2, \cdots, n_3 n_3\}$ and $S^{++}_\alpha = \{w_1^+ w_1^+, w_1^+ w_2^+, w_2^+ w_2^+, \cdots, w_3^+ w_3^+\}$, an exchange $n \leftrightarrow w^+$ suffices to go from one matrix to another. Thus, we only need to show that V_4 is invariant under the exchange in the doublet space. Indeed, we have for every pair $(\Phi_i^\dagger \Phi_j)$

$$\begin{aligned}\Phi_i^\dagger \Phi_j = w_i^- w_j^+ + n_i^* n_j &= \begin{pmatrix} w_i^- & n_i^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_j^+ \\ n_j \end{pmatrix} \\ &= \begin{pmatrix} n_i^* & w_i^- \end{pmatrix} \begin{pmatrix} n_j \\ w_j^+ \end{pmatrix} = \tilde{\Phi}_i^\dagger \tilde{\Phi}_j \,, \end{aligned} \tag{B.3}$$

where $\tilde{\Phi}$ is the doublet after the exchange of $n \leftrightarrow w^+$. Thus, for every NHDM we have $M_2^0 = M_2^{++}$.

C A generalized algorithm for block diagonalization

There is a procedure in which we may not even care about the hypercharge and electric charge. In section 5 we first separate the matrices into its hypercharge and electric charge charges. Then, we use an algorithm to put the matrices into block diagonal form using only permutations.

The method described in this appendix block diagonalizes an Hermitian matrix of arbitrary size. Let M be the matrix created with all possible combinations of quadratic forms $(w_i^- n_j)$, as we have done so far. The procedure is as follows.

- Build the matrix M from all combinations.
- Build a matrix P of the same size with zeros everywhere.
- Go to the first line of M and for every $M_{1j} \neq 0$, put $P_{kj} = 1$ in consecutive lines (where k runs from 1 to the number of nonzero entries in M_{1j}).
- Repeat this process until every line of P has exactly one entry equal to 1.
- Compute $\tilde{M} = PMP^T$. This matrix $\tilde{M}$ is now block diagonalized up to permutations.

Let us consider as an explicit example the matrix in eq. (5.3):

$$M = \begin{pmatrix} 2r_1 & 2\sqrt{2}c_1 & 0 & 2c_3 & 0 & 2c_5 \\ 2\sqrt{2}c_1^* & 2\,(r_4 + r_7) & 0 & 2\sqrt{2}c_7 & 0 & 2\sqrt{2}c_{11} \\ 0 & 0 & 2\,(r_5 + r_8) & 0 & 2\,(c_{13} + c_{14}) & 0 \\ 2c_3^* & 2\sqrt{2}c_7^* & 0 & 2r_2 & 0 & 2c_{17} \\ 0 & 0 & 2\,(c_{13}^* + c_{14}^*) & 0 & 2\,(r_6 + r_9) & 0 \\ 2c_5^* & 2\sqrt{2}c_{11}^* & 0 & 2c_{17}^* & 0 & 2r_3 \end{pmatrix}, \tag{C.1}$$

which arises in the M_2^{++} scattering matrix of the $\mathbb{Z}_2$-symmetric 3HDM. Then,

$$P = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0 \end{pmatrix}, \tag{C.2}$$

where:

- The first line of M has non-zero entries in columns $\{1,2,4,6\}$. Then for every line in P we put 1 for the columns $\{1,2,4,6\}$.
- The second line is equal to the first.
- The third line of M has non-zero entries in columns $\{3,5\}$. Then for every remaining line in P we put 1 for the columns $\{3,5\}$.
- We are done as there are no other unique lines in M or (equivalently) more lines in P.
- Now we compute $\tilde{M} = PMP^T$. This matrix $\tilde{M}$ is now block diagonalized up to permutations.

Thus,

$$\tilde{M} = \begin{pmatrix} r_1 & \sqrt{2}c_1 & c_3 & c_5 & 0 & 0\\ \sqrt{2}c_1^* & r_4+r_7 & \sqrt{2}c_7 & \sqrt{2}c_{11} & 0 & 0\\ c_3^* & \sqrt{2}c_7^* & r_2 & c_{17} & 0 & 0\\ c_5^* & \sqrt{2}c_{11}^* & c_{17}^* & r_3 & 0 & 0\\ 0 & 0 & 0 & 0 & r_5+r_8 & c_{13}+c_{14}\\ 0 & 0 & 0 & 0 & c_{13}^*+c_{14}^* & r_6+r_9 \end{pmatrix}. \tag{C.3}$$

This technique allows us to separate the diagonal blocks that arise from electric charge, hypercharge and global symmetries in general.

D Relating basis

The potentials are shown in section 5 choosing some particular representation for the respective symmetry. Typically, for each symmetry, we made the choice which simplifies the presentation of the quartic part of the respective symmetry-constrained potential. For example, eq. (5.39) for the $\mathbb{Z}_3$-symmetric 3HDM was written in the basis where the $\mathbb{Z}_3$ generator is represented by $\mathrm{diag}(e^{\frac{2\pi i}{3}}, e^{\frac{-2\pi i}{3}}, 1)$.

But we see from figure 1 that the A_4-symmetric 3HDM can be obtained from the $\mathbb{Z}_3$-symmetric 3HDM. When the $\mathbb{Z}_3$-symmetric 3HDM is written as in eq. (5.39), the A_4 limit arises from a complicated relation among the parameters in eq. (5.39), and, moreover, it does *not* have the simple form in eq. (5.203).

In contrast, had we written the $\mathbb{Z}_3$-symmetric 3HDM potential in the basis where the generator is written as in eq. (D.1) below, then, imposing invariance under the appropriate additional diagonal generator, $\mathrm{diag}(1, -1, -1)$, the A_4 potential would have the simple form in eq. (5.203). This is what we show next. The remaining subsections are intended to facilitate the interpretation of other limiting cases shown in figure 1. The limiting cases present in figure 1 and not covered in this appendix, are trivially found using the basis choices made in section 5.

D.1 A_4 from $\mathbb{Z}_3$

Going to A_4 from $\mathbb{Z}_3$ is easier to see with a good choice for the basis of the latter symmetry.

Let us choose the generator of $\mathbb{Z}_3$ to be

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{D.1}$$

instead of the usual diagonal form $\mathrm{diag}(\omega, \omega^2, 1)$. Then, the quartic potential is given by

$$\begin{aligned} V_{\mathbb{Z}_3} = {} & r_1 \left[(\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2 + (\phi_3^\dagger\phi_3)^2\right] \\ & + 2r_4 \left[(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + (\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + (\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)\right] \\ & + 2r_7 \left[|\phi_1^\dagger\phi_2|^2 + |\phi_2^\dagger\phi_3|^2 + |\phi_3^\dagger\phi_1|^2\right] \\ & + \Big[2c_1 \left[(\phi_1^\dagger\phi_1)(\phi_1^\dagger\phi_2) + (\phi_2^\dagger\phi_2)(\phi_2^\dagger\phi_3) + (\phi_3^\dagger\phi_3)(\phi_3^\dagger\phi_1)\right] \\ & + 2c_2 \left[(\phi_1^\dagger\phi_1)(\phi_1^\dagger\phi_3) + (\phi_2^\dagger\phi_2)(\phi_2^\dagger\phi_1) + (\phi_3^\dagger\phi_3)(\phi_3^\dagger\phi_2)\right] \\ & + c_3 \left[(\phi_1^\dagger\phi_2)^2 + (\phi_2^\dagger\phi_3)^2 + (\phi_3^\dagger\phi_1)^2\right] \\ & + 2c_4 \left[(\phi_1^\dagger\phi_2)(\phi_1^\dagger\phi_3) + (\phi_2^\dagger\phi_3)(\phi_2^\dagger\phi_1) + (\phi_3^\dagger\phi_1)(\phi_3^\dagger\phi_2)\right] \\ & + 2c_6 \left[(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_3) + (\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_1) + (\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_2)\right] \\ & + 2c_8 \left[(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_3) + (\phi_2^\dagger\phi_3)(\phi_3^\dagger\phi_1) + (\phi_3^\dagger\phi_1)(\phi_1^\dagger\phi_2)\right] + h.c.\Big]. \end{aligned} \tag{D.2}$$

This is, of course, equivalent to the usual basis for the symmetry. By enforcing, in addition, the generator $\mathrm{diag}(1, -1, -1)$, or equivalently, further removing the complex coefficients $\{c_1, c_2, c_4, c_6, c_8\}$, we get

$$\begin{aligned} V_{A_4} = {} & r_1 \left[(\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2 + (\phi_3^\dagger\phi_3)^2\right] \\ & + 2r_4 \left[(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + (\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + (\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)\right] \\ & + 2r_7 \left[|\phi_1^\dagger\phi_2|^2 + |\phi_1^\dagger\phi_3|^2 + |\phi_2^\dagger\phi_3|^2\right] \\ & + \left[c_3 \left((\phi_1^\dagger\phi_2)^2 + (\phi_2^\dagger\phi_3)^2 + (\phi_3^\dagger\phi_1)^2\right) + h.c.\right], \end{aligned} \tag{D.3}$$

which coincides with eq. (5.203).

D.2 S_4 from S_3

Going to S_4 from S_3 is easier to see with a good choice for the basis of the latter symmetry.

Let us choose the generators of S_3 to be

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \quad c = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{D.4}$$

instead of the usual diagonal form $\mathrm{diag}(\omega, \omega^2, 1)$ and c.[10] Then, the quartic potential is given by

$$\begin{aligned} V_{S_3} = {} & r_1 \left[(\phi_1^\dagger \phi_1)^2 + (\phi_2^\dagger \phi_2)^2 + (\phi_3^\dagger \phi_3)^2 \right] \\ & + 2r_4 \left[(\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + (\phi_1^\dagger \phi_1)(\phi_3^\dagger \phi_3) + (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3) \right] \\ & + 2r_7 \left[|\phi_1^\dagger \phi_2|^2 + |\phi_2^\dagger \phi_3|^2 + |\phi_3^\dagger \phi_1|^2 \right] \\ & + \Big[2c_1 \Big((\phi_1^\dagger \phi_1)(\phi_1^\dagger \phi_2) + (\phi_2^\dagger \phi_2)(\phi_2^\dagger \phi_3) + (\phi_3^\dagger \phi_3)(\phi_3^\dagger \phi_1) \\ & + (\phi_1^\dagger \phi_1)(\phi_1^\dagger \phi_3) + (\phi_2^\dagger \phi_2)(\phi_2^\dagger \phi_1) + (\phi_3^\dagger \phi_3)(\phi_3^\dagger \phi_2) \Big) + h.c. \Big] \\ & + r_{10} \left[(\phi_1^\dagger \phi_2)^2 + (\phi_2^\dagger \phi_3)^2 + (\phi_3^\dagger \phi_1)^2 + h.c. \right] \\ & + \Big[2c_4 \Big((\phi_1^\dagger \phi_2)(\phi_1^\dagger \phi_3) + (\phi_2^\dagger \phi_3)(\phi_2^\dagger \phi_1) + (\phi_3^\dagger \phi_1)(\phi_3^\dagger \phi_2) \Big) + h.c. \Big] \\ & + 2r_{11} \left[(\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_3) + (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_1) + (\phi_3^\dagger \phi_3)(\phi_1^\dagger \phi_2) + h.c. \right] \\ & + 2r_{12} \left[(\phi_1^\dagger \phi_2)(\phi_2^\dagger \phi_3) + (\phi_2^\dagger \phi_3)(\phi_3^\dagger \phi_1) + (\phi_3^\dagger \phi_1)(\phi_1^\dagger \phi_2) + h.c. \right] . \end{aligned} \tag{D.5}$$

By enforcing the generators $\mathrm{diag}(-1, 1, 1)$ and $\mathrm{diag}(1, 1, -1)$, or equivalently, removing the coefficients $\{c_1, c_4, r_{11}, r_{12}\}$ we get the potential of S_4

$$\begin{aligned} V_{S_4} = {} & r_1 \left[(\phi_1^\dagger \phi_1)^2 + (\phi_2^\dagger \phi_2)^2 + (\phi_3^\dagger \phi_3)^2 \right] \\ & + 2r_4 \left[(\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + (\phi_1^\dagger \phi_1)(\phi_3^\dagger \phi_3) + (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3) \right] \\ & + 2r_7 \left[|\phi_1^\dagger \phi_2|^2 + |\phi_2^\dagger \phi_3|^2 + |\phi_3^\dagger \phi_1|^2 \right] \\ & + r_{10} \left[(\phi_1^\dagger \phi_2)^2 + (\phi_2^\dagger \phi_3)^2 + (\phi_3^\dagger \phi_1)^2 + h.c. \right] . \end{aligned} \tag{D.6}$$

The unitarity of S_3 in the new basis, from which we go to S_4 is not trivial to compute. Although we know what the result should be, the new scattering matrices are rotated with an orthogonal transformation. Thus, they can not be trivially block diagonalized.

D.3 D_4 from $\mathbb{Z}_2 \times \mathbb{Z}_2$

Going to D_4 from $\mathbb{Z}_2 \times \mathbb{Z}_2$ is easier to see with a good choice for the basis of the latter symmetry.

[10] Notice that the rotation of the diagonal generator $\mathrm{diag}(\omega, \omega^2, 1)$ to b leaves c invariant.

Let us choose the generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$ to be

$$c' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad c = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{D.7}$$

instead of the usual diagonal forms $\mathrm{diag}(-1,-1,1)$ and $\mathrm{diag}(1,-1,-1)$. Then, the quartic potential is given by

$$\begin{aligned} V_{\mathbb{Z}_2\times\mathbb{Z}_2} = r_1 &\left[(\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2\right] + r_3|\phi_3|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) \\ &+ 2r_5(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) + 2r_7|\phi_1^\dagger\phi_2|^2 \\ &+ 2r_8\left[|\phi_1^\dagger\phi_3|^2 + |\phi_2^\dagger\phi_3|^2\right] \\ &+ \text{complex terms}\,. \end{aligned} \tag{D.8}$$

By enforcing the generator of $\mathbb{Z}_4$ given by $\mathrm{diag}(i,-i,1)$ we remove all complex coefficients except c_3 and c_{11}, which can be rephased to be real. Thus, we get the potential

$$\begin{aligned} V_{D_4} = r_1 &\left[(\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2\right] + r_3|\phi_3|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) \\ &+ 2r_5(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) + 2r_7|\phi_1^\dagger\phi_2|^2 \\ &+ 2r_8\left[|\phi_1^\dagger\phi_3|^2 + |\phi_2^\dagger\phi_3|^2\right] + r_{10}\left[(\phi_1^\dagger\phi_2)^2 + h.c.\right] \\ &+ 2r_{11}\left[(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3) + h.c.\right], \end{aligned} \tag{D.9}$$

D.4 S_3 from $\mathbb{Z}_2$

Going to S_3 from $\mathbb{Z}_2$ is easier to see with a good choice for the basis of the latter symmetry.

Let us choose the generator of $\mathbb{Z}_2$ to be

$$a_2' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{D.10}$$

instead of the usual diagonal form $\mathrm{diag}(1,1,-1)$. Then, the quartic potential is given by

$$\begin{aligned} V_{\mathbb{Z}_2} = r_1 &\left[(\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2\right] + r_3|\phi_3|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) \\ &+ 2r_5(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) + 2r_7|\phi_1^\dagger\phi_2|^2 \\ &+ 2r_8\left[|\phi_1^\dagger\phi_3|^2 + |\phi_2^\dagger\phi_3|^2\right] \\ &+ \text{complex terms}\,. \end{aligned} \tag{D.11}$$

By enforcing the generator of $\mathbb{Z}_3$ given by $\mathrm{diag}(\omega,\omega^2,1)$ we remove all remaining complex coefficients except c_{11} and c_{12}. Thus, we get the potential

$$\begin{aligned} V_{S_3} = r_1 &\left[(\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2\right] + r_3|\phi_3|^4 + 2r_4(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) \\ &+ 2r_5(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) + 2r_7|\phi_1^\dagger\phi_2|^2 + 2r_8\left[|\phi_1^\dagger\phi_3|^2 + |\phi_2^\dagger\phi_3|^2\right] \\ &+ \left[2c_{11}(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3) + 2c_{12}\Big((\phi_1^\dagger\phi_2)(\phi_3^\dagger\phi_2) + (\phi_2^\dagger\phi_1)(\phi_3^\dagger\phi_1)\Big) + h.c.\right], \end{aligned} \tag{D.12}$$

D.5 $\Sigma(36)$ from $\mathbb{Z}_4$

Going to $\Sigma(36)$ from $\mathbb{Z}_4$ is easier to see with a good choice for the basis of the latter symmetry.

Let us choose the generator of $\mathbb{Z}_4$ to be

$$d = \frac{i}{\sqrt{3}} \begin{pmatrix} \omega^2 & \omega & 1 \\ \omega & \omega^2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{D.13}$$

instead of the usual diagonal form $\mathrm{diag}(i, -i, 1)$. Then, by also using

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{D.14}$$

and $\mathrm{diag}(\omega, \omega^2, 1)$, we get the symmetry $\Sigma(36)$.

References

[1] ATLAS collaboration, *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, Phys. Lett. B **716** (2012) 1 [arXiv:1207.7214] [INSPIRE].

[2] CMS collaboration, *Observation of a New Boson at a Mass of 125 GeV with the CMS Experiment at the LHC*, Phys. Lett. B **716** (2012) 30 [arXiv:1207.7235] [INSPIRE].

[3] J.F. Gunion, H.E. Haber, G.L. Kane and S. Dawson, *The Higgs Hunter's Guide*, *Front. Phys.* **80** (2000) [INSPIRE].

[4] G.C. Branco, P.M. Ferreira, L. Lavoura, M.N. Rebelo, M. Sher and J.P. Silva, *Theory and phenomenology of two-Higgs-doublet models*, Phys. Rept. **516** (2012) 1 [arXiv:1106.0034] [INSPIRE].

[5] I.P. Ivanov, *Building and testing models with extended Higgs sectors*, Prog. Part. Nucl. Phys. **95** (2017) 160 [arXiv:1702.03776] [INSPIRE].

[6] S.L. Glashow and S. Weinberg, *Natural Conservation Laws for Neutral Currents*, Phys. Rev. D **15** (1977) 1958 [INSPIRE].

[7] E.A. Paschos, *Diagonal Neutral Currents*, Phys. Rev. D **15** (1977) 1966 [INSPIRE].

[8] I.P. Ivanov, *Minkowski space structure of the Higgs potential in 2HDM. II. Minima, symmetries, and topology*, Phys. Rev. D **77** (2008) 015017 [arXiv:0710.3490] [INSPIRE].

[9] I.P. Ivanov and E. Vdovin, *Classification of finite reparametrization symmetry groups in the three-Higgs-doublet model*, Eur. Phys. J. C **73** (2013) 2309 [arXiv:1210.6553] [INSPIRE].

JHEP08(2022)273

[10] I.P. Ivanov and C.C. Nishi, *Symmetry breaking patterns in 3HDM*, *JHEP* **01** (2015) 021 [arXiv:1410.6139] [INSPIRE].

[11] I. de Medeiros Varzielas and I.P. Ivanov, *Recognizing symmetries in a 3HDM in a basis-independent way*, *Phys. Rev. D* **100** (2019) 015008 [arXiv:1903.11110] [INSPIRE].

[12] N. Darvishi and A. Pilaftsis, *Classifying Accidental Symmetries in Multi-Higgs Doublet Models*, *Phys. Rev. D* **101** (2020) 095008 [arXiv:1912.00887] [INSPIRE].

[13] N.G. Deshpande and E. Ma, *Pattern of Symmetry Breaking with Two Higgs Doublets*, *Phys. Rev. D* **18** (1978) 2574 [INSPIRE].

[14] I.P. Ivanov, *Minkowski space structure of the Higgs potential in 2HDM*, *Phys. Rev. D* **75** (2007) 035001 [*Erratum ibid.* **76** (2007) 039902] [hep-ph/0609018] [INSPIRE].

[15] M. Maniatis, A. von Manteuffel, O. Nachtmann and F. Nagel, *Stability and symmetry breaking in the general two-Higgs-doublet model*, *Eur. Phys. J. C* **48** (2006) 805 [hep-ph/0605184] [INSPIRE].

[16] K. Kannike, *Vacuum Stability Conditions From Copositivity Criteria*, *Eur. Phys. J. C* **72** (2012) 2093 [arXiv:1205.3781] [INSPIRE].

[17] M. Maniatis and O. Nachtmann, *Stability and symmetry breaking in the general n-Higgs-doublet model*, *Phys. Rev. D* **92** (2015) 075017 [arXiv:1504.01736] [INSPIRE].

[18] I.P. Ivanov and J.P. Silva, *Tree-level metastability bounds for the most general two Higgs doublet model*, *Phys. Rev. D* **92** (2015) 055017 [arXiv:1507.05100] [INSPIRE].

[19] B.W. Lee, C. Quigg and H.B. Thacker, *Weak Interactions at Very High-Energies: The Role of the Higgs Boson Mass*, *Phys. Rev. D* **16** (1977) 1519 [INSPIRE].

[20] B.W. Lee, C. Quigg and H.B. Thacker, *The Strength of Weak Interactions at Very High-Energies and the Higgs Boson Mass*, *Phys. Rev. Lett.* **38** (1977) 883 [INSPIRE].

[21] I.F. Ginzburg and I.P. Ivanov, *Tree level unitarity constraints in the 2HDM with CP-violation*, hep-ph/0312374 [INSPIRE].

[22] S. Kanemura, T. Kubota and E. Takasugi, *Lee-Quigg-Thacker bounds for Higgs boson masses in a two doublet model*, *Phys. Lett. B* **313** (1993) 155 [hep-ph/9303263] [INSPIRE].

[23] I.F. Ginzburg and I.P. Ivanov, *Tree-level unitarity constraints in the most general 2HDM*, *Phys. Rev. D* **72** (2005) 115010 [hep-ph/0508020] [INSPIRE].

[24] S. Kanemura and K. Yagyu, *Unitarity bound in the most general two Higgs doublet model*, *Phys. Lett. B* **751** (2015) 289 [arXiv:1509.06060] [INSPIRE].

[25] D. Das and U.K. Dey, *Analysis of an extended scalar sector with S_3 symmetry*, *Phys. Rev. D* **89** (2014) 095025 [*Erratum ibid.* **91** (2015) 039905] [arXiv:1404.2491] [INSPIRE].

[26] M.P. Bento, H.E. Haber, J.C. Romão and J.P. Silva, *Multi-Higgs doublet models: physical parametrization, sum rules and unitarity bounds*, *JHEP* **11** (2017) 095 [arXiv:1708.09408] [INSPIRE].

[27] S. Moretti and K. Yagyu, *Constraints on Parameter Space from Perturbative Unitarity in Models with Three Scalar Doublets*, *Phys. Rev. D* **91** (2015) 055022 [arXiv:1501.06544] [INSPIRE].

[28] P.M. Ferreira and J.P. Silva, *Discrete and continuous symmetries in multi-Higgs-doublet models*, *Phys. Rev. D* **78** (2008) 116007 [arXiv:0809.2788] [INSPIRE].

[29] D.S. Bernstein, *Scalar, Vector, and Matrix Mathematics*, Princeton University Press (2018).

[30] J. Hernandez-Sanchez, V. Keus, S. Moretti, D. Rojas-Ciofalo and D. Sokolowska, *Complementary Probes of Two-component Dark Matter*, arXiv:2012.11621 [INSPIRE].

[31] S. Weinberg, *Gauge Theory of CP-violation*, Phys. Rev. Lett. **37** (1976) 657 [INSPIRE].

[32] G.C. Branco, *Spontaneous CP Nonconservation and Natural Flavor Conservation: A Minimal Model*, Phys. Rev. D **22** (1980) 2901 [INSPIRE].

[33] P.M. Ferreira, L. Lavoura and J.P. Silva, *Renormalization-group constraints on Yukawa alignment in multi-Higgs-doublet models*, Phys. Lett. B **688** (2010) 341 [arXiv:1001.2561] [INSPIRE].

[34] K. Yagyu, *Higgs boson couplings in multi-doublet models with natural flavour conservation*, Phys. Lett. B **763** (2016) 102 [arXiv:1609.04590] [INSPIRE].

[35] J.F. Gunion and H.E. Haber, *The CP conserving two Higgs doublet model: The Approach to the decoupling limit*, Phys. Rev. D **67** (2003) 075019 [hep-ph/0207010] [INSPIRE].

[36] S. Carrolo, J.C. Romão, J.P. Silva and F. Vazão, *Symmetry and decoupling in multi-Higgs boson models*, Phys. Rev. D **103** (2021) 075026 [arXiv:2102.11303] [INSPIRE].

[37] M.D. Goodsell and F. Staub, *Unitarity constraints on general scalar couplings with SARAH*, Eur. Phys. J. C **78** (2018) 649 [arXiv:1805.07306] [INSPIRE].

[38] J. Kalinowski, W. Kotlarski, M.N. Rebelo and I. de Medeiros Varzielas, *3HDM with* $\Delta(27)$ *symmetry and its phenomenological consequences*, arXiv:2112.12699 [INSPIRE].