Advanced Quantum Field Theory

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Preface

This is a text for an *Advanced Quantum Field Theory* course that I have been teaching for many years at Instituto Superior Técnico, in Lisbon. This course was first written in Portuguese. Then, at a latter stage, I added some text in one-loop techniques in English. Then, I realized that this text could be more useful if it was all in English. The process took some years but the text it is now almost all in English. An effort has been made to correct known misprints. However, I am certain that many more still remain. If you find errors or misprints, please send me an email.

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Chapter 1

Free Field Quantization

1.1 General formalism

1.1.1 Canonical quantization for particles

Before we study the canonical quantization of systems with an infinite number of degrees of freedom, as it is the case with fields, we will review briefly the quantization of systems with a finite number of degrees of freedom, like a system of particles.

Let us start with a system that consists of one particle with just one degree of freedom, like a particle moving in one space dimension. The classical equations of motion are obtained from the action,

$$S = \int_{t_1}^{t_2} dt L(q, \dot{q}) \ . \tag{1.1}$$

The condition for the minimization of the action, $\delta S = 0$, gives the Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{1.2}$$

which are the equations of motion.

Before proceeding to the quantization, it is convenient to change to the Hamiltonian formulation. We start by defining the conjugate momentum p, to the coordinate q, by

$$p = \frac{\partial L}{\partial \dot{q}} \tag{1.3}$$

Then we introduce the Hamiltonian using the Legendre transform

$$H(p, q) = p\dot{q} - L(q, \dot{q}) \tag{1.4}$$

In terms of H the equations of motion are,

$$\{H, q\}_{\rm PB} = \frac{\partial H}{\partial p} = \dot{q} \tag{1.5}$$

$$\{H, p\}_{\rm PB} = -\frac{\partial H}{\partial q} = \dot{p} \tag{1.6}$$

where the Poisson Bracket (PB) is defined by

$$\{f(p,q), g(p,q)\}_{\rm PB} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} \tag{1.7}$$

obviously satisfying

$$\{p, q\}_{\rm PB} = 1 \; . \tag{1.8}$$

The quantization is done by promoting p and q to hermitian operators that instead of Eq. (1.8) will satisfy the commutation relation ($\hbar = 1$),

$$[p, q] = -i \tag{1.9}$$

which is trivially satisfied in the coordinate representation where $p = -i\dfrac{\partial}{\partial q}$. The dynamics is the given by the Schrödinger equation

$$H(p,q) \left|\Psi_S(t)\right\rangle = i\frac{\partial}{\partial t} \left|\Psi_S(t)\right\rangle \tag{1.10}$$

If we know the state of the system in $t = 0, \left|\Psi_S(0)\right\rangle$, then Eq. (1.10) completely determines the state $\left|\Psi_s(t)\right\rangle$ and therefore the value of any physical observable. This description, where the states are time dependent and the operators, on the contrary, do not depend on time, is known as the Schrödinger representation. There exits and alternative description, where the time dependence goes to the operators and the states are time independent. This is called the Heisenberg representation. To define this representation, we formally integrate Eq. (1.10) to obtain

$$\left|\Psi_S(t)\right\rangle = e^{-iHt} \left|\Psi_S(0)\right\rangle = e^{-iHt} \left|\Psi_H\right\rangle \; . \tag{1.11}$$

The state in the Heisenberg representation, $\left|\Psi_H\right\rangle$, is defined as the state in the Schrödinger representation for $t = 0$. The unitary operator e^{-iHt} allows us to go from one representation to the other. If we define the operators in the Heisenberg representation as,

$$O_H(t) = e^{iHt} O_S e^{-iHt} \tag{1.12}$$

then the matrix elements are representation independent. In fact,

$$\left\langle\Psi_S(t)\right|O_S\left|\Psi_S(t)\right\rangle \;\; = \;\; \left\langle\Psi_S(0)\right|e^{iHt}O_S e^{-iHt}\left|\Psi_S(0)\right\rangle \tag{1.13}$$

$$= \langle \Psi_H | O_H(t) | \Psi_H \rangle \ . \tag{1.14}$$

The time evolution of the operator $O_H(t)$ is then given by the equation

$$\frac{dO_H(t)}{dt} = i[H, O_H(t)] + \frac{\partial O_H}{\partial t} \tag{1.15}$$

which can easily be obtained from Eq. (1.12). The last term in Eq. (1.15) is only present if O_S explicitly depends on time.

In the non-relativistic theory the difference between the two representations is very small if we work with energy eigenfunctions. If $\psi_n(q,t) = e^{-i\omega_n t} u_n(q)$ is a Schrödinger wave function, then the Heisenberg wave function is simply $u_n(q)$. For the relativistic theory, the Heisenberg representation is more convenient, because it is easier to describe the time evolution of operators than that of states. Also, Lorentz covariance is more easily handled in the Heisenberg representation, because time and spatial coordinates are together in the field operators.

In the Heisenberg representation the fundamental commutation relation is now

$$[p(t), q(t)] = -i \tag{1.16}$$

The dynamics is now given by

$$\frac{dp(t)}{dt} = i[H, p(t)] \quad ; \quad \frac{dq(t)}{dt} = i[H, q(t)] \tag{1.17}$$

Notice that in this representation the fundamental equations are similar to the classical equations with the substitution,

$$\{,\}_{PB} \Longrightarrow i[,] \tag{1.18}$$

In the case of a system with n degrees of freedom Eqs. (1.16) and (1.17) are generalized to

$$[p_i(t), q_j(t)] = -i\delta_{ij} \tag{1.19}$$

$$[p_i(t), p_j(t)] = 0 \tag{1.20}$$

$$[q_i(t), q_j(t)] = 0 \tag{1.21}$$

and

$$\dot{p}_i(t) = i[H, p_i(t)] \ ; \ \dot{q}_i(t) = i[H, q_i(t)] \tag{1.22}$$

Because it is an important example let us look at the harmonic oscillator. The Hamiltonian is

$$H = \frac{1}{2}(p^2 + \omega_0^2 q^2) \tag{1.23}$$

The equations of motion are

$$\dot{p} = i[H, p] = -\omega_0^2 q \tag{1.24}$$

$$\dot{q} = i[H, q] = p \Longrightarrow \ddot{q} + \omega_0^2 q = 0 \ . \tag{1.25}$$

It is convenient to introduce the operators

$$a = \frac{1}{\sqrt{2\omega_0}}(\omega_0 q + ip) \ ; \ a^\dagger = \frac{1}{\sqrt{2\omega_0}}(\omega_0 q - ip) \tag{1.26}$$

The equations of motion for a and $a^\dagger$ are very simple:

$$\dot{a}(t) = -i\omega_0 a(t) \text{ e } \dot{a}^\dagger(t) = i\omega_0 a^\dagger(t) \ . \tag{1.27}$$

They have the solution

$$a(t) = a_0 e^{-i\omega_0 t} \ ; \ a^\dagger(t) = a_0^\dagger e^{i\omega_0 t} \tag{1.28}$$

and obey the commutation relations

$$[a, a^\dagger] = [a_0, a_0^\dagger] = 1 \tag{1.29}$$

$$[a, a] = [a_0, a_0] = 0 \tag{1.30}$$

$$[a^\dagger, a^\dagger] = [a_0^\dagger, a_0^\dagger] = 0 \tag{1.31}$$

In terms of $a, a^\dagger$ the Hamiltonian reads

$$H = \frac{1}{2} \, \omega_0 (a^\dagger a + a a^{\dagger)} = \frac{1}{2} \, \omega_0 (a_0^\dagger a_0 + a_0 a_0^\dagger) \tag{1.32}$$

$$= \omega_0 a_0^\dagger a_0 + \frac{1}{2} \, \omega_0 \tag{1.33}$$

where we have used

$$[H, a_0] = -\omega_0 a_0, \quad [H, a_0^\dagger] = \omega_0 a_0^\dagger \tag{1.34}$$

We see that a_0 decreases the energy of a state by the quantity ω_0 while $a_0^\dagger$ increases the energy by the same amount. As the Hamiltonian is a sum of squares the eigenvalues must be positive. Then it should exist a ground state (state with the lowest energy), $|0\rangle$, defined by the condition

$$a_0 \, |0\rangle = 0 \tag{1.35}$$

The state $|n\rangle$ is obtained by the application of $\left(a_0^\dagger\right)^n$. If we define

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(a_0^\dagger\right)^n |0\rangle \tag{1.36}$$

then

$$\langle m|n\rangle = \delta_{mn} \tag{1.37}$$

and

$$H \, |n\rangle = \left(n + \frac{1}{2}\right) \omega_0 \, |n\rangle \tag{1.38}$$

We will see that, in the quantum field theory, the equivalent of a_0 and $a_0^\dagger$ are the creation and annihilation operators.

1.1.2 Canonical quantization for fields

Let us move now to field theory, that is, systems with an infinite number of degrees of freedom. To specify the state of the system, we must give for all space-time points one number (or more if we are not dealing with a scalar field). The equivalent of the coordinates $q_i(t)$ and velocities, $\dot{q}_i$, are here the fields $\varphi(\vec{x},t)$ and their derivatives, $\partial^\mu\varphi(\vec{x},t)$. The action is now

$$S=\int d^4x\mathcal{L}(\varphi,\partial^\mu\varphi) \tag{1.39}$$

where the Lagrangian density $\mathcal{L}$, is a functional of the fields φ and their derivatives $\partial^\mu\varphi$. Let us consider closed systems for which $\mathcal{L}$ does not depend explicitly on the coordinates x^μ (energy and linear momentum are therefore conserved). For simplicity let us consider systems described by n scalar fields $\varphi_r(x), r=1,2,\cdots n$. The stationarity of the action, $\delta S=0$, implies the equations of motion, the so-called Euler-Lagrange equations,

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}-\frac{\partial\mathcal{L}}{\partial\varphi_r}=0 \qquad r=1,\cdots n \tag{1.40}$$

For the case of real scalar fields with no interactions that we are considering, we can easily see that the Lagrangian density should be,

$$\mathcal{L}=\sum_{r=1}^{n}\left[\frac{1}{2}\partial^\mu\varphi_r\partial_\mu\varphi_r-\frac{1}{2}m^2\varphi_r\varphi_r\right] \tag{1.41}$$

in order to obtain the Klein-Gordon equations as the equations of motion,

$$(\Box+m^2)\varphi_r=0 \quad ; \quad r=1,\cdots n \tag{1.42}$$

To define the canonical quantization rules we have to change to the Hamiltonian formalism, in particular we need to define the conjugate momentum $\pi(x)$ for the field $\varphi(x)$. To make an analogy with systems with n degrees of freedom, we divide the 3-dimensional space in cells with elementary volume ΔV_i. Then we introduce the coordinate $\varphi_i(t)$ as the average of $\varphi(\vec{x},t)$ in the volume element ΔV_i, that is,

$$\varphi_i(t)\equiv\frac{1}{\Delta V_i}\int_{(\Delta V_i)}d^3x\varphi(\vec{x},t) \tag{1.43}$$

and also

$$\dot{\varphi}_i(t)\equiv\frac{1}{\Delta V_i}\int_{(\Delta V_i)}d^3x\dot{\varphi}(\vec{x},t)\ . \tag{1.44}$$

Then

$$L=\int d^3x\mathcal{L}\rightarrow\sum_i\Delta V_i\overline{\mathcal{L}}_i\ . \tag{1.45}$$

Therefore the canonical momentum is now

$$p_i(t)=\frac{\partial L}{\partial\dot{\varphi}_i(t)}=\Delta V_i\frac{\partial\overline{\mathcal{L}}_i}{\partial\dot{\varphi}_i(t)}\equiv\Delta V_i\pi_i(t) \tag{1.46}$$

and the Hamiltonian

$$H = \sum_i p_i \dot{\varphi}_i - L = \sum_i \Delta V_i (\pi_i \dot{\varphi}_i - \overline{\mathcal{L}}_i) \tag{1.47}$$

Going now into the limit of the continuum, we define the conjugate momentum,

$$\pi(\vec{x}, t) \equiv \frac{\partial \mathcal{L}(\varphi, \dot{\varphi})}{\partial \dot{\varphi}(\vec{x}, t)} \tag{1.48}$$

in such a way that its average value in ΔV_i is $\pi_i(t)$ defined in Eq. (1.46). Eq. (1.47) suggests the introduction of an Hamiltonian density such that

$$H = \int d^3x \mathcal{H} \tag{1.49}$$

$$\mathcal{H} = \pi \dot{\varphi} - \mathcal{L} \; . \tag{1.50}$$

To define the rules of the canonical quantization we start with the coordinates $\varphi_i(t)$ and conjugate momenta $p_i(t)$. We have

$$\begin{aligned} &[p_i(t), \varphi_j(t)] = -i\delta_{ij} \\ &[\varphi_i(t), \varphi_j(t)] = 0 \\ &[p_i(t), p_j(t)] = 0 \end{aligned} \tag{1.51}$$

In terms of momentum $\pi_i(t)$ we have

$$[\pi_i(t), \varphi_j(t)] = -i \frac{\delta_{ij}}{\Delta V_i} \; . \tag{1.52}$$

Going into the continuum limit, $\Delta V_i \to 0$, we obtain

$$[\varphi(\vec{x}, t), \varphi(\vec{x}', t)] = 0 \tag{1.53}$$

$$[\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0 \tag{1.54}$$

$$[\pi(\vec{x}, t), \varphi(\vec{x}', t)] = -i\delta(\vec{x} - \vec{x}') \tag{1.55}$$

These relations are the basis of the canonical quantization. For the case of n scalar fields, the generalization is:

$$[\varphi_r(\vec{x}, t), \varphi_s(\vec{x}', t)] = 0 \tag{1.56}$$

$$[\pi_r(\vec{x}, t), \pi_s(\vec{x}', t)] = 0 \tag{1.57}$$

$$[\pi_r(\vec{x}, t), \varphi_s(\vec{x}', t)] = -i\delta_{rs}\delta(\vec{x} - \vec{x}') \tag{1.58}$$

where

$$\pi_r(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_r(\vec{x}, t)} \tag{1.59}$$

and the Hamiltonian is

$$H = \int d^3x \mathcal{H} \tag{1.60}$$

with

$$\mathcal{H} = \sum_{r=1}^{n} \pi_r \dot{\varphi}_r - \mathcal{L} \; . \tag{1.61}$$

1.1.3 Symmetries and conservation laws

The Lagrangian formalism gives us a powerful method to relate symmetries and conservation laws. At the classical level the fundamental result is the following theorem.

Noether's Theorem

To each continuous symmetry transformation that leaves $\mathcal{L}$ and the equations of motion invariant, corresponds one conservation law.

Proof:

We will make a generic proof for the most general case, and then consider particular cases. We take a general change of inertial frame, including Lorentz transformations and translations. For infinitesimal transformations we have

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu} \equiv x^{\mu} + \delta x^{\mu}, \quad \delta x^{\mu} = \varepsilon^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu} \tag{1.62}$$

where ε^{μ} and $\omega^{\mu\nu}$ are infinitesimal constant parameters. For the fields, under such a transformation we have two types of variations.

$$\delta\varphi_r(x) \equiv \varphi'_r(x) - \varphi_r(x) \tag{1.63}$$

$$\delta_{\mathrm{T}}\varphi_r(x) \equiv \varphi'_r(x') - \varphi_r(x) \tag{1.64}$$

They are related because (we neglect second order terms)

$$\begin{aligned} \delta_T\varphi_r(x) =& \left[\varphi'_r(x') - \varphi_r(x')\right] + \left[\varphi_r(x') - \varphi_r(x)\right] \\ =& \delta\varphi_r(x') + \frac{\partial\varphi_r}{\partial x_\beta}\delta x_\beta = \delta\varphi_r(x) + \frac{\partial\varphi_r}{\partial x_\beta}\delta x_\beta \end{aligned} \tag{1.65}$$

Now the invariance of the Lagrangian

$$\mathcal{L}(\varphi'_r(x'), \partial_\alpha\varphi'(x')) = \mathcal{L}(\varphi_r(x), \partial_\alpha\varphi(x)) \tag{1.66}$$

can be written as

$$\begin{aligned} 0 =& \mathcal{L}(\varphi'_r(x'), \partial_\alpha\varphi'(x')) - \mathcal{L}(\varphi_r(x), \partial_\alpha\varphi(x)) \\ =& \delta\mathcal{L} + \frac{\partial\mathcal{L}}{\partial x^\beta}\delta x^\beta \end{aligned} \tag{1.67}$$

Now we calculate $\delta\mathcal{L}$ (using the equations of motion)

$$\begin{aligned} \delta\mathcal{L} =& \frac{\partial\mathcal{L}}{\partial\varphi_r}\delta\varphi_r + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\varphi_r)}\delta(\partial_\alpha\varphi_r) \\ =& \frac{\partial}{\partial x_\alpha}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\varphi_r)}\delta\varphi_r\right] \\ =& \frac{\partial}{\partial x_\alpha}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\varphi_r)}\delta_T\varphi_r - \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\varphi_r)}\frac{\partial\varphi_r}{\partial x^\beta}\delta x^\beta\right] \end{aligned} \tag{1.68}$$

where we have used Eq. (1.65). Introducing now Eq. (1.68) in Eq. (1.67) we obtain

$$
\begin{aligned}
0 =& \frac{\partial}{\partial x_\alpha}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi_r)}\delta_T\varphi_r - \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi_r)}\frac{\partial \varphi_r}{\partial x^\beta} - \mathcal{L}g^{\alpha\beta}\right)\delta x^\beta\right]\\
=& \frac{\partial}{\partial x_\alpha}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi_r)}\delta_T\varphi_r - T^{\alpha\beta}\delta x^\beta\right]\\
=& \partial_\alpha J^\alpha
\end{aligned}
\tag{1.69}
$$

We have defined the conserved current J^α and the tensor $T^{\alpha\beta}$ by

$$
T^{\alpha\beta} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi_r)}\frac{\partial \varphi_r}{\partial x^\beta} - \mathcal{L}g^{\alpha\beta} \tag{1.70}
$$

$$
J^\alpha \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi_r)}\delta_T\varphi_r - T^{\alpha\beta}\delta x_\beta \tag{1.71}
$$

This ends the proof of Noether's theorem.

Before we apply it to particular cases let us also introduce some useful notation. We define for infinitesimal transformations

$$
\varphi'_r(x') = S_{rs}(a)\varphi_s(x), \quad S_{rs}(\omega) = \delta_{rs} + \frac{1}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta}_{rs} \rightarrow \delta_T\varphi_r(x) = \frac{1}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta}_{rs}\varphi_s \tag{1.72}
$$

1) Translations

First we consider the case of translations. For this case we have

$$
\delta_T\varphi_r = 0, \quad \delta x^\mu = \varepsilon^\mu \tag{1.73}
$$

From the above and using the fact that ε^μ is arbitrary and constant we get from Eq. (1.71)

$$
\partial_\mu J^\mu = 0 \longrightarrow \partial_\mu T^{\mu\nu} = 0 \tag{1.74}
$$

where $T^{\mu\nu}$ is the *energy-momentum tensor* defined above in Eq. (1.70),

$$
T^{\mu\nu} = -g^{\mu\nu}\mathcal{L} + \sum_r \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)}\partial^\nu\, \varphi_r \tag{1.75}
$$

Using these relations we can define the conserved quantities

$$
P^\mu \equiv \int d^3x T^{0\mu} \quad \Rightarrow \quad \frac{dP^\mu}{dt} = 0 \tag{1.76}
$$

Noticing that $T^{00} = \mathcal{H}$, it is easy to realize that P^μ should be the 4-momentum vector. Therefore we conclude that invariance for translations leads to the conservation of energy and momentum.

2) Lorentz transformations

Consider the infinitesimal Lorentz transformations

$$x'^{\mu} = x^{\mu} + \omega^{\mu}{}_{\nu}\, x^{\nu}, \quad \delta_T \varphi_r(x) = \frac{1}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta}_{rs}\varphi_s \tag{1.77}$$

where we have indicated the total variation of the fiels using the conventions of Eq. (1.72). For instance for scalar fields we have $\delta_T\varphi_r(x) = 0$ and for spinors

$$\delta_T\psi_r(x) = \frac{1}{8}[\gamma_\mu, \gamma_\nu]_{rs}\omega^{\mu\nu}\psi_s(x) \tag{1.78}$$

Inserting these variations in the conserved current, Eq. (1.71), and factoring out the constant parameters $\omega_{\alpha\beta}$ we obtain

$$\partial_\mu M^{\mu\alpha\beta} = 0 \quad \text{with} \quad M^{\mu\alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi_r)}\Sigma^{\alpha\beta}_{rs}\varphi_s \tag{1.79}$$

The conserved angular momentum is then

$$M^{\alpha\beta} = \int d^3x M^{0\alpha\beta} = \int d^3x \left[x^\alpha T^{0\beta} - x^\beta T^{0\alpha} + \sum_{r,s}\pi_r\Sigma^{\alpha\beta}_{rs}\varphi_s\right] \tag{1.80}$$

with

$$\frac{dM^{\alpha\beta}}{dt} = 0 \tag{1.81}$$

3) Internal Symmetries

Let us consider that the Lagrangian is invariant for an infinitesimal internal symmetry transformation

$$\delta_T\varphi_r(x) = -i\varepsilon\lambda_{rs}\varphi_s(x), \quad \delta x^\mu = 0 \tag{1.82}$$

where we explicitely indicate that there are no chnage in the coordinates, only in the fields. Then substituting in the current we easily obtain

$$\partial_\mu J^\mu = 0 \quad \text{where} \quad J^\mu = -i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}\lambda_{rs}\varphi_s \tag{1.83}$$

This leads to the conserved charge

$$Q(\lambda) = -i\int d^3x\pi_r\lambda_{rs}\varphi_s \quad ; \quad \frac{dQ}{dt} = 0 \tag{1.84}$$

These relations between symmetries and conservation laws were derived for the classical theory. Let us see now what happens when we quantize the theory. In the quantum theory the fields $\varphi_r(x)$ become operators acting on the Hilbert space of the states. The physical observables are related with the matrix elements of these operators. We have therefore to require Lorentz covariance for those matrix elements. This in turn requires that the operators have to fulfill certain conditions.

This means that the classical fields relation

$$\varphi_r'(x') = S_{rs}(a)\varphi_s(x) \tag{1.85}$$

should be in the quantum theory[1]

$$\left\langle \Phi_\alpha' | \varphi_r(x') | \Phi_\beta' \right\rangle = S_{rs}(a) \left\langle \Phi_\alpha | \varphi_s(x) | \Phi_\beta \right\rangle \tag{1.86}$$

It should exist an unitary transformation $U(a,b)$ that should relate the two inertial frames

$$\left| \Phi' \right\rangle = U(a,b) \left| \Phi \right\rangle \tag{1.87}$$

where $a^\mu{}_\nu$ e b^μ are defined by

$$x'^\mu = a^\mu{}_\nu x^\nu + b^\mu \tag{1.88}$$

Using Eq. (1.87) in Eq. (1.86) we get that the field operators should transform as

$$U(a,b)\varphi_r(x)U^{-1}(a,b) = S^{(-1)}_{rs}(a)\varphi_s(ax+b) \tag{1.89}$$

Let us look at the consequences of this relation for translations and Lorentz transformations. We consider first the translations. Eq. (1.89) is then

$$U(b)\varphi_r(x)U^{-1}(b) = \varphi_r(x+b) \tag{1.90}$$

For infinitesimal translations we can write

$$U(\varepsilon) \equiv e^{i\varepsilon_\mu \mathcal{P}^\mu} \simeq 1 + i\varepsilon_\mu \mathcal{P}^\mu \tag{1.91}$$

where $\mathcal{P}^\mu$ is an hermitian operator. Then Eq. (1.90) gives

$$i[\mathcal{P}^\mu, \varphi_r(x)] = \partial^\mu \varphi_r(x) \tag{1.92}$$

The correspondence with classical mechanics and non relativistic quantum theory suggests that we identify $\mathcal{P}^\mu$ with the 4-momentum, that is, $\mathcal{P}^\mu \equiv P^\mu$ where P^μ has been defined in Eq. (1.76).

As we have an explicit expression for P^μ and we know the commutation relations of the quantum theory, the Eq. (1.92) becomes an additional requirement that the theory has to verify in order to be invariant under translations. We will see explicitly that this is indeed the case for the theories in which we are interested.

For Lorentz transformations $x'^\mu = a^\mu{}_\nu\, x^\nu$, we write for an infinitesimal transformation

$$a^\mu{}_\nu = g^\mu_\nu + \omega^\mu{}_\nu + O(\omega^2) \tag{1.93}$$

and therefore

$$U(\omega) \equiv 1 - \frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu} \tag{1.94}$$

We then obtain from Eq. (1.89) the requirement

$$i[\mathcal{M}^{\mu\nu}, \varphi_r(x)] = x^\mu \partial^\nu \varphi_r - x^\nu \partial^\mu \varphi_r + \Sigma^{\mu\nu}_{rs}\varphi_s(x) \tag{1.95}$$

Once more the classical correspondence lead us to identify $\mathcal{M}^{\mu\nu} = M^{\mu\nu}$ where the angular momentum $M^{\mu\nu}$ is defined in Eq. (1.80). For each theory we will have to verify Eq. (1.95) for the theory to be invariant under Lorentz transformations. We will see that this is true for the cases of interest.

[1] This is a definition not a derivation from Eq. (1.85). See[1, 2]

1.2 Quantization of scalar fields

1.2.1 Real scalar field

The real scalar field described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi \varphi \tag{1.96}$$

to which corresponds the Klein-Gordon equation

$$(\Box + m^2)\varphi = 0 \tag{1.97}$$

is the simplest example, and in fact was already used to introduce the general formalism. As we have seen the conjugate momentum is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi} \tag{1.98}$$

and the commutation relations are

$$\begin{aligned} &[\varphi(\vec{x},t), \varphi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0 \\ &[\pi(\vec{x},t), \varphi(\vec{x}',t)] = -i\delta^3(\vec{x}-\vec{x}') \end{aligned} \tag{1.99}$$

The Hamiltonian is given by,

$$\begin{aligned} H &= P^0 = \int d^3x \mathcal{H} \\ &= \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}|\vec{\nabla}\varphi|^2 + \frac{1}{2}m^2\varphi^2\right] \end{aligned} \tag{1.100}$$

and the linear momentum is

$$\vec{P} = -\int d^3x \pi \vec{\nabla}\varphi \tag{1.101}$$

Using Eqs. (1.100) and (1.101) it is easy to verify that

$$i[P^\mu, \varphi] = \partial^\mu \varphi \tag{1.102}$$

showing the invariance of the theory for the translations. In the same way we can verify the invariance under Lorentz transformations, Eq. (1.95), with $\Sigma^{\mu\nu}_{rs} = 0$ (spin zero).

In order to define the states of the theory it is convenient to have eigenstates of energy and momentum. To build these states we start by making a spectral Fourier decomposition of $\varphi(\vec{x},t)$ in plane waves:

$$\varphi(\vec{x},t) = \int \widetilde{dk} \ \left[a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right] \tag{1.103}$$

where

$$\widetilde{dk} \equiv \frac{d^3k}{(2\pi)^3 2\omega_k} \ ; \ \omega_k = +\sqrt{|\vec{k}|^2 + m^2} \tag{1.104}$$

is the Lorentz invariant integration measure. As in the quantum theory φ is an operator, also $a(k)$ e $a^\dagger(k)$ should be operators. As φ is real, then $a^\dagger(k)$ should be the hermitian conjugate to $a(k)$. In order to determine their commutation relations we start by solving Eq. (1.103) in order to $a(k)$ and $a^\dagger(k)$. Using the properties of the delta function, we get

$$a(k) = i\int d^3x e^{ik\cdot x}\overleftrightarrow{\partial}_0\varphi(x)$$

$$a^\dagger(k) = -i\int d^3x e^{-ik\cdot x}\overleftrightarrow{\partial}_0\varphi(x) \tag{1.105}$$

where we have introduced the notation

$$a\overleftrightarrow{\partial}_0 b = a\frac{\partial b}{\partial t} - \frac{\partial a}{\partial t}b \tag{1.106}$$

The second member of Eq. (1.105) is time independent as can be checked explicitly (see Problem 1.3). This observation is important in order to be able to choose equal times in the commutation relations. We get

$$\begin{aligned}[a(k), a^\dagger(k')] =& \int d^3x \int d^3y \ \left[e^{ik\cdot x}\overleftrightarrow{\partial}_0\varphi(\vec{x},t), e^{-ik'\cdot y}\overleftrightarrow{\partial}_0\varphi(\vec{y},t)\right]\\ =&(2\pi)^3 2\omega_k\delta^3(\vec{k}-\vec{k}')\end{aligned} \tag{1.107}$$

and

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0 \tag{1.108}$$

We then see that, except for a small difference in the normalization, $a(k)$ e $a^\dagger(k)$ should be interpreted as annihilation and creation operators of states with momentum k^μ. To show this, we observe that

$$H = \frac{1}{2}\int \widetilde{dk}\ \omega_k \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] \tag{1.109}$$

$$\vec{P} = \frac{1}{2}\int \widetilde{dk}\ \vec{k} \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] \tag{1.110}$$

Using these explicit forms we can then obtain

$$[P^\mu, a^\dagger(k)] = k^\mu a^\dagger(k) \tag{1.111}$$

$$[P^\mu, a(k)] = -k^\mu a(k) \tag{1.112}$$

showing that $a^\dagger(k)$ adds momentum k^μ and that $a(k)$ destroys momentum k^μ. That the quantization procedure has produced an infinity number of oscillators should come as no surprise. In fact $a(k), a^\dagger(k)$ correspond to the quantization of the normal modes of the classical Klein-Gordon field.

By analogy with the harmonic oscillator, we are now in position of finding the eigenstates of H. We start by defining the base state, that in quantum field theory is called the vacuum. We have

$$a(k)\left|0\right\rangle_k = 0 \quad ; \quad \forall_k \tag{1.113}$$

Then the vacuum, that we will denote by $|0\rangle$, will be formally given by

$$|0\rangle = \Pi_k \, |0\rangle_k \tag{1.114}$$

and we will assume that it is normalized, that is $\langle 0|0\rangle = 1$. If now we calculate the vacuum energy, we find immediately the first problem with infinities in Quantum Field Theory (QFT). In fact

$$\begin{aligned}
\langle 0|H|0\rangle &= \frac{1}{2}\int \widetilde{dk}\;\omega_k \left\langle 0\right| \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] \left|0\right\rangle \\
&= \frac{1}{2}\int \widetilde{dk}\;\omega_k \left\langle 0\right| \left[a(k), a^\dagger(k)\right] \left|0\right\rangle \\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\;\omega_k (2\pi)^3 2\omega_k \delta^3(0) \\
&= \frac{1}{2}\int d^3k\;\omega_k \delta^3(0) = \infty
\end{aligned} \tag{1.115}$$

This infinity can be understood as the the (infinite) sum of the zero point energy of all quantum oscillators. In the discrete case we would have, $\sum_k \frac{1}{2}\omega_k = \infty$. This infinity can be easily removed. We start by noticing that we only measure energies as differences with respect to the vacuum energy, and those will be finite. We will then define the energy of the vacuum as being zero. Technically this is done as follows. We define a new operator $P^\mu_{N.O.}$ as

$$\begin{aligned}
P^\mu_{N.O.} &\equiv \frac{1}{2}\int \widetilde{dk}\;k^\mu \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] \\
&\quad -\frac{1}{2}\int \widetilde{dk}\;k^\mu \left\langle 0\right| \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] \left|0\right\rangle \\
&= \int \widetilde{dk}\;k^\mu a^\dagger(k)a(k)
\end{aligned} \tag{1.116}$$

Now $\left\langle 0|P^\mu_{N.O.}|0\right\rangle = 0$. The ordering of operators where the annihilation operators appear on the right of the creation operators is called *normal ordering* and the usual notation is

$$:\frac{1}{2}(a^\dagger(k)a(k) + a(k)a^\dagger(k)) :\equiv a^\dagger(k)a(k) \tag{1.117}$$

Therefore to remove the infinity of the energy and momentum corresponds to choose the normal ordering to our operators. We will adopt this convention in the following dropping the subscript "$N.O.$" to simplify the notation. This should not appear as an *ad hoc* procedure. In fact, in going from the classical theory where we have products of fields into the quantum theory where the fields are operators, we should have a prescription for the correct ordering of such products. We have just seen that this should be the normal ordering.

Once we have the vacuum we can build the states by applying the the creation operators $a^\dagger(k)$. As in the case of the harmonic oscillator, we can define the *number* operator,

$$N = \int \widetilde{dk}\;a^\dagger(k)a(k) \tag{1.118}$$

It is easy to see that N commutes with H and therefore the eigenstates of H are also eigenstates of N. The state with one particle of momentum k^μ is obtained as $a^\dagger(k)\left|0\right\rangle$. In fact we have

$$\begin{aligned} P^\mu a^\dagger(k)\left|0\right\rangle &= \int \widetilde{dk}' k'^\mu a^\dagger(k') a(k') a^\dagger(k)\left|0\right\rangle \\ &= \int d^3k' k'^\mu \delta^3(\vec{k}-\vec{k}') a^\dagger(k)\left|0\right\rangle \\ &= k^\mu a^\dagger(k)\left|0\right\rangle \end{aligned} \tag{1.119}$$

and

$$N a^\dagger(k)\left|0\right\rangle = a^\dagger(k)\left|0\right\rangle \tag{1.120}$$

In a similar way, the state $a^\dagger(k_1)...a^\dagger(k_n)\left|0\right\rangle$ would be a state with n particles. However, the sates that we have just defined have a problem. They are not normalizable and therefore they can not form a basis for the Hilbert space of the quantum field theory, the so-called Fock space. The origin of the problem is related to the use of plane waves and states with exact momentum. This can be solved forming states that are superpositions of plane waves

$$\left|1\right\rangle = \lambda \int \widetilde{dk} C(k) a^\dagger(k)\left|0\right\rangle \tag{1.121}$$

Then

$$\begin{aligned} \left\langle 1|1\right\rangle &= \lambda^2 \int \widetilde{dk_1}\widetilde{dk_2} C^*(k_1) C(k_2) \left\langle 0|a(k_1)a^\dagger(k_2)|0\right\rangle \\ &= \lambda^2 \int \widetilde{dk} |C(k)|^2 = 1 \end{aligned} \tag{1.122}$$

and therefore

$$\lambda = \left(\int \widetilde{dk}\ |C(k)|^2\right)^{-1/2} \tag{1.123}$$

with the condition that $\int \widetilde{dk}\ |C(k)|^2 < \infty$. If k is only different from zero in a neighborhood of a given 4-momentum k^μ, then the state will have a well defined momentum (within some experimental error).

A basis for the Fock space can then be constructed from the n–particle normalized states

$$\begin{aligned} \left|n\right\rangle &= \left(n! \int \widetilde{dk_1}\cdots\widetilde{dk_n} |C(k_1,\cdots k_n)|^2\right)^{-1/2} \\ & \int \widetilde{dk_1}\cdots\widetilde{dk_n} C(k_1,\cdots k_n) a^\dagger(k_1)\cdots a^\dagger(k_n)\left|0\right\rangle \end{aligned} \tag{1.124}$$

that satisfy

$$\left\langle n|n\right\rangle = 1 \tag{1.125}$$

$$N\left|n\right\rangle = n\left|n\right\rangle \tag{1.126}$$

Due to the commutation relations of the operators $a^\dagger(k)$ in Eq. (1.124), the functions $C(k_1 \cdots k_n)$ are symmetric, that is,

$$C(\cdots k_i, \cdots k_j, \cdots) = C(\cdots k_j \cdots k_i \cdots) \tag{1.127}$$

This shows that the quanta that appear in the canonical quantization of real scalar fields obey the Bose–Einstein statistics. This interpretation in terms of particles, with creation and annihilation operators, that results from the canonical quantization, is usually called *second quantization*, as opposed to the description in terms of wave functions (the *first quantization*.

1.2.2 Microscopic causality

Classically, the fields can be measured with an arbitrary precision. In a relativistic quantum theory we have several problems. The first, results from the fact that the fields are now operators. This means that the observables should be connected with the matrix elements of the operators and not with the operators. Besides this question, we can only speak of measuring φ in two space-time points x and y if $[\varphi(x), \varphi(y)]$ vanishes. Let us look at the conditions needed for this to occur.

$$\begin{aligned}
[\varphi(x), \varphi(y)] &= \int \widetilde{dk}_1 \widetilde{dk}_2 \left\{ \left[a(k_1), a^\dagger(k_2)\right] e^{-ik_1\cdot x + ik_2 \cdot y} + \left[a^\dagger(k_1), a(k_2)\right] e^{ik_1 x - ik_2 \cdot y} \right\} \\
&= \int \widetilde{dk}_1 \left(e^{-ik_1\cdot(x-y)} - e^{ik_1\cdot(x-y)} \right) \\
&\equiv i\Delta(x-y)
\end{aligned} \tag{1.128}$$

The function $\Delta(x-y)$ is Lorentz invariant and satisfies the relations

$$(\Box_x + m^2)\Delta(x-y) = 0 \tag{1.129}$$

$$\Delta(x-y) = -\Delta(y-x) \tag{1.130}$$

$$\Delta(\vec{x}-\vec{y}, 0) = 0 \tag{1.131}$$

The last relation ensures that the equal time commutator of two fields vanishes. Lorentz invariance implies then,

$$\Delta(x-y) = 0 \quad ; \quad \forall\ (x-y)^2 < 0 \tag{1.132}$$

This means that for two points that can not be physically connected, that is for which $(x-y)^2 < 0$, the fields interpreted as physical observables, can then be independently measured. This result is known as *Microscopic Causality*. We note that

$$\partial^0 \Delta(x-y)|_{x^0=y^0} = -\delta^3(\vec{x}-\vec{y}) \tag{1.133}$$

which ensures the canonical commutation relation, Eq. (1.99).

1.2.3 Vacuum fluctuations

It is well known from Quantum Mechanics that, in an harmonic oscillator, the coordinate is not well defined for the energy eigenstates, that is

$$\left\langle n|q^2|n\right\rangle \;>\; (\langle n|q|n\rangle)^2 = 0 \tag{1.134}$$

In Quantum Field Theory, we deal with an infinite set of oscillators, and therefore we will have the same behavior, that is,

$$\langle 0|\varphi(x)\varphi(y)|0\rangle \neq 0 \tag{1.135}$$

although

$$\langle 0|\varphi(x)|0\rangle = 0 \tag{1.136}$$

We can calculate Eq. (1.135). We have

$$\begin{aligned} \langle 0|\varphi(x)\varphi(y)|0\rangle &= \int \widetilde{dk}_1 \widetilde{dk}_2 e^{-ik_1\cdot x} e^{ik_2\cdot y} \left\langle 0|a(k_1)a^\dagger(k_2)|0\right\rangle \\ &= \int \widetilde{dk}_1 e^{-ik\cdot(x-y)} \equiv \Delta_+(x-y) \end{aligned} \tag{1.137}$$

The function $\Delta_+(x-y)$ corresponds to the positive frequency part of $\Delta(x-y)$. When $y \to x$ this expression diverges quadratically,

$$\left\langle 0|\varphi^2(x)|0\right\rangle = \Delta_+(0) = \int \widetilde{dk}_1 = \int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \tag{1.138}$$

This divergence can not be eliminated in the way we did with the energy of the vacuum. In fact these vacuum fluctuations, as they are known, do have observable consequences like, for instance, the *Lamb shift*. We will be less worried with the result of Eq. (1.138), if we notice that for measuring the square of the operator φ at x we need frequencies arbitrarily large, that is, an infinite amount of energy. Physically only *averages* over a finite space-time region have meaning.

1.2.4 Charged scalar field

The description in terms of real fields does not allow the distinction between particles and anti-particles. It applies only the those cases were the particle and anti-particle are identical, like the π^0. For the more usual case where particles and anti-particles are distinct, it is necessary to have some charge (electric or other) that allows us to distinguish them. For this we need complex fields.

The theory for the scalar complex field can be easily obtained from two real scalar fields φ_1 and φ_2 with the same mass. If we denote the complex field φ by,

$$\varphi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \tag{1.139}$$

then

$$\mathcal{L} = \mathcal{L}(\varphi_1) + \mathcal{L}(\varphi_2) =: \partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi : \tag{1.140}$$

which leads to the equations of motion

$$(\Box + m^2)\varphi = 0 \; ; \; (\Box + m^2)\varphi^\dagger = 0 \tag{1.141}$$

The classical theory given in Eq. (1.140) has, at the classical level, a conserved current, $\partial_\mu J^\mu = 0$, with

$$J^\mu = i\varphi^\dagger \overleftrightarrow{\partial}^\mu \varphi \tag{1.142}$$

Therefore we expect, at the quantum level, the charge Q

$$Q = \int d^3x : i(\varphi^\dagger \dot{\varphi} - \dot{\varphi}^\dagger \varphi) : \tag{1.143}$$

to be conserved, that is, $[H, Q] = 0$. To show this we need to know the commutation relations for the field φ. The definition Eq. (1.139), and the commutation relations for φ_1 and φ_2 allow us to obtain the following relations for φ and $\varphi^\dagger$:

$$[\varphi(x), \varphi(y)] = [\varphi^\dagger(x), \varphi^\dagger(y)] = 0 \tag{1.144}$$

$$[\varphi(x), \varphi^\dagger(y)] = i\Delta(x - y) \tag{1.145}$$

For equal times we can get from Eq. (1.145)

$$[\pi(\vec{x}, t), \varphi(\vec{y}, t)] = [\pi^\dagger(\vec{x}, t), \varphi^\dagger(\vec{y}, t)] = -i\delta^3(\vec{x} - \vec{y}) \tag{1.146}$$

where

$$\pi = \dot{\varphi}^\dagger \quad ; \quad \pi^\dagger = \dot{\varphi} \tag{1.147}$$

The plane waves expansion is then

$$\begin{aligned} \varphi(x) &= \int \widetilde{dk} \; \left[a_+(k) e^{-ik\cdot x} + a_-^\dagger(k) e^{ik\cdot x} \right] \\ \varphi^\dagger(x) &= \int \widetilde{dk} \; \left[a_-(k) e^{-ik\cdot x} + a_+^\dagger(k) e^{ik\cdot x} \right] \end{aligned} \tag{1.148}$$

where the definition of $a_\pm(k)$ is

$$a_\pm(k) = \frac{a_1(k) \pm i a_2(k)}{\sqrt{2}} \; ; \; a_\pm^\dagger = \frac{a_1^\dagger(k) \mp i a_2^\dagger(k)}{\sqrt{2}} \tag{1.149}$$

The algebra of the operators $a_\pm$ it is easily obtained from the algebra of the operators $a_{i'}s$. We get the following non-vanishing commutators:

$$[a_+(k), a_+^\dagger(k')] = [a_-(k), a_-^\dagger(k')] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}') \tag{1.150}$$

therefore allowing us to interpret a_+ and $a_+^\dagger$ as annihilation and creation operators of quanta of type $+$, and similarly for the quanta of type $-$. We can construct the number operators for those quanta:

$$N_\pm = \int \widetilde{dk} \; a_\pm^\dagger(k) a_\pm(k) \tag{1.151}$$

One can easily verify that

$$N_+ + N_- = N_1 + N_2 \tag{1.152}$$

where

$$N_i = \int \widetilde{dk}\; a_i^\dagger(k) a_i(k) \tag{1.153}$$

The energy-momentum operator can be written in terms of the + and − operators,

$$P^\mu = \int \widetilde{dk}\; k^\mu \left[a_+^\dagger(k) a_+(k) + a_-^\dagger(k) a_-(k) \right] \tag{1.154}$$

where we have already considered the normal ordering. Using the decomposition in Eq. (1.148), we obtain for the charge Q:

$$\begin{aligned} Q &= \int d^3x : i(\varphi^\dagger \dot\varphi - \dot\varphi^\dagger \varphi) : \\ &= \int \widetilde{dk}\; \left[a_+^\dagger(k) a_+(k) - a_-^\dagger(k) a_-(k) \right] \\ &= N_+ - N_- \end{aligned} \tag{1.155}$$

Using the commutation relation in Eq. (1.150) one can easily verify that

$$[H, Q] = 0 \tag{1.156}$$

showing that the charge Q is conserved. The Eq. (1.155) allows us to interpret the $\pm$ quanta as having charge ± 1. However, before introducing interactions, the theory is symmetric, and we can not distinguish between the two types of quanta. From the commutation relations (1.150) we obtain,

$$\begin{aligned} &[P^\mu, a_+^\dagger(k)] = k^\mu a_+^\dagger(k) \\ &[Q, a_+^\dagger(k)] = +a_+^\dagger(k) \end{aligned} \tag{1.157}$$

showing that $a_+^\dagger(k)$ creates a quanta with 4-momentum k^μ and charge $+1$. In a similar way we can show that $a_-^\dagger$ creates a quanta with charge -1 and that $a_\pm(k)$ annihilate quanta of charge ± 1, respectively.

1.2.5 Time ordered product and the Feynman propagator

The operator $\varphi^\dagger$ creates a particle with charge $+1$ or annihilates a particle with charge -1. In both cases it adds a total charge $+1$. In a similar way φ annihilates one unit of charge. Let us construct a state of one particle (not normalized) with charge $+1$ by application of $\varphi^\dagger$ in the vacuum:

$$|\Psi_+(\vec{x}, t)\rangle \equiv \varphi^\dagger(\vec{x}, t)\,|0\rangle \tag{1.158}$$

The amplitude to propagate the state $|\Psi_+\rangle$ into the future to the point $(\vec{x}\,', t')$ with $t' > t$ is given by

$$\theta(t'-t)\left\langle \Psi_+(\vec{x}\,', t') | \Psi_+(\vec{x}, t) \right\rangle = \theta(t'-t) \left\langle 0 | \varphi(\vec{x}\,', t') \varphi^\dagger(\vec{x}, t) | 0 \right\rangle \tag{1.159}$$

In $\varphi^\dagger(\vec{x},t)\left|0\right>$ only the operator $a_+^\dagger(k)$ is active, while in $\left<0\right|\varphi(\vec{x}',t')$ the same happens to $a_+(k)$. Therefore Eq. (1.159) is the matrix element that creates a quanta of charge $+1$ in $(\vec{x},t)$ and annihilates it in $(\vec{x}',t')$ with $t'>t$.

There exists another way of increasing the charge by $+1$ unit in $(\vec{x},t)$ and decreasing it by -1 in $(\vec{x}',t')$. This is achieved if we create a quanta of charge -1 in $\vec{x}'$ at time t' and let it propagate to $\vec{x}$ where it is absorbed at time $t>t'$. The amplitude is then,

$$\theta(t-t')\left<\Psi_-(\vec{x},t)|\Psi_-(\vec{x}',t')\right> = \left<0|\varphi^\dagger(\vec{x},t)\varphi(\vec{x}',t')|0\right>\theta(t-t') \tag{1.160}$$

Since we can not distinguish the two paths we must sum of the two amplitudes in Eqs. (1.159) and (1.160). This is the so-called *Feynman propagator*. It can be written in a more compact way if we introduce the time ordered product. Given two operators $a(x)$ and $b(x')$ we define the time ordered product T by,

$$Ta(x)b(x') = \theta(t-t')a(x)b(x') + \theta(t'-t)b(x')a(x) \tag{1.161}$$

In this prescription the older times are always to the right of the more recent times. It can be applied to an arbitrary number of operators. With this definition, the Feynman propagator reads,

$$\Delta_F(x'-x) = \left<0|T\varphi(x')\varphi^\dagger(x)|0\right> \tag{1.162}$$

Using the φ and $\varphi^\dagger$ decomposition we can calculate Δ_F (for free fields, of course)

$$\begin{aligned}
\Delta_F(x'-x) &= \int \widetilde{dk}\ \left[\theta(t'-t)e^{-ik\cdot(x'-x)} + \theta(t-t')e^{ik\cdot(x'-x)}\right] && (1.163)\\
&= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}e^{-ik\cdot(x'-x)} && (1.164)\\
&\equiv \int \frac{d^4k}{(2\pi)^4}\Delta_F(k)e^{-ik\cdot(x'-x)}
\end{aligned}$$

where

$$\Delta_F(k) \equiv \frac{i}{k^2-m^2+i\varepsilon} \tag{1.165}$$

$\Delta_F(k)$ is the propagator in momenta space (Fourier transform). The equivalence between Eq. (1.164) and Eq. (1.163) is done using integration in the complex plane of the time component k^0, with the help of the residue theorem. The contour is defined by the $i\varepsilon$ prescription, as indicated in Fig. (1.1). Applying the operator $(\Box'_x+m^2)$ to $\Delta_F(x'-x)$ in any of the forms of Eq. (1.163) one can show that

$$(\Box'_x+m^2)\Delta_F(x'-x) = -i\delta^4(x'-x) \tag{1.166}$$

that is, $\Delta_F(x'-x)$ is the Green's function for the Klein-Gordon equation with Feynman boundary conditions.

In the presence of interactions, Feynman propagator looses the simple form of Eq. (1.165). However, as we will see, the free propagator plays a key role in perturbation theory.

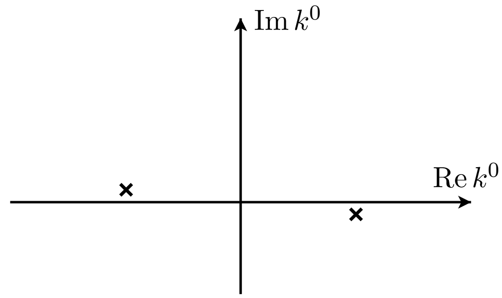


Figure 1.1: Integration in the complex k^0 plane.

1.3 Second quantization of the Dirac field

Let us now apply the formalism of second quantization to the Dirac field. As we will see, something has to be changed, otherwise we would be led to a theory obeying Bose statistics, while we know that electrons have spin 1/2 and obey Fermi statistics.

1.3.1 Canonical formalism for the Dirac field

The Lagrangian density that leads to the Dirac equation is

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi \tag{1.167}$$

The conjugate momentum to ψ_α is

$$\pi_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha} = i\psi^\dagger_\alpha \tag{1.168}$$

while the conjugate momentum to $\psi^\dagger_\alpha$ vanishes. The Hamiltonian density is then

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi \tag{1.169}$$

The requirement of translational and Lorentz invariance for $\mathcal{L}$ leads to the tensors $T^{\mu\nu}$ and $M^{\mu\nu\lambda}$. Using the obvious generalizations of Eqs. (1.75) and (1.79) we get

$$T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\mathcal{L} \tag{1.170}$$

and

$$M^{\mu\nu\lambda} = i\overline{\psi}\gamma^\mu(x^\nu\partial^\lambda - x^\lambda\partial^\nu + \Sigma^{\nu\lambda})\psi - \left(x^\nu g^{\mu\lambda} - x^\mu g^{\nu\lambda}\right)\mathcal{L} \tag{1.171}$$

where

$$\Sigma^{\nu\lambda} \equiv \frac{1}{4}[\gamma^\nu, \gamma^\lambda] \tag{1.172}$$

The 4-momentum P^μ and the angular momentum tensor $M^{\nu\lambda}$ are then given by,

$$P^\mu \equiv \int d^3x T^{0\mu}$$

$$M^{\mu\lambda} \equiv \int d^3x M^{0\nu\lambda} \tag{1.173}$$

or

$$\begin{aligned} H &\equiv \int d^3x \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi \\ \vec{P} &\equiv \int d^3x \psi^\dagger(-i\vec{\nabla})\psi \end{aligned} \tag{1.174}$$

If we define the angular momentum vector $\vec{J} \equiv (M^{23}, M^{31}, M^{12})$ we get

$$\vec{J} = \int d^3x \psi^\dagger \left(\vec{r} \times \frac{1}{i}\vec{\nabla} + \frac{1}{2}\vec{\Sigma} \right) \psi \tag{1.175}$$

which has the familiar aspect $\vec{J} = \vec{L} + \vec{S}$. We can also identify a conserved current, $\partial_\mu j^\mu = 0$, with $j^\mu = \overline{\psi}\gamma^\mu\psi$, which will give the conserved charge

$$Q = \int d^3x \psi^\dagger \psi \tag{1.176}$$

All that we have done so far is at the classical level. To apply the canonical formalism we have to enforce commutation relations and verify the Lorentz invariance of the theory. This will lead us into problems. To see what are the problems and how to solve them, we will introduce the plane wave expansions,

$$\psi(x) = \int \widetilde{dp} \sum_s \left[b(p,s)u(p,s)e^{-ip\cdot x} + d^\dagger(p,s)v(p,s)e^{ip\cdot x} \right] \tag{1.177}$$

$$\psi^\dagger(x) = \int \widetilde{dp} \sum_s \left[b^\dagger(p,s)u^\dagger(p,s)e^{+ip\cdot x} + d(p,s)v^\dagger(p,s)e^{-ip\cdot x} \right] \tag{1.178}$$

where $u(p,s)$ and $v(p,s)$ are the spinors for positive and negative energy, respectively, introduced in the study of the Dirac equation and $b, b^\dagger, d$ and $d^\dagger$ are operators. To see what are the problems with the canonical quantization of fermions, let us calculate P^μ. We get

$$P^\mu = \int \widetilde{dk}\; k^\mu \sum_s \left[b^\dagger(k,s)b(k,s) - d(k,s)d^\dagger(k,s) \right] \tag{1.179}$$

where we have used the orthogonality and closure relations for the spinors $u(p,s)$ and $v(p,s)$. From Eq. (1.179) we realize that if we define the vacuum as $b(k,s)\left|0\right\rangle = d(k,s)\left|0\right\rangle = 0$ and if we quantize with commutators then particles b and particles d will contribute with opposite signs to the energy and the theory will not have a stable ground state. In fact, this was the problem already encountered in the study of the negative energy solutions of the Dirac equation, and this is the reason for the negative sign in Eq. (1.179). Dirac's hole theory required Fermi statistics for the electrons and we will see how spin and statistics are related.

To discover what are the relations that $b, b^\dagger, d$ and $d^\dagger$ should obey, we recall that at the quantum level it is always necessary to verify Lorentz invariance. This gives,

$$i[P_\mu, \psi(x)] = \partial_\mu \psi \; ; \; i[P_\mu, \overline{\psi}(x)] = \partial_\mu \overline{\psi} \tag{1.180}$$

Instead of imposing canonical quantization commutators and, as a consequence, verifying Eq. (1.180) we will do the other way around. We start with Eq. (1.180) and we will discover the appropriate relations for the operators. Using the expansions Eqs. (1.177) and (1.178) we can show that Eq. (1.180) leads to

$$[P_\mu, b(k,s)] = -k_\mu b(k,s) \; ; \; [P_\mu, b^\dagger(k,s)] = k_\mu b^\dagger(k,s) \tag{1.181}$$

$$[P_\mu, d(k,s)] = -k_\mu d(k,s) \; ; \; [P_\mu, d^\dagger(k,s)] = k_\mu d^\dagger(k,s) \tag{1.182}$$

Using Eq. (1.179) for P_μ we get

$$\sum_{s'} \left[\left(b^\dagger(p,s')b(p,s') - d(p,s')d^\dagger(p,s') \right), b(k,s) \right] = -(2\pi)^3 2k^0 \delta^3(\vec{k}-\vec{p}) b(k,s) \tag{1.183}$$

and three other similar relations. If we assume that

$$[d^\dagger(p,s')d(p,s'), b(k,s)] = 0 \tag{1.184}$$

the condition from Eq. (1.183) reads

$$\begin{aligned} &\sum_{s'} \left[b^\dagger(p,s')\{b(p,s'), b(k,s)\} - \{b^\dagger(p,s'), b(k,s)\} b(p,s') \right] = \\ &= -(2\pi)^3 2k^0 \delta^3(\vec{p}-\vec{k}) b(k,s) \end{aligned} \tag{1.185}$$

where the parenthesis $\{,\}$ denote anti-commutators. It is easy to see that Eq. (1.185) is verified if we impose the canonical commutation relations. We should have

$$\begin{aligned} \{b^\dagger(p,s), b(k,s)\} &= (2\pi)^3 2k^0 \delta^3(\vec{p}-\vec{k}) \delta_{ss'} \\ \{d^\dagger(p,s'), d(k,s)\} &= (2\pi)^3 2k^0 \delta^3(\vec{p}-\vec{k}) \delta_{ss'} \end{aligned} \tag{1.186}$$

and all the other anti-commutators vanish. Note that as b anti-commutes with d and $d^\dagger$, then it commutes with $d^\dagger d$ and therefore Eq. (1.184) is verified.

With the anti-commutator relations both contributions to P^μ in Eq. (1.179) are positive. As in boson case we have to subtract the zero point energy. This is done, as usual, by taking all quantities normal ordered. Therefore we have for P^μ,

$$\begin{aligned} P^\mu &= \int \widetilde{dk}\; k^\mu \sum_s : \left(b^\dagger(k,s)b(k,s) - d(k,s)d^\dagger(k,s) \right) : \\ &= \int \widetilde{dk}\; k^\mu \sum_s : \left(b^\dagger(k,s)b(k,s) + d^\dagger(k,s)d(k,s) \right) : \end{aligned} \tag{1.187}$$

and for the charge

$$\begin{aligned} Q &= \int d^3x : \psi^\dagger(x)\psi(x) : \\ &= \int \widetilde{dk} \sum_s \left[b^\dagger(k,s)b(k,s) - d^\dagger(k,s)d(k,s) \right] \end{aligned} \tag{1.188}$$

which means that the quanta of b type have charge $+1$ while those of d type have charge -1. It is interesting to note that was the second quantization of the Dirac field that introduced the $-$ sign in Eq. (1.188), making the charge operator without a definite sign, while in Dirac theory was the probability density that was positive defined. The reverse is true for bosons. We can easily show that

$$\begin{array}{ll} [Q, b^\dagger(k,s)] = b^\dagger(k,s) & [Q, d(k,s)] = d(k,s) \\ [Q, b(k,s)] = -b(k,s) & [Q, d^\dagger(k,s)] = -d^\dagger(k,s) \end{array}$$

and then

$$[Q, \psi] = -\psi \; ; \; [Q, \overline{\psi}] = \overline{\psi} \tag{1.189}$$

In QED the charge is given by eQ ($e < 0$). Therefore we see that ψ creates positrons and annihilates electrons and the opposite happens with $\overline{\psi}$.

We can introduce the number operators

$$N^+(p,s) = b^\dagger(p,s)b(p,s) \quad ; \quad N^-(p,s) = d^\dagger(p,s)d(p,s) \tag{1.190}$$

and we can rewrite

$$P^\mu = \int \widetilde{dk}\; k^\mu \sum_s (N^+(k,s) + N^-(k,s)) \tag{1.191}$$

$$Q = \int \widetilde{dk}\; \sum_s (N^+(k,s) - N^-(k,s)) \tag{1.192}$$

Using the anti-commutator relations in Eq. (1.186) it is now easy to verify that the theory is Lorentz invariant, that is (see Problem 1.5),

$$i[M^{\mu\nu}, \psi] = (x^\mu \partial^\nu - x^\nu \partial^\mu)\psi + \Sigma^{\mu\nu}\psi \; . \tag{1.193}$$

1.3.2 Microscopic causality

The anti-commutation relations in Eq. (1.186) can be used to find the anti-commutation relations at equal times for the fields. We get

$$\{\psi_\alpha(\vec{x},t), \psi^\dagger_\beta(\vec{y},t)\} = \delta^3(\vec{x}-\vec{y})\delta_{\alpha\beta} \tag{1.194}$$

and

$$\{\psi_\alpha(\vec{x},t), \psi_\beta(\vec{y},t)\} = \{\psi^\dagger_\alpha(\vec{x},t), \psi^\dagger_\beta(\vec{y},t)\} = 0 \tag{1.195}$$

These relations can be generalized to unequal times

$$\begin{aligned} \{\psi_\alpha(x), \psi^\dagger_\beta(y)\} &= \int \widetilde{dp}\; \Big[\big[(\not{p}+m)\gamma^0\big]_{\alpha\beta}\; e^{-ip\cdot(x-y)} - \big[(-\not{p}+m)\gamma^0\big]_{\alpha\beta}\; e^{ip\cdot(x-y)} \Big] \\ &= \big[(i\not{\partial}_x + m)\gamma^0\big]_{\alpha\beta}\; i\Delta(x-y) \end{aligned} \tag{1.196}$$

where the $\Delta(x-y)$ function was defined in Eq. (1.128) for the scalar field. The fact that γ^0 appears in Eq. (1.196) is due to the fact that in Eq. (1.196) we took $\psi^\dagger$ and not $\overline{\psi}$. In fact, if we multiply on the right by γ^0 we get

$$\{\psi_\alpha(x), \overline{\psi}_\beta(y)\} = (i\not{\partial}_x + m)_{\alpha\beta} i\Delta(x-y) \tag{1.197}$$

and

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \{\overline{\psi}_\alpha(x), \overline{\psi}_\beta(y)\} = 0 \tag{1.198}$$

We can easily verify the covariance of Eq. (1.197). We use

$$\begin{aligned} &U(a,b)\psi(x)U^{-1}(a,b) = S^{-1}(a)\psi(ax+b) \\ &U(a,b)\overline{\psi}(x)U^{-1}(a,b) = \overline{\psi}(ax+b)S(a) \\ &S^{-1}\gamma^\mu S = a^\mu{}_\nu \gamma^\nu \end{aligned} \tag{1.199}$$

to get

$$\begin{aligned} U(a,b)\{\psi_\alpha(x), \overline{\psi}_\beta(y)\}U^{-1}(a,b) &= \\ &= S^{-1}_{\alpha\tau}(a)\{\psi_\tau(ax+b), \overline{\psi}_\lambda(ay+b)\}S_{\lambda\beta}(a) \\ &= S^{-1}_{\alpha\tau}(a)(i\partial\!\!\!/_{ax} + m)_{\tau\lambda} i\Delta(ax-ay)S_{\lambda\beta}(a) \\ &= (i\partial\!\!\!/ + m)_{\alpha\beta} i\Delta(x-y) \end{aligned} \tag{1.200}$$

where we have used the invariance of $\Delta(x-y)$ and the result $S^{-1} i\partial\!\!\!/_{ax} S = i\partial\!\!\!/_x$. For $(x-y)^2 < 0$ the anti-commutators vanish, because $\Delta(x-y)$ also vanishes. This result allows us to show that any two observables built as bilinear products of $\overline{\psi}$ e ψ commute for two spacetime points for which $(x-y)^2 < 0$. Therefore

$$\begin{aligned} \left[\overline{\psi}_\alpha(x)\psi_\beta(x), \overline{\psi}_\lambda(y)\psi_\tau(y)\right] &= \\ &= \overline{\psi}_\alpha(x)\{\psi_\beta(x), \overline{\psi}_\lambda(y)\}\psi_\tau(y) - \{\overline{\psi}_\alpha(x), \overline{\psi}_\lambda(y)\}\psi_\beta(x)\psi_\tau(y) \\ &\quad + \overline{\psi}_\lambda(y)\overline{\psi}_\alpha(x)\{\psi_\beta(x), \psi_\tau(y)\} - \overline{\psi}_\lambda(y)\{\psi_\tau(y), \overline{\psi}_\alpha(x)\}\psi_\beta(x) \\ &= 0 \end{aligned} \tag{1.201}$$

for $(x-y)^2 < 0$. In this way the microscopic causality is satisfied for the physical observables, such as the charge density or the momentum density.

1.3.3 Feynman propagator

For the Dirac field, as in the case of the charged scalar field, there are two ways of increasing the charge by one unit in x' and decrease it by one unit in x (note that the electron has negative charge). These ways are

$$\theta(t'-t)\left\langle 0|\psi_\beta(x')\psi^\dagger_\alpha(x)|0\right\rangle \tag{1.202}$$

$$\theta(t-t')\left\langle 0|\psi^\dagger_\alpha(x)\psi_\beta(x')|0\right\rangle \tag{1.203}$$

In Eq. (1.202) an electron of positive energy is created at $\vec{x}$ in the instant t, propagates until $\vec{x}'$ where is annihilated at time $t' > t$. In Eq. (1.203) a positron of positive energy is created in x' and annihilated at x with $t > t'$. The Feynman propagator is obtained summing the two amplitudes. Due the exchange of ψ_β and $\overline{\psi}_\alpha$ there must be a minus sign between these two amplitudes. Multiplying by γ^0, in order to get $\overline{\psi}$ instead of $\psi^\dagger$, we get for the Feynman propagator,

$$\begin{aligned} S_F(x'-x)_{\alpha\beta} &= \theta(t'-t)\left\langle 0|\psi_\alpha(x')\overline{\psi}_\beta(x)|0\right\rangle \\ &\quad -\theta(t-t')\left\langle 0|\overline{\psi}_\beta(x)\psi_\alpha(x')|0\right\rangle \\ &\equiv \left\langle 0|T\psi_\alpha(x')\overline{\psi}_\beta(x)|0\right\rangle \end{aligned} \tag{1.204}$$

where we have defined the time ordered product for fermion fields,

$$T\eta(x)\chi(y) \equiv \theta(x^0-y^0)\eta(x)\chi(y) - \theta(y^0-x^0)\chi(y)\eta(x)\ . \tag{1.205}$$

Inserting in Eq. (1.204) the expansions for ψ and $\overline{\psi}$ we get,

$$\begin{aligned} S_F(x'-x)_{\alpha\beta} &= \int \widetilde{dk}\ \left[(\not{k}+m)_{\alpha\beta}\theta(t'-t)e^{-ik\cdot(x'-x)} + (-\not{k}+m)_{\alpha\beta}\theta(t-t')e^{ik\cdot(x'-x)}\right] \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{i(\not{k}+m)_{\alpha\beta}}{k^2-m^2+i\varepsilon}e^{-ik\cdot(x'-x)} \\ &\equiv \int \frac{d^4k}{(2\pi)^4} S_F(k)_{\alpha\beta}e^{-ik\cdot(x'-x)} \end{aligned} \tag{1.206}$$

where $S_F(k)$ is the Feynman propagator in momenta space. We can also verify that Feynman's propagator is the Green function for the Dirac equation, that is (see Problem 1.7),

$$(i\not{\partial}-m)_{\lambda\alpha}\, S_F(x'-x)_{\alpha\beta} = i\delta_{\lambda\beta}\delta^4(x'-x) \tag{1.207}$$

1.4 Electromagnetic field quantization

1.4.1 Introduction

The free electromagnetic field is described by the classical Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.208}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.209}$$

The free field Maxwell equations are

$$\partial_\alpha F^{\alpha\beta} = 0 \tag{1.210}$$

that corresponds to the usual equations in 3-vector notation,

$$\vec{\nabla}\cdot\vec{E} = 0 \quad ; \quad \vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t} \tag{1.211}$$

The other Maxwell equations are a consequence of Eq. (1.209) and can written as,

$$\partial_\alpha \widetilde{F}^{\alpha\beta} = 0 \quad ; \quad \widetilde{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu} F_{\mu\nu} \tag{1.212}$$

corresponding to

$$\vec{\nabla} \cdot \vec{B} = 0 \quad ; \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.213}$$

Classically, the quantities with physical significance are the fields $\vec{E}$ e $\vec{B}$, and the potentials A^μ are auxiliary quantities that are not unique due to the gauge invariance of the theory. In quantum theory the potentials A_μ are the ones playing the leading role as, for instance in the minimal prescription. We have therefore to formulate the quantum fields theory in terms of A^μ and not of $\vec{E}$ and $\vec{B}$.

When we try to apply the canonical quantization to the potentials A^μ we immediately run into difficulties. For instance, if we define the conjugate momentum as,

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\dot{A}_{\mu)}} \tag{1.214}$$

we get

$$\begin{aligned} \pi^k &= \frac{\partial \mathcal{L}}{\partial(\dot{A}_k)} = -\dot{A}^k - \frac{\partial A^0}{\partial x^k} = E^k \\ \pi^0 &= \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \end{aligned} \tag{1.215}$$

Therefore the conjugate momentum to the coordinate A^0 vanishes and does not allow us to use directly the canonical formalism. The problem has its origin in the fact that the photon, that we want to describe, has only two degrees of freedom (positive or negative helicity) but we are using a field A^μ with four degrees of freedom. In fact, we have to impose constraints on A^μ in such a way that it describes the photon. This problem can be addressed in three different ways:

i) *Radiation Gauge*

Historically, this was the first method to be used. It is based in the fact that it is always possible to choose a gauge, called the *radiation gauge*, where

$$A^0 = 0 \quad ; \quad \vec{\nabla} \cdot \vec{A} = 0 \tag{1.216}$$

that is, the potential $\vec{A}$ is transverse. The conditions in Eq. (1.216) reduce the number of degrees of freedom to two, the transverse components of $\vec{A}$. It is then possible to apply the canonical formalism to these transverse components and quantize the electromagnetic field in this way. The problem with this method is that we loose explicit Lorentz covariance. It is then necessary to show that this is recovered in the final result. This method is followed in many text books, for instance in Bjorken and Drell [3].

ii) *Quantization of systems with constraints*

It can be shown that the electromagnetism is an example of an Hamilton generalized system, that is a system where there are constraints among the variables. The way to quantize these systems was developed by Dirac for systems of particles with n degrees of freedom. The generalization to quantum field theories is done using the formalism of path integrals. We will study this method in Chapter 6, where it will be shown, this is the only method that can be applied to non-abelian gauge theories, like the Standard Model.

iii) *Undefined metric formalism*

There is another method that works for the electromagnetism, called the formalism of the *undefined metric*, developed by Gupta and Bleuler [4, 5]. In this formalism, that we will study below, Lorentz covariance is kept, that is we will always work with the 4-vector A_μ, but the price to pay is the appearance of states with negative norm. We have then to define the Hilbert space of the physical states as a sub-space where the norm is positive. We see that in all cases, in order to maintain the explicit Lorentz covariance, we have to complicate the formalism. We will follow the book of Silvan Schweber [6].

1.4.2 Undefined metric formalism

To solve the difficulty of the vanishing of π^0, we will start by modifying the Maxwell Lagrangian introducing a new term,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial \cdot A)^2 \tag{1.217}$$

where ξ is a dimensionless parameter. The equations of motion are now,

$$\Box A^\mu - \left(1 - \frac{1}{\xi}\right) \partial^\mu (\partial \cdot A) = 0 \tag{1.218}$$

and the conjugate momenta

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} - \frac{1}{\xi} g^{\mu 0} (\partial \cdot A) \tag{1.219}$$

that is

$$\begin{cases} \pi^0 & = -\frac{1}{\xi}(\partial \cdot A) \\ \pi^k & = E^k \end{cases}$$

We remark that the Lagrangian of Eq. (1.216) and the equations of motion, Eq. (1.218), reduce to Maxwell theory in the gauge $\partial \cdot A = 0$. This why we say that the choice of Eq. (1.216) corresponds to a class of Lorenz gauges with parameter ξ. With this abuse of language (in fact we are not setting $\partial \cdot A = 0$, otherwise the problems would come back) the value of $\xi = 1$ is known as the *Feynman gauge* and $\xi = 0$ as the *Landau gauge*.

From Eq. (1.218) we get

$$\Box (\partial \cdot A) = 0 \tag{1.220}$$

implying that $(\partial \cdot A)$ is a massless scalar field. Although it would be possible to continue with a general ξ, from now on we will take the case of the so-called *Feynman gauge*, where $\xi = 1$. Then the equation of motion coincide with the Maxwell theory in the Lorenz gauge. As we do not have anymore $\pi^0 = 0$, we can impose the canonical commutation relations at equal times:

$$
\begin{aligned}
&[\pi^\mu(\vec{x},t), A_\nu(\vec{y},t)] = -ig^\mu{}_\nu\, \delta^3(\vec{x}-\vec{y}) \\
&[A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = [\pi_\mu(\vec{x},t), \pi_\nu(\vec{y},t)] = 0
\end{aligned}
\tag{1.221}
$$

Knowing that $[A_\mu(\vec{x},t), A_\mu(\vec{y},t)] = 0$ at equal times, we can conclude that the space derivatives of A_μ also commute at equal times. Then, noticing that

$$
\pi^\mu = -\dot{A}^\mu + \text{ space derivatives} \tag{1.222}
$$

we can write instead of Eq. (1.221)

$$
\begin{aligned}
&[A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = [\dot{A}_\mu(\vec{x},t), \dot{A}_\mu(\vec{y},t)] = 0 \\
&[\dot{A}_\mu(\vec{x},t), A_\nu(\vec{y},t)] = ig_{\mu\nu}\delta^3(\vec{x}-\vec{y})
\end{aligned}
\tag{1.223}
$$

If we compare these relations with the corresponding ones for the real scalar field, where the only one non-vanishing is,

$$
[\dot{\varphi}(\vec{x},t), \varphi(\vec{y},t)] = -i\delta^3(\vec{x}-\vec{y}) \tag{1.224}
$$

we see ($g_{\mu\nu} = \text{diag}(+,-,-,-)$ that the relations for space components are equal but they differ for the time component. This sign will be the source of the difficulties previously mentioned.

If, for the moment, we do not worry about this sign, we expand $A_\mu(x)$ in plane waves,

$$
A^\mu(x) = \int \widetilde{dk} \sum_{\lambda=0}^{3} \left[a(k,\lambda)\varepsilon^\mu(k,\lambda)e^{-ik\cdot x} + a^\dagger(k,\lambda)\varepsilon^{\mu *}(k,\lambda)e^{ik\cdot x} \right] \tag{1.225}
$$

where $\varepsilon^\mu(k,\lambda)$ are a set of four independent 4-vectors that we assume to real, without loss of generality. We will now make a choice for these 4-vectors. We choose $\varepsilon^\mu(1)$ and $\varepsilon^\mu(2)$ orthogonal to k^μ and n^μ, such that

$$
\varepsilon^\mu(k,\lambda)\varepsilon_\mu(k,\lambda') = -\delta_{\lambda\lambda'} \text{ for } \lambda, \lambda' = 1, 2 \tag{1.226}
$$

After, we choose $\varepsilon^\mu(k,3)$ in the plane (k^μ, n^μ) orthogonal to n^μ and normalized, that is

$$
\varepsilon^\mu(k,3)n_\mu = 0 \quad ; \quad \varepsilon^\mu(k,3)\varepsilon_\mu(k,3) = -1 \tag{1.227}
$$

Finally we choose $\varepsilon^\mu(k,0) = n^\mu$. The vectors $\varepsilon^\mu(k,1)$ and $\varepsilon^\mu(k,2)$ are called transverse polarizations, while $\varepsilon^\mu(k,3)$ and $\varepsilon^\mu(k,0)$ longitudinal and scalar polarizations, respectively. We can give an example. In the frame where $n^\mu = (1,0,0,0)$ and $\vec{k}$ is along the z axis we have

$$
\varepsilon^\mu(k,0) \equiv (1,0,0,0) \ ; \ \varepsilon^\mu(k,1) \equiv (0,1,0,0)
$$

$$\varepsilon^\mu(k,2) \equiv (0,0,1,0) \ ; \ \varepsilon^\mu(k,3) \equiv (0,0,0,1) \tag{1.228}$$

In general we can show that

$$\begin{aligned} &\varepsilon(k,\lambda)\cdot\varepsilon^*(k,\lambda') = g^{\lambda\lambda'} \\ &\sum_\lambda g^{\lambda\lambda}\varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda) = g^{\mu\nu} \end{aligned} \tag{1.229}$$

Inserting the expansion (1.225) in (1.223) we get consistency only if

$$[a(k,\lambda), a^\dagger(k',\lambda')] = -g^{\lambda\lambda'} 2k^0 (2\pi)^3 \delta^3(\vec{k}-\vec{k}') \tag{1.230}$$

showing, once more, that the quanta associated with $\lambda = 0$ has a commutation relation with the wrong sign. Before addressing this problem, we can verify that the generalization of Eq. (1.223) for arbitrary times is

$$[A_\mu(x), A_\nu(y)] = -i g_{\mu\nu} \Delta(x,y) \tag{1.231}$$

showing the covariance of the theory. The function $\Delta(x-y)$ is the same that was introduced before for scalar fields.

Therefore, up to this point, everything is as if we had 4 scalar fields. There is, however, the problem of the sign difference in one of the commutators. Let us now see what are the consequences of this sign. For that we introduce the vacuum state defined by

$$a(k,\lambda)\,|0\rangle = 0 \qquad \lambda = 0,1,2,3 \tag{1.232}$$

To see the problem with the sign we construct the one-particle state with scalar polarization, that is

$$|1\rangle = \int \widetilde{dk}\ f(k) a^\dagger(k,0)\,|0\rangle \tag{1.233}$$

and calculate its norm

$$\begin{aligned} \langle 1|1\rangle &= \int \widetilde{dk}_1 \widetilde{dk}_2 f^*(k_1) f(k_2) \left\langle 0|a(k_1,0)a^\dagger(k_2,0)|0\right\rangle \\ &= -\langle 0|0\rangle \int \widetilde{dk}\ |f(k)|^2 \end{aligned} \tag{1.234}$$

where we have used Eq. (1.230) for $\lambda = 0$. The state $|1\rangle$ has a negative norm. The same calculation for the other polarization would give well behaved positive norms. We therefore conclude that the Fock space of the theory has indefinite metric. What happens then to the probabilistic interpretation of quantum mechanics?

To solve this problem we note that we are not working anymore with the classical Maxwell theory because we modified the Lagrangian. What we would like to do is to impose the condition $\partial \cdot A = 0$, but that is impossible as an equation for operators, as that would bring us back to the initial problems with $\pi^0 = 0$. We can, however, require that condition on a weaker form, as a condition only to be verified by the physical states.

More specifically, we require that the part of $\partial \cdot A$ that contains the annihilation operator (positive frequencies) annihilates the physical states,

$$\partial^\mu A_\mu^{(+)} |\psi\rangle = 0 \tag{1.235}$$

The states $|\psi\rangle$ can be written in the form

$$|\psi\rangle = |\psi_T\rangle \, |\phi\rangle \tag{1.236}$$

where $|\psi_T\rangle$ is obtained from the vacuum with creation operators with transverse polarization and $|\phi\rangle$ with scalar and longitudinal polarization. This decomposition depends, of course, on the choice of polarization vectors. To understand the consequences of Eq. (1.235) is enough to analyze the states $|\phi\rangle$ as $\partial^\mu A_\mu^{(+)}$ contains only scalar and longitudinal polarizations,

$$i\partial \cdot A^{(+)} = \int \widetilde{dk} \; e^{-ik\cdot x} \sum_{\lambda=0,3} a(k,\lambda) \; \varepsilon(k,\lambda) \cdot k \tag{1.237}$$

and therefore Eq. (1.235) becomes

$$\sum_{\lambda=0,3} k \cdot \varepsilon(k,\lambda) \; a(k,\lambda) \, |\phi\rangle = 0 \tag{1.238}$$

Condition (1.238) does not determine completely $|\phi\rangle$. In fact, there is much arbitrariness in the choice of the transverse polarization vectors, to which we can always add a term proportional to k^μ because $k \cdot k = 0$. This arbitrariness must reflect itself on the choice of $|\phi\rangle$. Condition (1.238) is equivalent to,

$$[a(k,0) - a(k,3)] \, |\phi\rangle = 0 \; . \tag{1.239}$$

We can construct $|\phi\rangle$ as a linear combination of states $|\phi_n\rangle$ with n scalar or longitudinal photons:

$$\begin{aligned} &|\phi\rangle = C_0 \, |\phi_0\rangle + C_1 \, |\phi_1\rangle + \cdots + C_n \, |\phi_n\rangle + \cdots \\ &|\phi_0\rangle \equiv |0\rangle \end{aligned} \tag{1.240}$$

The states $|\phi_n\rangle$ are eigenstates of the operator number for scalar or longitudinal photons,

$$N' \, |\phi_n\rangle = n \, |\phi_n\rangle \tag{1.241}$$

where

$$N' = \int \widetilde{dk} \; \left[a^\dagger(k,3) a(k,3) - a^\dagger(k,0) a(k,0) \right] \tag{1.242}$$

Then

$$n \, \langle \phi_n | \phi_n \rangle = \left\langle \phi_n | N' | \phi_n \right\rangle = 0 \tag{1.243}$$

where we have used Eq. (1.239). This means that

$$\langle \phi_n | \phi_n \rangle = \delta_{n0} \tag{1.244}$$

that is, for $n \neq 0$, the state $|\phi_n\rangle$ has zero norm. We have then for the general state $|\phi\rangle$,

$$\langle\phi|\phi\rangle = |C_0|^2 \geq 0 \tag{1.245}$$

and the coefficients $C_i, i = 1, \cdots n \cdots$ are arbitrary. We have to show that this arbitrariness does not affect the physical observables. The Hamiltonian is

$$\begin{aligned} H &= \int d^3x : \pi^\mu \dot{A}_\mu - \mathcal{L} : \\ &= \frac{1}{2}\int d^3x : \sum_{i=1}^{3}\left[\dot{A}_i^2 + (\vec{\nabla}A_i)^2\right] - \dot{A}_0^2 - (\vec{\nabla}A_0)^2 : \\ &= \int \widetilde{dk}\ k^0 \left[\sum_{\lambda=1}^{3} a^\dagger(k,\lambda)a(k,\lambda) - a^\dagger(k,0)a(k,0)\right] \end{aligned} \tag{1.246}$$

It is easy to check that if $|\psi\rangle$ is a physical state we have

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\left\langle\psi_T|\int \widetilde{dk}\ k^0 \sum_{\lambda=1}^{2} a^\dagger(k,\lambda)a(k,\lambda)|\psi_T\right\rangle}{\langle\psi_T|\psi_T\rangle} \tag{1.247}$$

and the arbitrariness on the physical states completely disappears when we take average values. Besides that, only the physical transverse polarizations contribute to the result. One can show (see Problem 1.10) that the arbitrariness in $|\phi\rangle$ is related with a gauge transformation within the class of Lorenz gauges.

It is important to note that although for the average values of the physical observables only the transverse polarizations contribute, the scalar and longitudinal polarizations are necessary for the consistency of the theory. In particular they show up when we consider complete sums over the intermediate states.

Invariance for translations is readily verified. For that we write,

$$P^\mu = \int \widetilde{dk}\ k^\mu \sum_{\lambda=0}^{3}(-g^{\lambda\lambda})a^\dagger(k,\lambda)a(k,\lambda) \tag{1.248}$$

Then

$$\begin{aligned} i[P^\mu, A^\nu] &= \int \widetilde{dk}\ \widetilde{dk}'ik^\mu \sum_{\lambda,\lambda'}(-g^{\lambda\lambda})\left\{\left[a^\dagger(k,\lambda)a(k,\lambda), a(k',\lambda')\right]\varepsilon^\nu(k',\lambda')e^{-ik\cdot x}\right. \\ &\qquad \left.+\left[a^\dagger(x,\lambda)a(k,\lambda), a^\dagger(k',\lambda')\right]\varepsilon^{*\nu}(k',\lambda')e^{ik'\cdot x}\right\} \\ &= \int \widetilde{dk}\ ik^\mu \sum_\lambda \left[a(k,\lambda)\varepsilon^\nu(k,\lambda)e^{-ik\cdot x} - a^\dagger(k,\lambda)\varepsilon^\nu(k,\lambda)e^{ik\cdot x}\right] \\ &= \partial^\mu A^\nu \end{aligned} \tag{1.249}$$

showing the invariance under translations. In a similar way, it can be shown the invariance for Lorentz transformations (see Problem 1.11). For that we have to show that

$$M^{jk} = \int d^3x : \left[x^j T^{0k} - x^k T^{0j} + E^j A^k - E^k A^j\right] : \tag{1.250}$$

$$M^{0i} = \int d^3x : \left[x^0 T^{0i} - x^i T^{00} - (\partial \cdot A) A^i - E^i A^0\right] : \tag{1.251}$$

where $(\xi = 1)$

$$\begin{aligned} T^{0i} &= -(\partial \cdot A)\, \partial^i A^0 - E^k \partial^i A^k \\ T^{00} &= \sum_{i=1}^{3} \left[\dot{A}_i^2 + (\vec{\nabla} A_j)^2\right] - \dot{A}_0^2 - (\vec{\nabla} A_0)^2 \end{aligned} \tag{1.252}$$

Using these expressions one can show that the photon has helicity ± 1, corresponding therefore to spin one. For that we start by choosing the direction of $\vec{k}$ along the axis 3 (z axis) and take the polarization vector with the choice of Eq. (1.228). A one-photon physical state will then be (not normalized),

$$|k, \lambda\rangle = a^\dagger(k, \lambda)\, |0\rangle \qquad \lambda = 1, 2 \tag{1.253}$$

Let us now calculate the angular momentum along the axis 3. This is given by

$$\begin{aligned} M^{12}\, |k, \lambda\rangle &= M^{12} a^\dagger(k, \lambda)\, |0\rangle \\ &= [M^{12}, a^\dagger(k, \lambda)]\, |0\rangle \end{aligned} \tag{1.254}$$

where we have used the fact that the vacuum state satisfies $M^{12}\, |0\rangle = 0$. The operator M^{12} has one part corresponding the orbital angular momenta and another corresponding to the spin. The contribution of the orbital angular momenta vanishes (angular momenta in the direction of motion) as one can see calculating the commutator. In fact the commutator with the orbital angular momenta is proportional to k^1 or k^2, which are zero by hypothesis. Let us then calculate the spin part. Using the notation,

$$A^\mu = A^{\mu(+)} + A^{\mu(-)} \tag{1.255}$$

where $A^{\mu(+)}(A^{\mu(-)})$ correspond to the positive (negative) frequencies, we get

$$: E^1 A^2 - E^2 A^1 := E^{1(+)} A^{2(+)} + E^{1(-)} A^{2(+)} + A^{2(-)} E^{1(+)} + E^{1(-)} A^{2(-)} - (1 \leftrightarrow 2) \tag{1.256}$$

Then

$$\begin{aligned} &\left[: E^1 A^2 - E^2 A^1 :, a^\dagger(k, \lambda)\right] = \\ &\quad = E^{1(+)} \left[A^2(+), a^\dagger(k, \lambda)\right] + \left[E^{1(+)}, a^\dagger(k, \lambda)\right] A^{2(+)} \\ &\qquad + E^1(-) \left[A^2(+), a^\dagger(k, \lambda)\right] + A^{2(-)} \left[E^{1(+)}, a^\dagger(k, \lambda)\right] - (1 \leftrightarrow 2) \\ &\quad = E^1 \left[A^{2(+)}, a^\dagger(k, \lambda)\right] + A^2 \left[E^{1(+)}, a^\dagger(k, \lambda)\right] - (1 \leftrightarrow 2) \end{aligned} \tag{1.257}$$

Now (recall that $\lambda = 1, 2$)

$$[A^{2(+)}, a^\dagger(k, \lambda)] = \int \widetilde{dk}' \sum_{\lambda'} \varepsilon^2(k', \lambda') \, \left[a(k', \lambda'), a^\dagger(k, \lambda)\right] e^{-ik' \cdot x}$$

$$
\begin{aligned}
&= \varepsilon^2(k,\lambda) e^{-ik\cdot x} \\
[E^{1(+)}, a^\dagger(k,\lambda)] &= \int \widetilde{dk}' \sum_{\lambda'} \left(ik'^0 \varepsilon^0(k',\lambda') + ik'^1 \varepsilon^0(k',\lambda') \right) \left[a(k',\lambda'), a^\dagger(k,\lambda) \right] \ e^{-ik'\cdot x} \\
&= ik^0 \varepsilon^1(k,\lambda) e^{-ik\cdot x} \qquad (1.258)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int d^3x \left[: E^1 A^2 - E^2 A^1 :, a^\dagger(k,\lambda) \right] \\
&\quad = \int d^3x e^{-ik\cdot x} \left[E^1 \varepsilon^2(k,\lambda) + A^2 ik^0 \varepsilon^1(k,\lambda) - E^2 \varepsilon^1(k,\lambda) + A^1 ik^0 \varepsilon^2(k,\lambda) \right] \\
&\quad = \int d^3x e^{-ik\cdot x} \left[\varepsilon^1(k,\lambda) \overleftrightarrow{\partial}_0 A^2(x) - \varepsilon^2(k,\lambda) \overleftrightarrow{\partial}_0 A^1(x) \right] \qquad (1.259)
\end{aligned}
$$

where we have used the fact that $E^i = -\dot{A}^i, \quad i = 1,2$, for our choice of frame and polarization vectors. On the other hand

$$
\begin{aligned}
a(k,\lambda) &= -i \int d^3x e^{ik\cdot x} \ \overleftrightarrow{\partial}_0 \ \varepsilon^\mu(k,\lambda) A_\mu(x) \\
a^\dagger(k,\lambda) &= i \int d^3x e^{-ik\cdot x} \ \overleftrightarrow{\partial}_0 \ \varepsilon^\mu(k,\lambda) A_\mu(x) \qquad (1.260)
\end{aligned}
$$

For our choice we get

$$
\begin{aligned}
a^\dagger(k,1) &= -i \int d^3x e^{-ik\cdot x} \ \overleftrightarrow{\partial}_0 \ A^1(x) \\
a^\dagger(k,2) &= -i \int d^3x e^{-ik\cdot x} \ \overleftrightarrow{\partial}_0 \ A^2(x) \qquad (1.261)
\end{aligned}
$$

and therefore

$$
[M^{12}, a^\dagger(k,\lambda)] = i\varepsilon^1(k,\lambda) a^\dagger(k,2) - i\varepsilon^2(k,\lambda) a^\dagger(k,1) \qquad (1.262)
$$

We find that the state $a^\dagger(k,\lambda)\left|0\right\rangle, \lambda = 1,2$ is not an eigenstate of the operator M^{12}. However the linear combinations,

$$
\begin{aligned}
a_R^\dagger(k) &= \frac{1}{\sqrt{2}} \left[a^\dagger(k,1) + i a^\dagger(k,2) \right] \\
a_L^\dagger(k) &= \frac{1}{\sqrt{2}} \left[a^\dagger(k,1) - i a^\dagger(k,2) \right] \qquad (1.263)
\end{aligned}
$$

which correspond to right and left circular polarization, verify

$$
[M^{12}, a_R^\dagger(k)] = a_R^\dagger(k) \ ; \ [M^{12}, a_L^\dagger(k)] = -a_L^\dagger(k) \qquad (1.264)
$$

showing that the photon has spin 1 with right or left circular polarization (negative or positive helicity).

1.4.3 Feynman propagator

The Feynman propagator is defined as the vacuum expectation value of the time ordered product of the fields, that is

$$\begin{aligned} G_{\mu\nu}(x,y) &\equiv \langle 0|TA_\mu(x)A_\nu(y)|0\rangle \\ &= \theta(x^0-y^0)\,\langle 0|A_\mu(x)A_\nu(y)|0\rangle + \theta(y^0-x^0)\,\langle 0|A_\nu(y)A_\mu(x)|0\rangle \end{aligned} \tag{1.265}$$

Inserting the expansions for $A_\mu(x)$ and $A_\nu(y)$ we get

$$\begin{aligned} G_{\mu\nu}(x-y) &= -g_{\mu\nu}\int \widetilde{dk}\,\left[e^{-ik\cdot(x-y)}\theta(x^0-y^0) + e^{ik\cdot(x-y)\theta(y^0-x^0)}\right] \\ &= -g_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\varepsilon}e^{-ik\cdot(x-y)} \\ &\equiv \int \frac{d^4k}{(2\pi)^4}G_{\mu\nu}(k)e^{-ik\cdot(x-y)} \end{aligned} \tag{1.266}$$

where $G_{\mu\nu}(k)$ is the Feynman propagator on the momentum space

$$G_{\mu\nu}(k) \equiv \frac{-ig_{\mu\nu}}{k^2+i\varepsilon} \tag{1.267}$$

It is easy to verify that $G_{\mu\nu}(x-y)$ is the Green's function of the equation of motion, that for $\xi=1$ is the wave equation, that is

$$\Box_x G_{\mu\nu}(x-y) = ig_{\mu\nu}\delta^4(x-y) \tag{1.268}$$

These expressions for $G_{\mu\nu}(x-y)$ and $G_{\mu\nu}(k)$ correspond to the particular case of $\xi=1$, the so-called *Feynman gauge*. For the general case where $\xi \neq 0$ the equation of motion reads

$$\left[\Box_x g^\mu_\rho - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial_\rho\right]A^\rho(x) = 0 \tag{1.269}$$

For this case the equal times commutation relations are more complicated (see Problem 1.12). Using those relations one can show that the Feynman propagator is still the Green's function of the equation of motion, that is

$$\left[\Box_x g^\mu_\rho - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial_\rho\right]\langle 0|TA^\rho(x)A^\nu(y)|0\rangle = ig^{\mu\nu}\delta^4(x-y) \tag{1.270}$$

Using this equation we can then obtain in an arbitrary ξ gauge (of the Lorenz type),

$$G_{\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2+i\varepsilon} + i(1-\xi)\frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2}\ . \tag{1.271}$$

1.5 Discrete Symmetries

We know from the study of the Dirac equation the transformations like space inversion (Parity) and charge conjugation, are symmetries of the Dirac equation. More precisely, if $\psi(x)$ is a solution of the Dirac equation, then

$$\psi'(x) = \psi'(-\vec{x}, t) = \gamma_0 \psi(\vec{x}, t) \tag{1.272}$$

$$\psi^c(x) = C\overline{\psi}^T(x) \tag{1.273}$$

are also solutions (if we take the charge $-e$ for ψ^c). Similar operations could also be defined for scalar and vector fields.

With second quantization the fields are no longer functions, they become operators. We have therefore to find unitary operators $\mathcal{P}$ and $\mathcal{C}$ that describe those operations within this formalism. There is another discrete symmetry, time reversal, that in second quantization will be described by an anti-unitary operator $\mathcal{T}$. We will exemplify with the scalar field how to get these operators. We will leave the Dirac and Maxwell fields as exercises.

1.5.1 Parity

To define the meaning of the Parity operation we have to put the system in interaction with the measuring system, considered to be classical. This means that we will consider the system described by

$$\mathcal{L} \longrightarrow \mathcal{L} - j_\mu(x) A^\mu_{ext}(x) \tag{1.274}$$

where we have considered that the interaction is electromagnetic. $j_\mu(x)$ is the electromagnetic current that has the form,

$$\begin{aligned} j_\mu(x) &= ie : \varphi^* \overleftrightarrow{\partial}_\mu \varphi : \quad \text{scalar field} \\ j_\mu(x) &= e : \overline{\psi} \gamma_\mu \psi : \ \text{Dirac field} \end{aligned} \tag{1.275}$$

In a Parity transformation we invert the coordinates of the measuring system, therefore the classical fields are now

$$A^\mu_{ext} = (A^0_{ext}(-\vec{x}, t)), -\vec{A}_{ext}(-\vec{x}, t) = A^{ext}_\mu(-\vec{x}, t) \tag{1.276}$$

For the dynamics of the new system to be identical to that of the original system, which should be the case if Parity is conserved, it is necessary that the equations of motion remain the same. This is true if

$$\mathcal{P}\mathcal{L}(\vec{x}, t)\mathcal{P}^{-1} = \mathcal{L}(-\vec{x}, t) \tag{1.277}$$

$$\mathcal{P} j_\mu(\vec{x}, t)\mathcal{P}^{-1} = j^\mu(-\vec{x}, t) \tag{1.278}$$

Eqs. (1.277) and (1.278) are the conditions that a theory should obey in order to be invariant under Parity. Furthermore $\mathcal{P}$ should leave the commutation relations unchanged, so that the quantum dynamics is preserved. For each theory that conserves Parity should be possible to find an unitary operator $\mathcal{P}$ that satisfies these conditions.

Now we will find such an operator $\mathcal{P}$ for the scalar field. It is easy to verify that the condition

$$\mathcal{P}\varphi(\vec{x},t)\mathcal{P}^{-1} = \pm\varphi(-\vec{x},t) \tag{1.279}$$

satisfies all the requirements. The sign $\pm$ is the *intrinsic* parity of the particle described by the field φ, ($+$ for scalar and $-$ for pseudo-scalar). In terms of the expansion of the momentum, Eq. (1.279) requires

$$\mathcal{P}a(k)\mathcal{P}^{-1} = \pm a(-k) \quad ; \quad \mathcal{P}a^\dagger(k)\mathcal{P}^{-1} = \pm a^\dagger(-k) \tag{1.280}$$

where $-k$ means that we have changed $\vec{k}$ into $-\vec{k}$ (but k^0 remains intact, that is, $k^0 = +\sqrt{|\vec{k}|^2+m^2}$). It is easier to solve Eq. (1.280) in the momentum space. As $\mathcal{P}$ should be unitary, we write

$$\mathcal{P} = e^{iP} \tag{1.281}$$

Then

$$\begin{aligned}\mathcal{P}a(k)\mathcal{P}^{-1} &= a(k) + i[P,a(k)] + \cdots + \frac{i^n}{n!}[P,[\cdots,[P,a(k)]\cdots] + \cdots \\ &= -a(-k)\end{aligned} \tag{1.282}$$

where we have chosen the case of the pseudo-scalar field.

Eq. (1.282) suggests the form

$$[P,a(k)] = \frac{\lambda}{2}[a(k) + \varepsilon a(-k)] \tag{1.283}$$

where λ and $\varepsilon = \pm 1$ are to be determined. We get

$$[P,[P,a(k)]] = \frac{\lambda^2}{2}[a(k) + \varepsilon a(-k)] \tag{1.284}$$

and therefore

$$\begin{aligned}\mathcal{P}a(k)\mathcal{P}^{-1} &= a(k) + \frac{1}{2}\left[i\lambda + \frac{(i\lambda)^2}{2!} + \cdots + \frac{(i\lambda)^4}{n!} + \cdots\right](a(k) + \varepsilon a(-k)) \\ &= \frac{1}{2}[a(k) - \varepsilon a(-k)] + \frac{1}{2}e^{i\lambda}[a(k) + \varepsilon a(-k)] \\ &= -a(-k)\end{aligned} \tag{1.285}$$

We solve Eq. (1.285) if we choose $\lambda = \pi$ and $\varepsilon = +1$ ($\lambda = \pi$ and $\varepsilon = -1$ for the scalar case). It is easy to check that

$$P_{ps} = -\frac{\pi}{2}\int \widetilde{dk}\ \left[a^\dagger(k)a(k) + a^\dagger(k)a(-k)\right] = P^\dagger_{ps} \tag{1.286}$$

and it is solution of Eq. (1.283) for $\lambda = \pi$ and $\varepsilon = +1$. Therefore,

$$\mathcal{P}_{ps} = \exp\left\{-i\frac{\pi}{2}\int \widetilde{dk}\ \left[a^\dagger(k)a(k) + a^\dagger(k)a(-k)\right]\right\} \tag{1.287}$$

and for the scalar field

$$\mathcal{P}_s = \exp\left\{-i\frac{\pi}{2}\int \widetilde{dk}\ \left[a^\dagger(k)a(k) - a^\dagger(k)a(-k)\right]\right\} \tag{1.288}$$

For the case of the Dirac field, the condition equivalent to Eq. (1.279) is now

$$\mathcal{P}\psi(\vec{x},t)\mathcal{P}^{-1} = \gamma^0\psi(-\vec{x},t) \tag{1.289}$$

Repeating the same steps we get

$$\begin{aligned}\mathcal{P}_{Dirac} \quad = \quad & \exp\left\{-i\frac{\pi}{2}\int \widetilde{dp}\ \sum_s \left[b^\dagger(p,s)b(p,s) - b^\dagger(p,s)b(-p,s)\right.\right. \\ & \left.\left. + d^\dagger(p,s)d(p,s) + d^\dagger(p,s)d(-p,s)\right]\right\}\end{aligned} \tag{1.290}$$

The case of the Maxwell field is left as an exercise.

1.5.2 Charge conjugation

The conditions for charge conjugation invariance are now

$$\mathcal{C}\mathcal{L}(x)\mathcal{C}^{-1} = \mathcal{L} \quad ; \quad \mathcal{C}j_\mu\mathcal{C}^{-1} = -j_\mu \tag{1.291}$$

where j_μ is the electromagnetic current. Conditions (1.291) are verified for the charged scalar fields if

$$\mathcal{C}\varphi(x)\mathcal{C}^{-1} = \varphi^*(x) \quad ; \quad \mathcal{C}\varphi^*(x)\mathcal{C}^{-1} = \varphi(x) \tag{1.292}$$

and for the Dirac field if

$$\begin{aligned}\mathcal{C}\psi_\alpha(x)\mathcal{C}^{-1} &= C_{\alpha\beta}\ \overline{\psi}_\beta(x) \\ \mathcal{C}\overline{\psi}_\alpha(x)\mathcal{C}^{-1} &= -\psi_\beta(x)C^{-1}_{\beta\alpha}\end{aligned} \tag{1.293}$$

where C is the charge conjugation matrix.

Finally from the invariance of $j_\mu A^\mu$ we obtain the condition for the electromagnetic field,

$$\mathcal{C}A_\mu\mathcal{C}^{-1} = -A_\mu \tag{1.294}$$

By using a method similar to the one used in the case of the Parity we can get the operator $\mathcal{C}$ for the different theories. For instance, for the scalar field we get

$$\mathcal{C}_s = \exp\left\{i\frac{\pi}{2}\int \widetilde{dk}\ (a_+^\dagger - a_-^\dagger)(a_+ - a_-)\right\} \tag{1.295}$$

and for the Dirac field

$$\mathcal{C} = \mathcal{C}_1\mathcal{C}_2 \tag{1.296}$$

with

$$
\begin{aligned}
\mathcal{C}_1 &= \exp\left\{-i\int \widetilde{dp}\ \sum_s \phi(p,s)\left[b^\dagger(p,s)b(p,s) - d^\dagger(p,s)d(p,s)\right]\right\} \\
\mathcal{C}_2 &= \exp\left\{i\frac{\pi}{2}\int \widetilde{dp}\ \sum_s \left[b^\dagger(p,s) - d^\dagger(p,s)\right]\left[b(p,s) - d(p,s)\right]\right\}
\end{aligned}
\tag{1.297}
$$

with

$$
\begin{aligned}
v(p,s) &= e^{i\phi(p,s)}\ u^c(p,s) \\
u(p,s) &= e^{i\phi(p,s)}\ v^c(p,s)
\end{aligned}
\tag{1.298}
$$

where the phase $\phi(p,s)$ is arbitrary (see [7]).

1.5.3 Time reversal

Classically the meaning of the time reversal invariance it is clear. We change the sign of the time, the velocities change direction and the system goes from what was the final state to the initial state. This exchange between the initial and final state has as consequence, in quantum mechanics, that the corresponding operator must be anti-linear or anti-unitary. In fact $\langle f|i\rangle = \langle i|f\rangle^*$ and therefore if we want $\langle \mathcal{T}\varphi_f|\mathcal{T}\varphi_i\rangle = \langle\varphi_i|\varphi_f\rangle$ then $\mathcal{T}$ must include the complex conjugation operation. We can write

$$
\mathcal{T} = \mathcal{U}K \tag{1.299}
$$

where $\mathcal{U}$ is unitary and K is the instruction to tale the complex conjugate of all *c-numbers*. Then

$$
\begin{aligned}
\langle T\varphi_f|T\varphi_i\rangle &= \langle \mathcal{U}K\varphi_f|\mathcal{U}K\varphi_i\rangle \\
&= \langle \mathcal{U}\varphi_f|\mathcal{U}\varphi_i\rangle^* \\
&= \langle \varphi_f|\varphi_i\rangle^* = \langle\varphi_i|\varphi_f\rangle
\end{aligned}
\tag{1.300}
$$

as we wanted. A theory will be invariant under time reversal if

$$
\begin{aligned}
&\mathcal{T}\mathcal{L}(\vec{x},t)\mathcal{T}^{-1} = \mathcal{L}(\vec{x},-t) \\
&\mathcal{T}j_\mu(\vec{x},t)\mathcal{T}^{-1} = j^\mu(\vec{x},-t)
\end{aligned}
\tag{1.301}
$$

For the scalar field this condition will be verified if

$$
\mathcal{T}\varphi(\vec{x},t)\mathcal{T}^{-1} = \pm\varphi(\vec{x},-t) \tag{1.302}
$$

and for the electromagnetic field we must have.

$$
\mathcal{T}A^\mu(\vec{x},t)\mathcal{T}^{-1} = A_\mu(\vec{x},-t) \tag{1.303}
$$

making $j^\mu A_\mu$ invariant. For the case of the Dirac field the transformation is

$$\mathcal{T}\psi_\alpha(\vec{x},t)\mathcal{T}^{-1} = T_{\alpha\beta}\psi_\beta(\vec{x},-t) \tag{1.304}$$

In order that Eq. (1.301) is satisfied, the T matrix must satisfy

$$T\gamma_\mu T^{-1} = \gamma_\mu^T = \gamma^{\mu *} \tag{1.305}$$

with a solution, in the Dirac representation,

$$T = i\gamma^1\gamma^3 \tag{1.306}$$

Applying the same type of reasoning already used for $\mathcal{P}$ and $\mathcal{C}$ we can find $\mathcal{T}$, or equivalently, $\mathcal{U}$. For the Dirac field, noticing that

$$\begin{aligned} Tu(p,s) &= u^*(-p,-s)e^{i\alpha_+(p,s)} \\ Tv(p,s) &= v^*(-p,-s)e^{i\alpha_-(p,s)} \end{aligned} \tag{1.307}$$

we can write $\mathcal{U} = \mathcal{U}_1\mathcal{U}_2$ and obtain

$$\mathcal{U}_1 = \exp\left\{ -i\int \widetilde{dp}\ \sum_s \left[\alpha_+ b^\dagger(p,s)b(p,s) - \alpha_- d^\dagger(p,s)d(p,s)\right]\right\} \tag{1.308}$$

and

$$\begin{aligned} \mathcal{U}_2 &= \exp\left\{ -i\frac{\pi}{2}\int \widetilde{dp}\ \sum_s \left[b^\dagger(p,s)b(p,s) + b^\dagger(p,s)b(-p-s) \right.\right. \\ &\qquad \left.\left. - d^\dagger(p,s)d(p,s) - d^\dagger(p,s)d(-p,-s) \right]\right\} \end{aligned} \tag{1.309}$$

1.5.4 The $\mathcal{TCP}$ theorem

It is a fundamental theorem in Quantum Field Theory that the product $\mathcal{TCP}$ is an invariance of any theory that satisfies the following general conditions:

- The theory is local and covariant for Lorentz transformations.
- The theory is quantized using the usual relation between spin and statistics, that is, commutators for bosons and anti-commutators for fermions.

This theorem due to Lüdus, Zumino, Pauli e Schwinger has an important consequence that if one of the discrete symmetries is not preserved then another one must also be violated to preserve the invariance of the product. For a proof of the theorem see the books of Bjorken and Drell[3, 1] and Itzykson and Zuber[2].

Problems for Chapter 1

1.1 Verify, for the scalar field, the covariant relations for translations and Lorentz transformations,

$$
\begin{aligned}
i[P^\mu, \varphi] &= \partial^\mu \varphi \\
i[M^{\mu\nu}, \varphi] &= (x^\mu \partial^\nu - x^\nu \partial^\mu)\varphi
\end{aligned}
\tag{1.310}
$$

Hints:

- Consider the theory already normal ordered. Then use

$$
P^\mu = \int \widetilde{dk}\; k^\mu a^\dagger(k) a(k) \ , \tag{1.311}
$$

Eq. (1.103) and Eq. (1.107).

- Show that

$$
M^{0i} = -\, i \int \widetilde{dk}\; a^\dagger(k) \left(k^0 \frac{\partial}{\partial k^i} - k^i \frac{\partial}{\partial k^0} \right) a(k) \tag{1.312}
$$

$$
M^{ij} = -i \int \widetilde{dk}\; a^\dagger(k) \left(k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) a(k) \tag{1.313}
$$

and then use again Eq. (1.103) and Eq. (1.107).

1.2 Show that

$$
\partial^0 \Delta(x-y)|_{x^0=y^0} = -\delta^3(\vec{x} - \vec{y}) \tag{1.314}
$$

1.3 Show that

$$
\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4}\; \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik\cdot(x-y)} = \\
&\quad = \int \widetilde{dk}\; \left[\theta(x^0 - y^0) e^{-ik\cdot(x-y)} + \theta(y^0 - x^0) e^{ik\cdot(x-y)} \right]
\end{aligned}
\tag{1.315}
$$

where $\widetilde{dk} \ \equiv \dfrac{d^3k}{(2\pi)^3 2k^0}$.

Hint: Integrate in the complex plane of the variable dk^0 and use the prescription $i\varepsilon$ to define the contours.

1.4 Show that

$$\{\psi_\alpha(\vec{x},t), \psi^\dagger_\beta(\vec{y},t)\} = \delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y}) \tag{1.316}$$

Hint: You have to use the anti-commutation relations in Eq. (1.186) and the completion properties of the sum over spins for the u and v spinors [8].

1.5 Show that for the Dirac theory the requirements of Lorentz invariance are satisfied,

$$i[M^{\mu\nu}, \psi(x)] = (x^\mu\partial^\nu - x^\nu\partial^\mu)\psi + \Sigma^{\mu\nu}\psi \quad ; \quad \Sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] \tag{1.317}$$

Hint: Use the method of Problem 1.1 for the *orbital part*.

1.6 Show that

$$\begin{aligned} S_F(x-y)_{\alpha\beta} &= \theta(x^0-y^0)\left\langle 0|\psi_\alpha(x)\overline{\psi}_\beta(y)|0\right\rangle \\ &\quad -\theta(y^0-x^0)\left\langle 0|\overline{\psi}_\beta(y)\psi_\alpha(x)|0\right\rangle \end{aligned} \tag{1.318}$$

corresponds to

$$S_F(x-y)_{\alpha\beta} = \int \frac{d^4p}{(2\pi)^4}\, \frac{i(\not p + m)_{\alpha\beta}}{p^2-m^2+i\varepsilon}\, e^{-ip\cdot(x-y)} \tag{1.319}$$

Hint: Expand ψ_α e $\overline{\psi}_\beta$ in plane waves.

1.7 Show that

$$(i\not\partial_x - m)_{\alpha\beta} S_F(x-y)_{\beta\gamma} = i\delta_{\alpha\gamma}\delta^4(x-y) \tag{1.320}$$

1.8 Show that is is always possible to choose the electromagnetic potential A^μ such that

$$A^0 = 0 \;, \vec{\nabla}\cdot\vec{A} = 0 \qquad\qquad \text{(Radiation gauge)} \tag{1.321}$$

1.9 Show that we have

$$[A_\mu(x), A_\nu(y)] = -ig_{\mu\nu}\Delta(x-y) \tag{1.322}$$

1.10 Consider the indefinite metric formalism for the electromagnetic field.

a) Consider the expectation value of A_μ in the state $|\phi\rangle$. Show that

$$\begin{aligned} \langle\phi|A_\mu|\phi\rangle &= C_0^* C_1 \int \widetilde{dk}\; e^{-ik\cdot x}\,\langle 0|\left[\varepsilon_\mu(k,3)a(k,3) + \varepsilon_\mu(k,0)a(k,0)\right]|\phi_1\rangle \\ &\quad +\text{h.c.} \end{aligned} \tag{1.323}$$

b) Choose the state $|\phi_1\rangle$ in the form

$$|\phi_1\rangle = \int \widetilde{dk}\; f(k)\left[a^\dagger(k,3) - a^\dagger(k,0)\right]|0\rangle \tag{1.324}$$

Show that

$$\langle\phi|A_\mu|\phi\rangle = \int \widetilde{dk}\ \left[\varepsilon_\mu(k,3)+\varepsilon_\mu(k,0)\right]\left(C_0^* C_1 e^{-ik\cdot x} f(k) + c.c.\right) \tag{1.325}$$

c) Choose $\varepsilon^\mu(k,\lambda)$ to be real. Show that

$$\varepsilon^\mu(k,3)+\varepsilon^\mu(k,0) = \frac{k^\mu}{(n\cdot k)} \tag{1.326}$$

d) Show that

$$\langle\phi|A_\mu|\phi\rangle = \partial_\mu\Lambda(x) \tag{1.327}$$

where

$$\Box\Lambda = 0 \tag{1.328}$$

Comment the result.

1.11 Show the covariance of the electromagnetism for the Lorentz transformations,

$$i[M^{\mu\nu}, A^\lambda] = (x^\mu\partial^\nu - x^\nu\partial^\mu)A^\lambda + \Sigma^{\mu\nu,\lambda}{}_\sigma A^\sigma \tag{1.329}$$

where

$$\Sigma^{\mu\nu,\lambda\sigma} = g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\sigma}g^{\lambda\nu} \tag{1.330}$$

1.12 Show that for the general case of $\xi \neq 1$ we have

$$\begin{aligned}
&[A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = 0\\
&[\dot{A}_\mu(\vec{x},t), A_\nu(\vec{y},t)] = i g_{\mu\nu}\left[1-(1-\xi)g_{\mu 0}\right]\delta^3(\vec{x}-\vec{y})\\
&[\dot{A}_i(\vec{x},t), \dot{A}_j(\vec{y},t)] = [\dot{A}_0(\vec{x},t), \dot{A}_0(\vec{y},t)] = 0\\
&[\dot{A}_0(\vec{x},t), \dot{A}_i(\vec{y},t)] = i(1-\xi)\partial_i\delta^3(\vec{x}-\vec{y})
\end{aligned} \tag{1.331}$$

1.13 Use the results of Problem 1.12 to show that, in the general gauge with $\xi \neq 1$ we have

$$\left[\Box_x g^\mu{}_\rho - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial_\rho\right]\langle 0|T A^\rho(x) A^\nu(y)|0\rangle = i g^{\mu\nu}\delta^4(x-y) \tag{1.332}$$

where

$$\left(\Box g^\mu\rho - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial_\rho\right)A^\rho = 0 \tag{1.333}$$

1.14 Find the operator $\mathcal{P}$ for the Dirac and Maxwell fields.

1.15 Find the operator $\mathcal{C}$ for the Dirac and Maxwell fields.

1.16 Show that

$$\mathcal{T}\psi_\alpha(\vec{x},t)T^{-1} = \mathcal{T}_{\alpha\beta}\psi_\beta(\vec{x},-t) \tag{1.334}$$

ensures that

$$\mathcal{T}\mathcal{L}(\vec{x},t)\mathcal{T}^{-1} = \mathcal{L}(\vec{x},-t) \tag{1.335}$$

if there is a matrix T such that $T\gamma_\mu T^{-1} = \gamma^{\mu *}$. Find T in the Dirac representation.

1.17 Find the operator $\mathcal{T}$ for the Dirac and Maxwell fields.

1.18 Consider the Lagrangian

$$\mathcal{L} = \overline{\psi} i\gamma^\mu D_\mu P_L \psi - m\overline{\psi}\psi \tag{1.336}$$

where

$$\begin{aligned} D_\mu &= \partial_\mu + iA_\mu^a \frac{\tau^a}{2} \\ P_L &= \frac{1-\gamma_5}{2} \end{aligned} \tag{1.337}$$

Show that the theory is neither invariant under $\mathcal{P}$ nor under $\mathcal{C}$ but it is invariant for the product $\mathcal{CP}$.

Chapter 2

Physical States. S Matrix. LSZ Reduction.

2.1 Physical states

In the previous chapter we saw, for the case of free fields, how to construct the space of states, the so-called *Fock space* of the theory. When we consider the real physical case, with interactions, we are no longer able to solve the problem exactly. For instance, the interaction between electrons and photons is given by a set of nonlinear coupled equations,

$$(i\not\partial - m)\psi = e\not A\,\psi$$
$$\partial_\mu F^{\mu\nu} = e\overline{\psi}\gamma^\nu\psi \tag{2.1}$$

that do not have an exact solution. In practice we have to resort to approximation methods. In the following chapter we will learn how to develop a covariant perturbation theory. Here we are going just to study the general properties of the theory.

Let us start by the physical states. As we do not know how to solve the problem exactly, we can not prove the assumptions we are going to make about these states. However, these are *reasonable* assumptions, based essentially on Lorentz covariance. We choose our states to be eigenstates of energy and momentum, and of all the other observables that commute with P^μ. Besides that, we will also assume that

i) The eigenvalues of p^2 are non-negative and $p^0 > 0$.

ii) There exists one non-degenerate base state, with the minimum of energy, which is Lorentz invariant. This state is called the vacuum state $|0\rangle$ and by convention

$$p^\mu\,|0\rangle = 0 \tag{2.2}$$

iii) There exist one particle states $\left|p^{(i)}\right\rangle$, such that,

$$p^{(i)}_\mu p^{(i)\mu} = m_i^2 \tag{2.3}$$

for each stable particle with mass m_i.

iv) The vacuum and the one-particle states constitute the discrete spectrum of p^ν.

2.2 *In* states

As we are mainly interested in scattering problems, we should construct states that have a simple interpretation in the limit $t \to -\infty$. At that time, the particles that are going to participate in the scattering process have not interacted yet (we assume that the interactions are adiabatically switched off when $|t| \to \infty$ which is appropriate for scattering problems).

We look for operators that create one particle states with the physical mass. To be explicit, we start by an hermitian scalar field given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - V(x) \tag{2.4}$$

where $V(x)$ is an operator made of more than two interacting fields φ at point x. For instance, those interactions can be self-interactions of the type

$$V(x) = \frac{\lambda}{4!}\varphi^4(x) \tag{2.5}$$

The field φ satisfies the following equation of motion

$$(\Box + m^2)\varphi(x) = -\frac{\partial V}{\partial\varphi(x)} \equiv j(x) \tag{2.6}$$

and the equal time canonical commutation relations,

$$\begin{aligned} &[\varphi(\vec{x},t)\varphi(\vec{y},t)] = [\pi(\vec{x},t)\pi(\vec{y},t)] = 0 \\ &[\pi(\vec{x},t),\varphi(\vec{y},t)] = -i\delta^3(\vec{x}-\vec{y}) \end{aligned} \tag{2.7}$$

where

$$\pi(x) = \dot{\varphi}(x) \tag{2.8}$$

if we assume that $V(x)$ has no derivatives. We designate by $\varphi_{in}(x)$ the operator that creates one-particle states. It will be a functional of the fields $\varphi(x)$. Its existence will be shown by explicit construction. We require that $\varphi_{in}(x)$ must satisfy the conditions:

i) $\varphi_{in}(x)$ and $\varphi(x)$ transform in the same way for translations and Lorentz transformations. For translations we have then

$$i\,[P^\mu,\varphi_{in}(x)] = \partial^\mu\varphi_{in}(x) \tag{2.9}$$

ii) The spacetime evolution of $\varphi_{in}(x)$ corresponds to that of a free particle of mass m, that is

$$(\Box + m^2)\varphi_{in}(x) = 0 \tag{2.10}$$

From these definitions it follows that $\varphi_{in}(x)$ creates one-particle states from the vacuum. In fact, let us consider a state $|n\rangle$, such that,

$$P^\mu |n\rangle = p_n^\mu |n\rangle \ . \tag{2.11}$$

Then

$$\begin{aligned} \partial^\mu \langle n|\varphi_{in}(x)|0\rangle &= i \langle n| [P^\mu, \varphi_{in}(x)] |0\rangle \\ &= i p_n^\mu \langle n|\varphi_{in}(x)|0\rangle \end{aligned} \tag{2.12}$$

and therefore

$$\Box \langle n|\varphi_{in}(x)|0\rangle = -p_n^2 \langle n|\varphi_{in}(x)|0\rangle \tag{2.13}$$

Then

$$(\Box + m^2) \langle n|\varphi_{in}(x)|0\rangle = (m^2 - p_n^2) \langle n|\varphi_{in}(x)|0\rangle = 0 \tag{2.14}$$

where we have used the fact that $\varphi_{in}(x)$ is a free field, Eq. (2.10). Therefore the states created from the vacuum by φ_{in} are those for which $p_n^2 = m^2$, that is, the one-particle states of mass m.

The Fourier decomposition of $\varphi_{in}(x)$ is then the same as for free fields, that is,

$$\varphi_{in}(x) = \int \widetilde{dk} \ \left[a_{in}(k) e^{-ik\cdot x} + a_{in}^\dagger(k) e^{ik\cdot x} \right] \tag{2.15}$$

where $a_{in}(k)$ and $a_{in}^\dagger(k)$ satisfy the usual algebra for creation and annihilation operators. In particular, by repeated use of $a_{in}^\dagger(k)$ we can create one state of n particles.

To express $\varphi_{in}(x)$ in terms of $\varphi(x)$ we start by introducing the retarded Green's function of the Klein-Gordon operator,

$$(\Box_x + m^2)\Delta_{ret}(x-y;m) = \delta^4(x-y) \tag{2.16}$$

where

$$\Delta_{ret}(x-y;m) = 0 \ \text{ if } \ x^0 < y^0 \tag{2.17}$$

We can then write

$$\sqrt{Z}\varphi_{in}(x) = \varphi(x) - \int d^4y \Delta_{ret}(x-y;m) j(y) \tag{2.18}$$

The field $\varphi_{in}(x)$, defined by Eq. (2.15), satisfies the two initial conditions. The constant $\sqrt{Z}$ was introduced to normalize φ_{in} in such a way that it has amplitude 1 to create one-particle states from the vacuum. The fact that $\Delta_{ret} = 0$ for $x_0 \to -\infty$, suggests that $\sqrt{Z}\varphi_{in}(x)$ is, in some way, the limit of $\varphi(x)$ when $x_0 \to -\infty$. In fact, as φ and φ_{in} are operators, the correct asymptotic condition must be set on the matrix elements of the operators. Let $|\alpha\rangle$ and $|\beta\rangle$ be two normalized states. We define the operators

$$\begin{aligned} \varphi^f(t) &= i \int d^3x f^*(x) \overleftrightarrow{\partial}_0 \varphi(x) \\ \varphi^f_{in} &= i \int d^3x f^*(x) \overleftrightarrow{\partial}_0 \varphi_{in}(x) \end{aligned} \tag{2.19}$$

where $f(x)$ is a normalized solution of the Klein-Gordon equation. By Green's theorem, φ^f_{in} does not depend on time (for plane waves $f = e^{-ik\cdot x}$ and $\varphi^f_{in} = a_{in}$). Then the asymptotic condition of Lehmann, Symanzik e Zimmermann (LSZ) [9], is

$$\lim_{t\to-\infty} \langle\alpha|\, \varphi^f(t)\, |\beta\rangle = \sqrt{Z}\, \langle\alpha|\, \varphi^f_{in}\, |\beta\rangle \tag{2.20}$$

2.3 Spectral representation for scalar fields

We saw that Z had a physical meaning as the square of the amplitude for the field $\varphi(x)$ to create one-particle states from the vacuum. Let us now find a formal expression for Z and show that $0 \leq Z \leq 1$.

We start by calculating the expectation value in the vacuum of the commutator of two fields,

$$i\Delta'(x,y) \equiv \langle 0|\, [\varphi(x), \varphi(y)]\, |0\rangle \tag{2.21}$$

As we do not know how to solve the equations for the interacting fields φ, we can not solve exactly the problem of finding the Δ', in contrast with the free field case. We can, however, determine its form using general arguments of Lorentz invariance and the assumed spectra for the physical states. We introduce a complete set of states between the two operators in Eq. (2.21) and we use the invariance under translations in order to obtain,

$$\begin{aligned}
\langle n|\varphi(y)|m\rangle &= \langle n|\, e^{iP\cdot y}\varphi(0)e^{-iP\cdot y}\, |m\rangle \\
&= e^{i(p_n-p_m)\cdot y}\, \langle n|\varphi(0)|m\rangle
\end{aligned} \tag{2.22}$$

Therefore we get

$$\begin{aligned}
\Delta'(x,y) &= -i\sum_n \langle 0|\varphi(0)|n\rangle\, \langle n|\varphi(0)|0\rangle\, (e^{-ip_n\cdot(x-y)} - e^{ip_n\cdot(x-y)}) \\
&\equiv \Delta'(x-y)
\end{aligned} \tag{2.23}$$

that is, like in the free field case, Δ' is only a function of the difference $x-y$. Introducing now

$$1 = \int d^4q\ \delta^4(q-p_n) \tag{2.24}$$

we get

$$\begin{aligned}
\Delta'(x-y) &= -i\int \frac{d^4q}{(2\pi)^3}\left[(2\pi)^3\sum_n \delta^4(p_n-q)|\,\langle 0|\varphi(0)|n\rangle\,|^2\right](e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}) \\
&= -i\int \frac{d^4q}{(2\pi)^3}\rho(q)(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)})
\end{aligned} \tag{2.25}$$

where we have defined the density $\rho(q)$ (spectral amplitude),

$$\rho(q) = (2\pi)^3\sum_n \delta^4(p_n-q)|\,\langle 0|\varphi(0)|n\rangle\,|^2 \tag{2.26}$$

This spectral amplitude measures the contribution to Δ' of the states with 4-momentum q^μ. $\rho(q)$ is Lorentz invariant (as can be shown using the invariance of $\varphi(x)$ and the properties of the vacuum and of the states $|n\rangle$) and vanishes when q is not in future light cone, due the assumed properties of the physical states. Then we can write

$$\rho(q) = \overline{\rho}(q^2)\theta(q^0) \tag{2.27}$$

and we get

$$\begin{aligned}
\Delta'(x-y) &= -i\int \frac{d^4q}{(2\pi)^3}\,\overline{\rho}(q^2)\theta(q^0)(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}) \\
&= -i\int \frac{d^4q}{(2\pi)^3}\int d\sigma^2\delta(q^2-\sigma^2)\overline{\rho}(\sigma^2)\theta(q^0)\left[e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}\right] \\
&= \int_0^\infty d\sigma^2\overline{\rho}(\sigma^2)\Delta(x-y;\sigma)
\end{aligned} \tag{2.28}$$

where

$$\Delta(x-y;\sigma) = -i\int \frac{d^4q}{(2\pi)^3}\delta(q^2-\sigma^2)\theta(q^0)(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}) \tag{2.29}$$

is the invariant function defined for the commutator of free fields with mass σ.

The Eq. (2.28) is known as the spectral decomposition of the commutator of two fields. This expression will allow us to show that $0 \leq Z < 1$. To show that, we separate the states of one-particle from the sum in Eq. (2.26). Let $|p\rangle$ be a one-particle state with momentum p. Then

$$\begin{aligned}
\langle 0|\varphi(x)|p\rangle &= \sqrt{Z}\,\langle 0|\varphi_{in}(x)|p\rangle + \int d^4y\Delta_{ret}(x-y;m)\,\langle 0|j(y)|p\rangle \\
&= \sqrt{Z}\,\langle 0|\varphi_{in}(x)|p\rangle
\end{aligned} \tag{2.30}$$

where we have used

$$\begin{aligned}
\langle 0|j(y)|p\rangle &= \left\langle 0|(\Box+m^2)\varphi(y)|p\right\rangle = \\
&= (\Box+m^2)e^{-ip\cdot y}\,\langle 0|\varphi(0)|p\rangle \\
&= (m^2-p^2)e^{-ip\cdot y}\,\langle 0|\varphi(0)|p\rangle = 0
\end{aligned} \tag{2.31}$$

On the other hand

$$\begin{aligned}
\langle 0|\varphi_{in}(x)|p\rangle &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}e^{-ik\cdot x}\,\langle 0|a_{in}(k)|p\rangle \\
&= e^{-ip\cdot x}
\end{aligned} \tag{2.32}$$

and therefore

$$\rho(q) = (2\pi)^3\int \widetilde{dp}\;\delta^4(p-q)Z + \text{ contributions from more than one particle}$$

$$= Z\delta(q^2 - m^2)\theta(q^0) + \cdots \tag{2.33}$$

Therefore

$$\Delta'(x-y) = Z\Delta(x-y;m) + \int_{m_1^2}^{\infty} d\sigma^2 \overline{\rho}(\sigma^2)\Delta(x-y;\sigma) \tag{2.34}$$

where m_1 is the mass of the lightest state of two or more particles. Finally noticing that

$$\frac{\partial}{\partial x^0}\Delta'(x-y)|_{x^0=y^0} = \frac{\partial}{\partial x^0}\Delta(x-y;\sigma)|_{x^0=y^0} = -\delta^3(\vec{x}-\vec{y}) \tag{2.35}$$

we get the relation

$$1 = Z + \int_{m_1^2}^{\infty} d\sigma^2 \overline{\rho}(\sigma^2) \tag{2.36}$$

which means

$$0 \leq Z < 1 \tag{2.37}$$

where this last step results from the assumed positivity of $\overline{\rho}(\sigma^2)$.

2.4 *Out* states

In the same way as we reduced the dynamics of $t \to -\infty$ to the free fields φ_{in}, it is also possible to define in the limit $t \to +\infty$ the corresponding free fields, $\varphi_{out}(x)$. These free fields will be the final state of a scattering problem. The formalism is copied from the case of φ_{in}, and therefore we will present the results without going into the details of the derivations. $\varphi_{out}(x)$ obey the following relations:

$$\begin{aligned} i\,[P^\mu, \varphi_{out}] &= \partial^\mu \varphi_{out} \\ (\Box + m^2)\varphi_{out} &= 0 \end{aligned} \tag{2.38}$$

and has the expansion

$$\varphi_{out}(x) = \int \widetilde{dk}\ \left[a_{out}(k)e^{-ik\cdot x} + a^\dagger_{out}(k)e^{ik\cdot x}\right] \tag{2.39}$$

The asymptotic condition is now

$$\lim_{t\to\infty} \langle\alpha|\,\varphi^f(t)\,|\beta\rangle = \sqrt{Z}\,\langle\alpha|\,\varphi^f_{out}\,|\beta\rangle \tag{2.40}$$

and

$$\sqrt{Z}\varphi_{out}(x) = \varphi(x) - \int d^4y \Delta_{\rm adv}(x-y;m)j(y) \tag{2.41}$$

where the Green's functions $\Delta_{\rm adv}$ satisfy

$$\begin{aligned} &(\Box_x + m^2)\Delta_{\rm adv}(x-y;m) = \delta^4(x-y) \\ &\Delta_{\rm adv}(x-y;m) = 0 \quad ; \quad x^0 > y^0\ . \end{aligned} \tag{2.42}$$

For one-particle states we get

$$\begin{aligned} \langle 0|\varphi(x)|p\rangle &= \sqrt{Z}\,\langle 0|\varphi_{out}(x)|p\rangle \\ &= \sqrt{Z}\,\langle 0|\varphi_{in}(x)|p\rangle \\ &= \sqrt{Z}e^{-ip\cdot x} \end{aligned} \tag{2.43}$$

2.5 S matrix

We have now all the formalism needed to study the transition amplitudes from one initial state to a given final state, the so-called S matrix elements. Let us start by an initial state with n non-interacting particles (we suppose that initially they are well separated),

$$|p_1 \cdots p_n \ ; in\rangle \equiv |\alpha \ ; in\rangle \tag{2.44}$$

where $p_1 \cdots p_n$ are the 4-momenta of the n particles. Other quantum numbers are assumed but not explicitly written. The final state will be, in general, a state with m particles

$$\left|p'_1 \cdots p'_m \ ; out\right\rangle \equiv |\beta \ ; out\rangle \tag{2.45}$$

The S matrix element $S_{\beta\alpha}$ is defined by the amplitude

$$S_{\beta\alpha} \equiv \langle\beta \ ; out|\alpha \ ; in\rangle \tag{2.46}$$

The S matrix is an operator that induces an isomorphism between the *in* and *out* states, that by assumption are a complete set of states,

$$\begin{aligned}
&\langle\beta \ ; out| = \langle\beta \ ; in|\, S \\
&\langle\beta \ ; in| = \langle\beta \ ; out|\, S^{-1} \\
&\langle\beta \ ; out|\alpha \ ; in\rangle = \langle\beta \ ; in|S|\alpha \ ; in\rangle = \langle\beta \ ; out|S|\alpha ; out\rangle
\end{aligned} \tag{2.47}$$

From the assumed properties for the states we can show the following results for the S matrix.

i) $\langle 0|S|0\rangle = \langle 0|0\rangle = 1$ (stability and unicity of the vacuum)

ii) The stability of the one-particle states gives

$$\langle p \ ; in|S|p \ ; in\rangle = \langle p \ ; out|p \ ; in\rangle = \langle p \ ; in|p \ ; out\rangle = 1 \tag{2.48}$$

because $|p \ ; in\rangle = |p \ ; out\rangle$.

iii) $\varphi_{in}(x) = S\varphi_{out}(x)S^{-1}$

This relation results from the fact that we want that the matrix elements of operators do not depend on the basis *in* or *out*. In fact

$$\begin{aligned}
\langle\alpha; in|\varphi_{in}(x)|\beta; in\rangle &= \langle\alpha; out|\varphi_{out}(x)|\beta; out\rangle \\
&= \left\langle\alpha; in|S\varphi_{out}(x)S^{-1}|\beta; in\right\rangle
\end{aligned} \tag{2.49}$$

showing the above result.

iv) The S matrix is unitary. To show this we have

$$\delta_{\beta\alpha} = \langle \beta\ ; out | \alpha\ ; out \rangle = \left\langle \beta\ ; in | SS^{\dagger} | \alpha\ ; in \right\rangle \tag{2.50}$$

and therefore

$$SS^{\dagger} = 1 \tag{2.51}$$

v) The S matrix is Lorentz invariant, or more precisely the S matrix transition elements are Lorentz invariant. In fact we have

$$\begin{aligned}\varphi_{in}(ax+b) &= U(a,b)\varphi_{in}(x)U^{-1}(a,b) = US\varphi_{out}(x)S^{-1}U^{-1} \\ &= USU^{-1}\varphi_{out}(ax+b)US^{-1}U^{-1} \ .\end{aligned} \tag{2.52}$$

But

$$\varphi_{in}(ax+b) = S\varphi_{out}(ax+b)S^{-1} \ , \tag{2.53}$$

and therefore we get finally[1] for the S matrix in the transformed frame

$$S' = U(a,b)SU^{-1}(a,b) \ . \tag{2.54}$$

ensuring that

$$\left\langle \Phi'_{\alpha} | S' | \Phi'_{\beta} \right\rangle = \left\langle \Phi_{\alpha} | U^{-1} S' U | \Phi_{\beta} \right\rangle = \langle \Phi_{\alpha} | S | \Phi_{\beta} \rangle \tag{2.55}$$

2.6 Reduction formula for scalar fields

The S matrix elements are the quantities that are directly connected to the experiment. In fact, $|S_{\beta\alpha}|^2$ represents the transition probability from the initial state $|\alpha\ ; in\rangle$ to the final $|\beta\ ; out\rangle$. We are going in this section to use the previous formalism to express these matrix elements in terms of the so-called Green functions for the interacting fields. In this way the problem of the calculation of these probabilities is transferred to the problem of calculating these Green functions. These, of course, can not be evaluated exactly, but we will learn in the next chapter how to develop a covariant perturbation theory for that purpose.

Let us then proceed to the derivation of the relation between the S matrix elements and the the Green functions of the theory. This technique is known as the *LSZ reduction* from the names of Lehmann, Symanzik e Zimmermann [9] that have introduced it. By definition

$$\langle p_1 \cdots ; out | q_1 \cdots ; in \rangle = \left\langle p_1, \cdots ; out | a^{\dagger}_{in}(q_1) | q_2, \cdots ; in \right\rangle \tag{2.56}$$

Using

$$a^{\dagger}_{in}(q_1) = -i \int_t d^3x e^{-iq_1\cdot x} \overleftrightarrow{\partial}_0 \varphi_{in}(x) \tag{2.57}$$

[1] This proof is for scalar fields. For the other cases it is much more complicated to prove [10]

where the integral is time-independent, and therefore can be calculated for an arbitrary time t. If we take $t \to -\infty$ and use the asymptotic condition for the *in* fields, Eq. (2.20), we get

$$\langle p_1 \cdots ; out | q_1 \cdots ; in \rangle = - \lim_{t \to -\infty} i Z^{-1/2} \int_t d^3x e^{-iq_1 \cdot x} \overleftrightarrow{\partial}_0 \langle p_1 \cdots ; out | \varphi(x) | q_2 \cdots ; in \rangle \tag{2.58}$$

In a similar way one can show that

$$\begin{aligned} \left\langle p_1 \cdots ; out | a^\dagger_{out}(q_1) | q_2 \cdots ; in \right\rangle = \\ = - \lim_{t \to \infty} i Z^{-1/2} \int_t d^3x e^{-iq_1 \cdot x} \overleftrightarrow{\partial}_0 \langle p_1 \cdots ; out | \varphi(x) | q_2 \cdots ; in \rangle \ . \end{aligned} \tag{2.59}$$

Then, using the result,

$$\begin{aligned} \left(\lim_{t \to \infty} - \lim_{t \to -\infty} \right) \int d^3x f(\vec{x}, t) &= \lim_{t_f \to \infty, t_i \to -\infty} \int_{t_i}^{t_f} dt \frac{\partial}{\partial t} \int d^3x f(\vec{x}, t) \\ &= \int d^4x \partial_0 f(\vec{x}, t) \end{aligned} \tag{2.60}$$

and subtracting Eq. (2.59) from Eq. (2.58) we get

$$\begin{aligned} \langle p_1 \cdots ; out | q_1 \cdots ; in \rangle = & \left\langle p_1 \cdots ; out | a^\dagger_{out}(q_1) | q_2 \cdots ; in \right\rangle \\ & + i Z^{-1/2} \int d^4x \ \partial_0 \left[e^{-iq_1 \cdot x} \overleftrightarrow{\partial}_0 \langle p_1 \cdots ; out | \varphi(x) | q_2 \cdots ; in \rangle \right] \end{aligned} \tag{2.61}$$

The first term on the right-hand side of Eq. (2.61) corresponds to a sum of disconnected terms, in which at least one of the particles is not affected by the interaction (it will vanish if none of the initial momenta coincides with one of the final momenta). Its form is

$$\begin{aligned} \left\langle p_1 \cdots ; out | a^\dagger_{out}(q_1) | q_2 \cdots ; in \right\rangle = \\ = \sum_{k=1}^{n} (2\pi)^3 2 p_k^0 \, \delta^3(\vec{p}_k - \vec{q}_1) \langle p_1, \cdots, \widehat{p}_k, \cdots ; out | q_2, \cdots ; in \rangle \end{aligned} \tag{2.62}$$

where $\widehat{p}_k$ means that this momentum was taken out from that state. For the second term we write,

$$\begin{aligned} & \int d^4x \ \partial_0 \left[e^{-iq_i x} \overleftrightarrow{\partial}_0 \langle p_1 \cdots ; out | \varphi(x) | q_2 \cdots ; in \rangle \right] \\ &= \int d^4x \left[- \left(\partial_0^2 e^{-iq_1 \cdot x} \right) \langle \cdots \rangle + e^{-iq_1 \cdot x} \partial_0^2 \langle \cdots \rangle \right] \\ &= \int d^4x \left[\left((-\Delta^2 + m^2) \, e^{-iq_1 \cdot x} \right) \langle \cdots \rangle + e^{-iq_1 \cdot x} \partial_0^2 \langle \cdots \rangle \right] \end{aligned}$$

$$= \int d^4x e^{-iq_1\cdot x}(\Box + m^2)\, \langle p_1 \cdots ; out|\varphi(x)|q_2 \cdots ; in\rangle \tag{2.63}$$

where we have used $(\Box + m^2)e^{-iq_1\cdot x} = 0$, and have performed an integration by parts (whose justification would imply the substitution of plane waves by wave packets).

Therefore, after this first step in the reduction we get,

$$\begin{aligned}
&\langle p_1, \cdots p_n; out|q_1 \cdots q_\ell; in\rangle = \\
&= \sum_{k=1}^{n} 2p_k^0 (2\pi)^3 \delta^3(\vec{p}_k - \vec{q}_1)\, \langle p_1, \cdots \widehat{p}_k; \cdots p_n; out|q_2 \cdots q_2 \cdots q_\ell; in\rangle \\
&\quad + iZ^{-1/2} \int d^4x e^{-iq_1 x}(\Box + m^2)\, \langle p_1 \cdots p_n; out|\varphi(x)|q_2 \cdots q_\ell; in\rangle
\end{aligned} \tag{2.64}$$

We will proceed with the process until all the momenta in the initial and final state are exchanged by the field operators. To be specific, let us now remove one momentum in the final state. From now on we will no longer consider the disconnected terms, because in practice we are only interested in the cases where *all* the particles interact[2]. We have then

$$\begin{aligned}
&\langle p_1 \cdots p_n; out|\varphi(x_1)|q_2 \cdots q_\ell; in\rangle = \langle p_2 \cdots p_n; out|a_{out}(p_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle \\
&= \lim_{y_1^0 \to \infty} iZ^{-1/2} \int d^3y_1\, e^{ip_1\cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \langle p_2 \cdots p_n; out|\varphi(y_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle \\
&= \langle p_2 \cdots p_n; out|\varphi(x_1)a_{in}(p_1)|q_2 \cdots q_\ell; in\rangle \\
&\quad + \lim_{y_1^0 \to \infty} iZ^{-1/2} \int d^3y_1\, e^{ip_1\cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \langle p_2 \cdots p_n; out|\varphi(y_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle \\
&\quad - \lim_{y_1^0 \to -\infty} iZ^{-1/2} \int d^3y_1\, e^{ip_1\cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \langle p_2 \cdots p_n; out|\varphi(x_1)\varphi(y_1)|q_2 \cdots q_\ell; in\rangle \\
&= \langle p_2 \cdots p_n; out|\varphi(x_1)a_{in}(p_1)|q_2 \cdots q_\ell; in\rangle \\
&\quad + iZ^{-1/2} \left(\lim_{y_1^0 \to \infty} - \lim_{y_1^0 \to -\infty} \right) \int d^3y_1\, e^{ip_1\cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \langle p_2 \cdots p_n; out|T\varphi(y_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle
\end{aligned} \tag{2.65}$$

where we have used the properties of the time-ordered product, Eq. (1.161). Applying the same procedure that lead to Eq. (2.63) we obtain,

$$\begin{aligned}
&\langle p_1 \cdots p_n, ; out|\varphi(x_1)|q_2 \cdots q_\ell; in\rangle = \text{disconnected terms} \\
&\quad + iZ^{-1/2} \int d^4y_1 e^{ip_1\cdot y_1}(\Box_{y_1} + m^2)\, \langle p_2 \cdots p_n; out|T\varphi(y_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle
\end{aligned} \tag{2.66}$$

[2]Once we know the cases where all the particles interact, we can always calculate situations where some of the particles do not participate in the scattering.

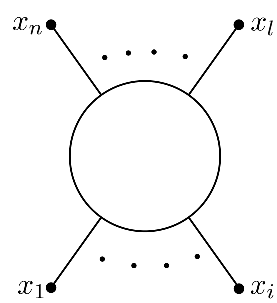
It is not very difficult to generalize this method to obtain the final reduction formula for scalar fields,

$$
\begin{aligned}
\langle p_1 \cdots p_n; out | q_1 \cdots q_\ell; in \rangle &= \text{disconnected terms} \\
&+ \left(\frac{i}{\sqrt{Z}} \right)^{n+\ell} \int d^4 y_1 \cdots d^4 y_n d^4 x_1 \cdots d^4 x_\ell e^{\left[i \sum_1^n p_k \cdot y_k - i \sum_1^\ell q_r \cdot x_r \right]} \\
&(\Box_{y_1} + m^2) \cdots (\Box_{x_\ell} + m^2) \langle 0 | T \varphi(y_1) \cdots \varphi(y_n) \varphi(x_1) \cdots \varphi(x_\ell) | 0 \rangle
\end{aligned} \tag{2.67}
$$

This last equation is the fundamental equation in quantum field theory. It allows us to relate the transition amplitudes to the Green functions of the theory. The quantity

$$
\langle 0 | T \varphi(x_1) \cdots \varphi(x_n) | 0 \rangle \equiv G(x_1 \cdots x_n) \tag{2.68}
$$

is known as the complete green function for $r = m + \ell$ particles and we will introduce the following diagrammatic representation for it,

$G(x_1 \cdots x_n) =$  (2.69)

The factors $(\Box + m^2)$ in Eq. (2.67) force the external particles to be on-shell. In fact, in momentum space $(\Box + m^2) \to (-p^2 + m^2)$. Therefore, Eq. (2.67) will vanish unless the propagators of the external legs are on-shell, as in that case they will have a pole, $\frac{1}{p^2 - m^2}$. Eq. (2.67) will then give the residue of that pole. We conclude that for the transition amplitudes only the truncated Green functions will contribute, that is the ones with the external legs removed. In the next chapter we will learn how to evaluate these Green functions in perturbation theory.

2.7 Reduction formula for fermions

2.7.1 States *in* and *out*

The definition of the *in* and *out* follows exactly the same steps as in the case of the scalar fields. We will therefore, for simplicity, just state the results with the details.

The states $\psi_{in}(x)$ satisfy the conditions,

$$
(i\partial\!\!\!/ - m)\psi_{in}(x) = 0
$$

$$[P_\mu, \psi_{in}(x)] = -i\partial_\mu \psi_{in}(x) \ . \tag{2.70}$$

The states $\psi_{in}(x)$ will create one-particle states and they are related with the fields $\psi(x)$ by,

$$\sqrt{Z_2}\psi_{in}(x) = \psi(x) - \int d^4y S_{\rm ret}(x-y,m) j(y) \tag{2.71}$$

where $\psi(x)$ satisfies the Dirac equation,

$$(i\partial\!\!\!/ - m)\psi(x) = j(x) \tag{2.72}$$

and $S_{\rm ret}$ is the retarded Green function for the Dirac equation,

$$\begin{aligned} &(i\partial\!\!\!/_x - m)S_{\rm ret}(x-y,m) = \delta^4(x-y) \\ &S_{ret}(x-y) = 0 \ ; \ x^0 < y^0 \end{aligned} \tag{2.73}$$

The fields $\psi_{in}(x)$, as free fields, have the Fourier expansion,

$$\psi_{in}(x) = \int \widetilde{dp} \sum_s \left[b_{in}(p,s)u(p,s)e^{-ip\cdot x} + d^\dagger_{in}(p,s)v(p,s)e^{ip\cdot x} \right] \tag{2.74}$$

where the operators b_{in}, d_{in} satisfy exactly the same algebra as in the free field case. The asymptotic condition is now,

$$\lim_{t\to -\infty} \langle\alpha|\, \psi^f(t)\, |\beta\rangle = \sqrt{Z_2}\, \langle\alpha|\, \psi^f_{in}\, |\beta\rangle \tag{2.75}$$

where $\psi^f(t)$ and ψ^f_{in} have a meaning similar to Eq. (2.19).

For the ψ_{out} fields we get essentially the same expressions with ψ_{in} substituted by ψ_{out}. The main difference is in the asymptotic condition that now reads,

$$\lim_{t\to \infty} \langle\alpha|\, \psi^f(t)\, |\beta\rangle = \sqrt{Z_2}\, \langle\alpha|\, \psi^f_{out}\, |\beta\rangle \tag{2.76}$$

implying the following relation between the fields ψ_{out} and ψ,

$$\sqrt{Z_2}\psi_{out} = \psi(x) - \int d^4y S_{adv}(x-y;m) j(y) \tag{2.77}$$

where

$$\begin{aligned} &(i\partial\!\!\!/_x - m)S_{adv}(x-y;m) = \delta^4(x-y) \\ &S_{adv}(x-y;m) = 0 \qquad x^0 > y^0 \ . \end{aligned} \tag{2.78}$$

2.7.2 Spectral representation fermions

Let us consider the vacuum expectation value of the anti-commutator of two Dirac fields,

$$S'_{\alpha\beta}(x,y) \quad \equiv \quad i\,\langle 0|\, \{\psi_\alpha(x), \overline{\psi}_\beta(y)\}\, |0\rangle$$

$$
\begin{aligned}
&= i\sum_n \Big[\langle 0|\,\psi_\alpha(0)\,|n\rangle\,\langle n|\,\overline{\psi}_\beta(0)\,|0\rangle\, e^{-ip_n(x-y)} \\
&\qquad + \langle 0|\,\overline{\psi}_\beta(0)\,|n\rangle\,\langle n|\,\psi_\alpha(0)\,|0\rangle\, e^{ip_n\cdot(x-y)} \Big] \\
&\equiv S'_{\alpha\beta}(x-y)
\end{aligned}
\tag{2.79}
$$

where we have introduced a complete set of eigen-states of the 4-momentum. As before we introduce the spectral amplitude $\rho_{\alpha\beta}(q)$,

$$
\rho_{\alpha\beta}(q) \equiv (2\pi)^3 \sum_n \delta^4(p_n-q)\,\langle 0|\,\psi_\alpha(0)\,|n\rangle\,\langle n|\,\overline{\psi}_\beta(0)\,|0\rangle \tag{2.80}
$$

We will now find the most general form for $\rho_{\alpha\beta}(q)$ using Lorentz invariance arguments. $\rho_{\alpha\beta}(q)$ is a 4×4 matrix in Dirac space, and can therefore be written as

$$
\rho_{\alpha\beta}(q) = \overline{\rho}(q)\delta_{\alpha\beta} + \rho_\mu(q)\gamma^\mu_{\alpha\beta} + \rho_{\mu\nu}(q)\sigma^{\mu\nu}_{\alpha\beta} + \tilde{\rho}(q)\gamma^5_{\alpha\beta} + \tilde{\rho}_\mu(q)(\gamma^\mu\gamma^5)_{\alpha\beta} \tag{2.81}
$$

Lorentz invariance arguments restrict the form of the coefficients $\overline{\rho}(q)$, $\rho_\mu(q)$, $\rho_{\mu\nu}(q)$, $\tilde{\rho}(q)$ and $\tilde{\rho}_\mu(q)$. Under Lorentz transformations the fields transform as

$$
\begin{aligned}
&U(a)\psi_\alpha(0)U^{-1}(a) = S^{-1}_{\alpha\lambda}(a)\psi_\lambda(0) \\
&U(a)\overline{\psi}_\alpha(0)U^{-1}(a) = \overline{\psi}_\lambda(0)S_{\lambda\alpha}(a) \\
&S^{-1}\gamma^\mu S = a^\mu{}_\nu\gamma^\nu
\end{aligned}
\tag{2.82}
$$

Then we can show that the matrix (in Dirac space), $\rho_{\alpha\beta}$ must obey the relation,

$$
\rho(q) = S^{-1}(a)\rho(qa^{-1})S(a) \tag{2.83}
$$

where we have used a matrix notation. This relation gives the properties of the different coefficients on Eq. (2.81). For instance,

$$
\rho^\mu(q) = a^\mu{}_\nu\rho^\nu(qa^{-1}) \tag{2.84}
$$

which means that ρ^μ transform as a $4-$vector.

Using the fact that $\rho_{\alpha\beta}$ is a function of q and vanishes outside the future light cone, we can finally write

$$
\rho_{\alpha\beta}(q) = \rho_1(q^2)\not{q}_{\alpha\beta} + \rho_2(q^2)\delta_{\alpha\beta} + \tilde{\rho}_1(q^2)(\not{q}\gamma^5)_{\alpha\beta} + \tilde{\rho}_2(q^2)\gamma^5_{\alpha\beta} \tag{2.85}
$$

that is, $\rho_{\alpha\beta}(q)$ is determined up to four scalar functions of q^2. Requiring invariance under parity transformations we get, instead of Eq. (2.83),

$$
\rho_{\alpha\beta}(\vec{q},q_0) = \gamma^0_{\alpha\lambda}\rho_{\lambda\delta}(-\vec{q},q^0)\gamma^0_{\delta\beta} \tag{2.86}
$$

and inserting in Eq. (2.85) we obtain,

$$
\widetilde{\rho}_1 = \widetilde{\rho}_2 = 0 \tag{2.87}
$$

Therefore for the Dirac theory, that preserves parity, we get,

$$\rho_{\alpha\beta}(q) = \rho_1(q^2)\not{q}_{\alpha\beta} + \rho_2(q^2)\delta_{\alpha\beta} \tag{2.88}$$

Repeating the steps of the scalar case we write,

$$\begin{aligned} S'_{\alpha\beta}(x-y) &= \int_0^\infty d\sigma^2 \left\{\rho_1(\sigma^2) S_{\alpha\beta}(x-y;\sigma) + \right. \\ &\qquad \left. + \left[\sigma\rho_1(\sigma^2) - \rho_2(\sigma^2)\right] \delta_{\alpha\beta}\Delta(x-y;\sigma)\right\} \end{aligned} \tag{2.89}$$

where Δ and $S_{\alpha\beta}$ are the functions defined for free fields. We can then show that

i) ρ_1 e ρ_2 are real

ii) $\rho_1(\sigma^2) \geq 0$

iii) $\sigma\rho_1(\sigma^2) - \rho_2(\sigma^2) \geq 0$

Using the previous relations we can extract of the one-particle states from Eq. (2.89). We get,

$$\begin{aligned} S'_{\alpha\beta}(x-y) &= Z_2 S_{\alpha\beta}(x-y;m) \\ &\quad + \int_{m_1^2}^\infty d\sigma^2 \left\{ \rho_1(\sigma^2) S_{\alpha\beta}(x-y;\sigma) \right. \\ &\qquad \left. + \left[\sigma\rho_1(\sigma^2) - \rho_2(\sigma^2)\right] \delta_{\alpha\beta}\Delta(x-y;\sigma) \right\} \end{aligned} \tag{2.90}$$

where m_1 is the threshold for the production of two or more particles. Evaluating Eq. (2.90) at equal times we can obtain

$$1 = Z_2 + \int_{m_1^2} d\sigma^2 \rho_1(\sigma^2) \tag{2.91}$$

that is

$$0 \leq Z_2 < 1 \tag{2.92}$$

2.7.3 Reduction formula fermions

To get the reduction formula for fermions we will proceed as in the scalar case. The only difficulty has to do with the spinor indices. The creation and annihilation operators can be expressed in terms of the fields ψ_{in} by the relations,

$$\begin{aligned} b_{in}(p,s) &= \int d^3x \overline{u}(p,s) e^{ip\cdot x} \gamma^0 \psi_{in}(x) \\ d^\dagger_{in}(p,s) &= \int d^3x \overline{v}(p,s) e^{-ip\cdot x} \gamma^0 \psi_{in}(x) \end{aligned}$$

$$
\begin{aligned}
b^{\dagger}_{in}(p,s) &= \int d^3x \overline{\psi}_{in}(x)\gamma^0 e^{-ip\cdot x}u(p,s) \\
d_{in}(p,s) &= \int d^3x \overline{\psi}_{in}(x)\gamma^0 e^{ip\cdot x}v(p,s)
\end{aligned}
\tag{2.93}
$$

with the integrals being time independent. In fact, to be more rigorous we should substitute the plane wave solutions by wave packets, but as in the scalar case, to simplify matter we will not do it. To establish the reduction formula we start by extracting one electron from the initial state,

$$
\begin{aligned}
\langle\beta; out|(ps)\alpha; in\rangle &= \left\langle \beta; out|b^{\dagger}_{in}(p,s)|\alpha, in\right\rangle \\
&= \langle\beta-(p,s); out|\alpha; in\rangle + \left\langle\beta; out|b^{\dagger}_{in}(p,s)-b^{\dagger}_{out}(p,s)|\alpha; in\right\rangle \\
&= \text{disconnected terms} \\
&\quad + \int d^3x \left\langle\beta; out|\overline{\psi}_{in}(x)-\overline{\psi}_{out}(x)|\alpha; in\right\rangle \gamma^0 e^{-ip\cdot x}u(p,s) \\
&= \text{disconnected terms} \\
&\quad - \left(\lim_{t\to+\infty} - \lim_{t\to-\infty}\right)\frac{1}{\sqrt{Z_2}}\int d^3x \left\langle\beta; out|\overline{\psi}(x)|\alpha; in\right\rangle \gamma^0 e^{-ip\cdot x}u(p,s) \\
&= \text{disconnected terms} \\
&\quad - Z_2^{-1/2}\int d^4x \left[\left\langle\beta; out|\partial_0\overline{\psi}(x)|\alpha; in\right\rangle \gamma^0 e^{-ip\cdot x}u(p,s)\right. \\
&\qquad\qquad \left. + \left\langle\beta; out|\overline{\psi}(x)|\alpha; in\right\rangle \gamma^0 \partial_0(e^{-ip\cdot x}u(p,s))\right]
\end{aligned}
\tag{2.94}
$$

Using now

$$
(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)(e^{-ip\cdot x}u(p,s)) = 0 \tag{2.95}
$$

we get, after an integration by parts,

$$
\begin{aligned}
\left\langle\beta; out|b^{\dagger}_{in}(\rho,s)|\alpha; in\right\rangle &= \text{disconnected terms} \\
&\quad -iZ_2^{-1/2}\int d^4x \left\langle\beta; out|\overline{\psi}(x)|\alpha; in\right\rangle (-i\overleftarrow{\not\partial}_x - m)e^{-ip\cdot x}u(p,s)
\end{aligned}
\tag{2.96}
$$

In a similar way the reduction of an anti-particle from the initial state gives,

$$
\begin{aligned}
\left\langle\beta; out|d^{\dagger}_{in}(p,s)|\alpha; in\right\rangle &= \text{disconnected terms} \\
&\quad + iZ_2^{-1/2}\int d^4x e^{-ip\cdot x}\overline{v}(p,s)(i\not\partial_x - m)\left\langle\beta; out|\psi(x)|\alpha; in\right\rangle
\end{aligned}
\tag{2.97}
$$

while the reduction of a particle or of an anti-particle from the final state give, respectively,

$$
\langle\beta; out|b_{out}(p,s)|\alpha; in\rangle = \text{disconnected terms}
$$

$$-iZ_2^{-1/2}\int d^4x e^{ip\cdot x}\overline{u}(p,s)(i\partial\!\!\!/_x - m)\left\langle \beta; out|\psi(x)|\alpha; in\right\rangle \tag{2.98}$$

and

$$\begin{aligned}\left\langle \beta; out|d_{out}(p,s)|\alpha; in\right\rangle &= \text{disconnected terms}\\ &+iZ_2^{-1/2}\int d^4x \left\langle \beta; out|\overline{\psi}(x)|\alpha; in\right\rangle (-i\overleftarrow{\partial\!\!\!/}_x - m)v(p,s)e^{ip\cdot x}\end{aligned} \tag{2.99}$$

Notice the formal relation between one electron in the initial state and a positron in the final state. To go from one to the other one just has to do,

$$u(p,s)e^{-ip\cdot x} \to -v(p,s)e^{ip\cdot x} \tag{2.100}$$

The following steps in the reduction are similar, one only has to pay attention to signs because of the anti-commutation relations for fermions. To write the final expression we denote the momenta in the state $\langle in|$ by p_i or $\overline{p_i}$, respectively for particles or anti-particles, and those in the state $\langle out|$ by $p'_i, \overline{p}'_i$. We also make the following conventions (needed to define the global sign),

$$|(p_1,s_1),\cdots,(\overline{p}_1,\overline{s}_1);\cdots;in\rangle = b^\dagger_{in}(p_1,s_1)\cdots d^\dagger_{in}(\overline{p}_1,\overline{s}_1)\cdots|0\rangle \tag{2.101}$$

and

$$\Big\langle out;(p'_1,s'_1)\cdots,(\overline{p}'_1,\overline{s}'_1)\cdots\Big| = \langle 0|\cdots d_{out}(\overline{p}'_1,\overline{s}'_1),\cdots b_{out}(p'_1,s'_1) \tag{2.102}$$

Then, if $n(n')$ denotes the total number of particles (anti-particles), we get

$$\begin{aligned}&\Big\langle out;(p'_1,s'_1)\cdots,(\overline{p}'_1,\overline{s}'_1)\cdots|(p_1,s_1),\cdots(\overline{p}_1,\overline{s}_1),\cdots;in\Big\rangle = \text{disconnected terms}\\ &\quad+(-iZ_2^{-1/2})^n(iZ_2^{-1/2})^{n'}\int d^4x_1\cdots d^4y_1\cdots d^4x'_1\cdots d^4y'_1\cdots\\ &\qquad e^{-i\sum(p_i\cdot x_i)-i\sum(\overline{p}_i\cdot y_i)}e^{+i\sum(p'_i\cdot x'_i)+i\sum(\overline{p}'_i\cdot y'_i)}\\ &\qquad\overline{u}(p'_1,s'_1)(i\partial\!\!\!/_{x'_1}-m)\cdots\overline{v}(\overline{p}_1,\overline{s}_1)(i\partial\!\!\!/_{y_1}-m)\\ &\qquad\langle 0|\,T(\cdots\overline{\psi}(y'_1)\cdots\psi(x'_1)\overline{\psi}(x_1)\cdots\psi(y_1)\cdots|0\rangle\\ &\qquad(-i\overleftarrow{\partial\!\!\!/}_{x_1}-m)u(p_1,s_1)\cdots(-i\overleftarrow{\partial\!\!\!/}_{y'_1}-m)v(\overline{p}'_1,\overline{s}'_1)\end{aligned} \tag{2.103}$$

Eq. (2.103) is the fundamental expression that allows to relate the elements of the S matrix with the Green functions of the theory. The operators within the time-ordered product can be reordered, modulo some minus sign. The sign and ordering shown correspond to the conventions in Eqs. (2.101) and (2.102). In terms of diagrams, we represent the Green function,

$$\langle 0|\,T\left[\overline{\psi}(y'_{m'})\cdots\overline{\psi}(y'_1)\psi(x'_{\ell'})\cdots\psi(x'_1)\overline{\psi}(x_1)\cdots\overline{\psi}(x_\ell)\psi(y_1)\cdots\psi(y_m)\right]|0\rangle \tag{2.104}$$

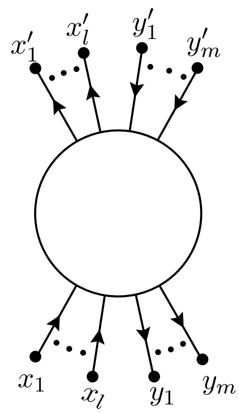


Figure 2.1: Green function for fermions.

by the diagram[3] of Fig. (2.1).

The operators $(i\partial\!\!\!/ - m)$ e $(-i\overleftarrow{\partial\!\!\!/} - m)$ force the particles to be on-shell and remove the propagators from the external lines (truncated Green functions). In the next chapter we will learn how to determine these functions in perturbation theory.

2.8 Reduction formula for photons

The *LSZ* formalism for photons, has some difficulties connected with the problems in quantizing the electromagnetic field. When one adopts a formalism (radiation gauge) where the only components of the field A^μ are transverse (as in Ref.[1]), the problems arise in showing the Lorentz and gauge invariance of the S matrix. In the formalism of the undefined metric, that we adopted in section 1.4.2, the difficulties are connected with the states of negative norm, besides the gauge invariance.

Here we are going to ignore these difficulties[4] and assume that we can define the *in* fields by the relation,

$$\sqrt{Z_3}A^\mu_{in}(x) = A^\mu(x) - \int d^4y D^{\mu\nu}_{\rm ret}(x-y)j_\nu(y) \tag{2.105}$$

and in the same way for the *out* fields,

$$\sqrt{Z_3}A^\mu_{out}(x) = A^\mu(x) - \int d^4y D^{\mu\nu}_{\rm adv}(x-y)j_\nu(y) \tag{2.106}$$

[3]With lepton number conservation, the number of particles minus anti-particles is conserved, that is

$$\ell - m = \ell' - m'$$

[4]We will see in chapter 6 a more satisfactory procedure to quantize all gauge theories, including Maxwell theory of the electromagnetic field. We will see that the resulting perturbation theory coincides with the one we get here. This is our justification to be less precise here.

where

$$
\begin{aligned}
&\Box A^\mu_{in} = \Box A^\mu_{out} = 0\\
&\Box A^\mu = j^\mu\\
&\Box D^{\mu\nu}_{\text{adv, ret}} = \delta^{\mu\nu}\delta^\mu(x-y)
\end{aligned}
\tag{2.107}
$$

The fields *in* and *out* are free fields, and therefore they have a Fourier expansion in plane waves and creation and annihilation operators of the form

$$
A^\mu_{in}(x) = \int \widetilde{dk} \sum_{\lambda=0}^{3} \left[a_{in}(k,\lambda)\varepsilon^\mu(k,\lambda)e^{-ik\cdot x} + a^\dagger_{in}(k,\lambda)\varepsilon^{\mu *}(k,\lambda)e^{ik\cdot x}\right] \tag{2.108}
$$

and therefore

$$
\begin{aligned}
a_{in}(k,\lambda) &= -i\int d^3x e^{ik\cdot x}\overleftrightarrow{\partial}_0 \varepsilon^\mu(k,\lambda)A^{in}_\mu(x)\\
a^\dagger_{in}(k,\lambda) &= i\int d^3x e^{-ik\cdot x}\overleftrightarrow{\partial}_0 \varepsilon^{\mu *}(k,\lambda)A^{in}_\mu(x)
\end{aligned}
\tag{2.109}
$$

where, as usual, $a_{in}(k,\lambda)$ and $a^\dagger_{in}(k,\lambda)$ are time independent. In Eq. (2.108) all the polarizations appear, but as the elements of the S matrix are between physical states, we are sure that the longitudinal and scalar polarizations do not contribute. In this formalism what is difficult to show is the spectral decomposition. We are not going to enter those details, just state that we can show that Z_3 is gauge independent and satisfies $0 \leq Z_3 < 1$. The reduction formula is easily obtained. We get

$$
\begin{aligned}
\langle \beta; out|(k\lambda)\alpha; in\rangle &= \langle \beta - (k,\lambda); out|\alpha; in\rangle + \left\langle \beta; out|a^\dagger_{in}(k,\lambda) - a^\dagger_{out}(k,\lambda)|\alpha; in\right\rangle\\
&= \text{disconnected terms}\\
&\quad + i\int d^3x e^{-ik\cdot x}\overleftrightarrow{\partial}_0 \varepsilon^*_\mu(k,\lambda)\langle \beta; out|A^\mu_{in}(x) - A^\mu_{out}(x)|\alpha; in\rangle\\
&= \text{disconnected terms}\\
&\quad -i(\lim_{t\to+\infty} - \lim_{t\to-\infty})Z_3^{-1/2}\int d^3x e^{-ik\cdot x}\overleftrightarrow{\partial}_0 \langle \beta; out|A^\mu(x)|\alpha; in\rangle\, \varepsilon^*_\mu(k,\lambda)\\
&= \text{disconnected terms}\\
&\quad -iZ_3^{-1/2}\int d^4x e^{-ik\cdot x}\overleftrightarrow{\partial}_0 \langle \beta; out|A^\mu(x)|\alpha; in\rangle\, \varepsilon^*_\mu(k,\lambda)\\
&= \text{disconnected terms}\\
&\quad -iZ_3^{-1/2}\int d^4x e^{-ik\cdot x}\overrightarrow{\Box}_x \langle \beta; out|A^\mu(x)|\alpha; in\rangle\, \varepsilon^*_\mu(k,\lambda)
\end{aligned}
\tag{2.110}
$$

The final formula for photons is then

$$
\left\langle k'_1 \cdots k'_n; out|k_1\cdots k_\ell; in\right\rangle = \text{disconnected terms}
$$

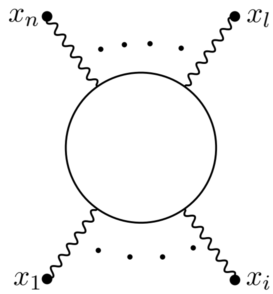


Figure 2.2: Green function for photons.

$$
\begin{aligned}
&+\left(\frac{-i}{\sqrt{Z_3}}\right)^{n+\ell} \int d^4y_1 \cdots d^4y_n d^4x_1 \cdots d^4x_\ell \; e^{\left[i\sum^n k_i'\cdot y_i - i\sum^\ell k_i\cdot x_i\right]}\\
&\qquad \varepsilon^{\mu_1}(k_1,\lambda_1)\cdots\varepsilon^{\mu_\ell}(k_\ell,\lambda_\ell)\varepsilon^{*\mu_1'}(k_1',\lambda_1')\cdots\varepsilon^{*\mu_n'}(k_n',\lambda_n')\\
&\qquad \Box_{y_1}\cdots\Box_{x_\ell}\langle 0|\,T(A_{\mu_1'}(y_1)\cdots A_{\mu_n'}(y_n)A_{\mu_1}(x_1)\cdots A_{\mu_\ell}(x_\ell)\,|0\rangle
\end{aligned}
\tag{2.111}
$$

and corresponds to the diagram of Fig. (2.2).

2.9 Cross sections

The reduction formulas, Eqs.(2.67), (2.103) and (2.111), are the fundamental results of this chapter. They relate the transition amplitudes from the initial to the final state with the Green functions of the theory. In the next chapter we will show how to evaluate these Green functions setting up the so-called covariant perturbation theory. Before we close this chapter, let us indicate how these transition amplitudes

$$S_{fi} \equiv \langle f; out|i; in\rangle \tag{2.112}$$

are related with the quantities that are experimentally accessible, the cross sections. Then the path between experiment (cross sections) and theory (Green functions) will be established.

As we have seen in the reduction formulas there is always a trivial contribution to the S matrix, that corresponds to the so-called *disconnected terms*, when the system goes from the initial to the final state without interaction. The subtraction of this trivial contribution leads us to introduce the T matrix with the relation,

$$S_{fi} = 1_{fi} + i(2\pi)^4\delta^4(P_f - P_i)T_{fi} \tag{2.113}$$

where we have factorized explicitly the delta function expressing the 4-momentum conservation. If we neglect the trivial contribution, the transition probability from the initial to the final state will be given by

$$W_{f\leftarrow i} = \left|(2\pi)^4\delta^4(P_f - P_i)T_{fi}\right|^2 \tag{2.114}$$

To proceed we have to deal with the meaning of a square of a delta function. This appears because we are using plane waves. To solve this problem we can normalize in a box of volume V and consider that the interaction has a duration of T. Then

$$(2\pi)^4\delta^4(P_f - P_i) = \lim_{\substack{V\to\infty \\ T\to\infty}} \int_V d^3x \int_{-T/2}^{T/2} dx^0 e^{i(P_f-P_i)\cdot x} \; . \tag{2.115}$$

However

$$F \equiv \int_V d^3x \int_{-T/2}^{T/2} dx^0 e^{i(P_f-P_i)\cdot x} = V\delta_{\vec{P}_f\vec{P}_j}\, \frac{2}{|E_f - E_i|}\sin\left|\frac{T}{2}(E_f - E_i)\right| \tag{2.116}$$

and the square of the last expression can be done, giving,

$$|F|^2 = V^2\delta_{\vec{P}_f,\vec{P}_i}\, \frac{4}{|E_f - E_i|^2}\sin^2\left|\frac{T}{2}(E_f - E_i)\right| \; . \tag{2.117}$$

If we want the transition rate by unit of volume (and unit of time) we divide by VT. Then

$$\Gamma_{fi} = \lim_{\substack{V\to\infty \\ T\to\infty}} V\delta_{\vec{P}_f,\vec{P}_i}\; 2\, \frac{\sin^2\frac{T}{2}(E_f - E_i)}{\frac{T}{2}(E_f - E_i)^2}|T_{fi}|^2 \tag{2.118}$$

Using now the results

$$\begin{aligned} &\lim_{V\to\infty} V\delta_{\vec{P}_f\vec{P}_j} = (2\pi)^3\delta^3(\vec{P}_f - \vec{P}_i) \\ &\lim_{T\to\infty} 2\, \frac{\sin^2\frac{T}{2}(E_f - E_i)}{\frac{T}{2}(E_f - E_i)^2} = (2\pi)\,\delta(E_f - E_i) \end{aligned} \tag{2.119}$$

we get for the transition rate by unit volume and unit time,

$$\Gamma_{fi} \equiv (2\pi)^4\delta^4(P_f - P_i)|T_{fi}|^2 \tag{2.120}$$

To get the cross section we have to further divide by the incident flux, and normalize the particle densities to one particle per unit volume. Finally, we sum (integrate) over all final states in a certain energy-momentum range. We get,

$$d\sigma = \frac{1}{\rho_1\rho_2}\frac{1}{|\vec{v}_{12}|}\Gamma_{fi}\prod_{j=3}^{n}\frac{d^3p_j}{2p_j^0(2\pi)^3} \tag{2.121}$$

where

$$\rho_1 = 2E_1 \quad ; \quad \rho_2 = 2E_2 \tag{2.122}$$

An equivalent way of writing this equation is

$$d\sigma = \frac{1}{4\left[(p_i\cdot p_2)^2 - m_1^2m_2^2\right]^{1/2}}(2\pi)^4\delta^4(P_f - P_i)|T_{fi}|^2\prod_{j=3}^{n}dp_j \tag{2.123}$$

that exhibits well the Lorentz invariance of each part that enters the cross section[5]. The incident flux and phase space factors are purely kinematics. The physics, with its interactions, is in the matrix element T_{fi}.

We note that with our conventions, fermion and boson fields have the same normalization, that is, the one-particle states obey

$$\langle p|p'\rangle = 2p^0(2\pi)^3\delta^3(\vec{p}-\vec{p}') \tag{2.124}$$

differing in this way from some older books like Ref.[3].

[5]It is assumed that, in the case of two beams they are in the same line. Then the cross section, being a *transverse area*, is invariant for Lorentz transformations along that direction.

Problems for Chapter 2

2.1 Show that the spectral representation for fermions, $\rho_{\alpha\beta}(q)$, satisfies,

a) $\rho(q) = S^{-1}(a)\rho(qa^{-1})S(a)$

b) $\rho_{\alpha\beta}(\vec{q}, q^0) = \gamma^0_{\alpha\lambda}\rho_{\lambda\delta}(-\vec{q}, q^0)\gamma^0_{\delta\beta}$

2.2 Use the results of the previous problem to show that, in a theory that preserves Parity, like QED, we have

$$\rho_{\alpha\beta}(q) = \rho_1(q^2)\not{q}_{\alpha\beta} + \rho_2(q^2)\delta_{\alpha\beta} \tag{2.125}$$

2.3 Show that the functions ρ_1 and ρ_2 defined in problem 2.2 satisfy the following properties:

i) ρ_1 and ρ_2 are real

ii) $\rho_1(\sigma^2) \geq 0$

iii) $\sigma\rho_1(\sigma^2) - \rho_2(\sigma^2) \geq 0$

2.4 Show that for the Dirac field we have

$$1 = Z_2 + \int_{m_1^2}^{\infty} d\sigma^2 \rho_1(\sigma^2) \tag{2.126}$$

2.5 Show that

$$\langle 0| \left[\varphi_{in}(x), \varphi_{out}(y)\right] |0\rangle = i\Delta(x - y; m) \tag{2.127}$$

Chapter 3

Covariant Perturbation Theory

3.1 U matrix

In this chapter we are going to develop a method to evaluate the Green functions of a given theory. From what we have seen in the two previous chapters, we realize that we only know how to calculate for free fields, like the *in* an *out* fields. However, the Green functions we are interested in, are given in terms of the physical interacting fields, and we do not know how to operate with these. We are going to see how to express the physical fields as perturbative series in terms of free *in* fields. In this way we will be able to evaluate the Green functions in perturbation theory.

We start by defining the U matrix. To simplify matters, we will be considering, for the moment, only scalar fields. In the end we will return to the other cases. The physical interacting fields $\varphi(\vec{x},t)$ and their conjugate momenta $\pi(\vec{x},t)$, satisfy the same equal time commutation relations than the *in* fields, $\varphi_{in}(\vec{x},t)$ and their $\pi_{in}(\vec{x},t)$. Also, both φ and φ_{in} form a complete set of operators, in the sense that any state, free or interacting, can be obtained by application of φ_{in} or φ in the vacuum. This implies that there should be an unitary transformation $U(t)$ that relates φ with φ_{in}, that is,

$$\begin{aligned}
\varphi(\vec{x},t) &= U^{-1}(t)\varphi_{in}(\vec{x},t)U(t) \\
\pi(\vec{x},t) &= U^{-1}(t)\pi_{in}(\vec{x},t)U(t)
\end{aligned} \tag{3.1}$$

The dynamics of U can be obtained from the equations of motion for $\varphi(x)$ and $\varphi_{in}(x)$. These are,

$$\begin{aligned}
\frac{\partial \varphi_{in}}{\partial t}(x) &= i[H_{in}(\varphi_{in},\pi_{in}),\varphi_{in}] \\
\frac{\partial \pi_{in}}{\partial t}(x) &= i[H_{in}(\varphi_{in},\pi_{in}),\pi_{in}]
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\frac{\partial \varphi}{\partial t}(x) &= i[H(\varphi,\pi),\varphi] \\
\frac{\partial \pi}{\partial t}(x) &= i[H(\varphi,\pi),\pi]
\end{aligned} \tag{3.3}$$

Then from Eqs. (3.2) and (3.1) we get,

$$\begin{aligned}
\dot{\varphi}_{in}(x) &= \frac{\partial}{\partial t}\left[U(t)\varphi(x)U^{-1}(t)\right] \\
&= \left[\dot{U}(t)U^{-1}(t), \varphi_{in}\right] + i\left[H(\varphi_{in}, \pi_{in}), \varphi_{in}(x)\right] \\
&= \dot{\varphi}_{in}(x) + \left[\dot{U}U^{-1} + iH_I(\varphi_{in}, \pi_{in}), \varphi_{in}\right] \qquad (3.4)
\end{aligned}$$

where

$$H_I(\varphi_{in}, \pi_{in}) = H(\varphi_{in}, \pi_{in}) - H_{in}(\varphi_{in}, \pi_{in}) \equiv H_I(t) \qquad (3.5)$$

and in a similar way

$$\dot{\pi}_{in}(x) = \dot{\pi}_{in} + \left[\dot{U}U^{-1} + iH_I(\varphi_{in}, \pi_{in}), \pi_{in}\right] \qquad (3.6)$$

From Eqs. (3.4) and (3.6) we obtain,

$$i\dot{U}U^{-1} = H_I(t) + E_0(t) \qquad (3.7)$$

where $E_0(t)$ commutes with φ_{in} and π_{in} and is therefore a time dependent c-number, not an operator. Defining

$$H'_I(t) = H_I(t) + E_0(t) \qquad (3.8)$$

we get a differential equation for $U(t)$, that reads,

$$i\frac{\partial U(t)}{\partial t} = H'_I(t)U(t) \qquad (3.9)$$

The solution of this equation in terms of the *in* fields, is the basis of the covariant perturbation theory.

To integrate Eq. (3.9) we need an initial condition. For that we introduce the operator

$$U(t, t') \equiv U(t)U^{-1}(t') \qquad (3.10)$$

where $t \geq t'$, and that obviously satisfies

$$U(t, t) = 1 \qquad (3.11)$$

It is easy to see that $U(t, t')$ also satisfies Eq. (3.9), that is,

$$i\frac{\partial U(t, t')}{\partial t} = H'_I(t)U(t, t') \qquad (3.12)$$

and has the initial condition, Eq. (3.11). To proceed we start by transforming Eq. (3.12) in an equivalent integral equation, that is,

$$U(t, t') = 1 - i\int_{t'}^{t} dt_1 H'_I(t_1)U(t_1, t') \qquad (3.13)$$

Notice that we have not solved the problem because $U(t,t')$ appears on both sides of the equation. However, we can iterate the equation to get the expansion,

$$
\begin{aligned}
U(t,t') &= 1 - i\int_{t'}^{t} dt_1 H_I'(t_1) + (-i)^2 \int_{t'}^{t} dt_1 H_I'(t_1)\int_{t'}^{t_1} dt_2 H_I'(t_2) \\
&\quad + \cdots + (-i)^n \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{n-1}} dt_n H_I'(t_1)\cdots H_I'(t_n) \\
&\quad + \cdots
\end{aligned}
\tag{3.14}
$$

Of course this expansion can only be useful if H_I contains a small parameter and, because of that, we can truncate the expansion at certain order in that parameter. Coming back to Eq. (3.14), as $t_1 \geq t_2 \geq, \cdots t_n$, the product is time-ordered and we can therefore write

$$
U(t,t') = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{n-1}} dt_n T(H_I'(t_1)\cdots H_I'(t_n)) \tag{3.15}
$$

Using the symmetry t_1, t_2 we can write,

$$
\begin{aligned}
\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 T(H_I'(t_1)H_I'(t_2)) &= \int_{t'}^{t} dt_2 \int_{t'}^{t_2} dt_1 T(H_I'(t_1)H_I'(t_2)) \\
&= \frac{1}{2}\int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 T(H_I'(t_1)H_2'(t_2))
\end{aligned}
\tag{3.16}
$$

which can be seen from the illustration in Fig. 3.1.

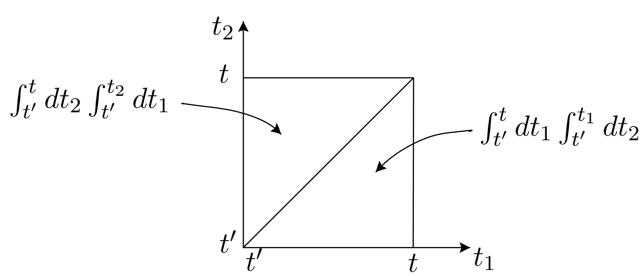


Figure 3.1: Integration regions in Eq.(3.16)

In general, for n integrations, instead of $\frac{1}{2}$ we will have $\frac{1}{n!}$, and we get,

$$
\begin{aligned}
U(t,t') &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t'}^{t} dt_1 \cdots \int_{t'}^{t} dt_n T(H_I'(t_1)\cdots H_I'(t_n)) \\
&\equiv T\left(\exp[-i\int_{t'}^{t} dt H_I'(t)]\right) \\
&= T\left(\exp[-i\int_{t'}^{t} d^4x \mathcal{H}_I(\varphi_{in})]\right)
\end{aligned}
\tag{3.17}
$$

where the time-ordered product is to be interpreted expanding the exponential.

The operators U satisfy the following multiplication rule

$$U(t,t') = U(t,t'')U(t'',t') \tag{3.18}$$

which can be seen using the definition, Eq. (3.10), or from the explicit expression, Eq. (3.17). From Eq. (3.18), we can obtain,

$$U(t,t') = U^{-1}(t',t) \tag{3.19}$$

3.2 Perturbative expansion of Green functions

As we saw in the previous chapter, the LSZ technique reduces the evaluation of the elements of the S matrix to a basic ingredient, the so-called Green functions of the theory. These are expectation values of time-ordered products of the Heisenberg fields, $\varphi(x)$,

$$G(x_1,\cdots,x_n) \equiv \langle 0|\, T\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\, |0\rangle \tag{3.20}$$

The basic idea for the evaluation of the Green functions consists in expressing the fields $\varphi(x)$ in terms of the fields $\varphi_{in}(x)$, using the operator U. We get

$$\begin{aligned} G(x_1,\cdots,x_n) &= \langle 0|\, T(U^{-1}(t_1)\varphi_{in}(x_1)U(t_1,t_2)\varphi_{in}(x_2)U(t_2,t_3)\cdots \\ &\quad \cdots U(t_{n-1},t_n)\varphi_{in}(x_n)U(t_n))\, |0\rangle \\ &= \langle 0|\, T(U^{-1}(t)U(t,t_1)\varphi_{in}(x_1)U(t_1,t_2)\cdots \\ &\quad \cdots U(t_{n-1},t_n)\varphi_{in}(x_n)U(t_n,-t)U(-t))\, |0\rangle \end{aligned} \tag{3.21}$$

where t is a time that we will let go to ∞. When $t \to \infty$, t is later than all the t_i and $-t$ is earlier than all the times t_i. Therefore we can take $U^{-1}(t)$ e $U(-t)$ out of the time-ordered product. Using the multiplicative property of the operator U we can then write,

$$G(x_1,\cdots,x_n) = \langle 0|\, U^{-1}(t)T\left(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n)\exp[-i\int_{-t}^{t} H'_I(t')dt']\right)U(-t)\,|0\rangle \tag{3.22}$$

where the time-ordered product T is meant to be applied after expanding the exponential. If it were not for the presence of the operators $U^{-1}(t)$ and $U(-t)$, we would have been successful in expressing the Green function $G(x_1\cdots x_n)$ completely in terms of the *in* fields. Now we are going to show that the vacuum is an eigenstate of the operator $U(t)$. For that we consider an arbitrary state $|\alpha p;\ in\rangle$ that contains one particle of momentum p, all the other quantum numbers being denoted collectively by α. To simplify, we continue considering the case of the scalar field. We have then,

$$\begin{aligned} \langle \alpha p; in|U(-t)|0\rangle &= \langle \alpha; in|a_{in}(p)U(-t)|0\rangle \\ &= -i\int d^3x f_p^*(\vec{x},-t')\left(\frac{\overrightarrow{\partial}}{\partial t'} - \frac{\overleftarrow{\partial}}{\partial t'}\right)\langle \alpha; in|\, \varphi_{in}(\vec{x},-t')U(-t)\,|0\rangle \end{aligned}$$

$$(3.23)$$

where $f_p(\vec{x},t) = e^{-ip\cdot x}$. We use now Eq. (3.1) to express $\varphi_{in}(\vec{x},-t)$ in terms of $\varphi(\vec{x},-t)$. We get,

$$\begin{aligned}
\langle \alpha p; in|U(-t)|0\rangle &= \\
&= -i\int d^3x f_p^*(\vec{x},-t')\overleftrightarrow{\partial}_0' \langle\alpha; in|U(-t')\varphi(\vec{x},-t')U^{-1}(-t')U(-t)|0\rangle \\
&= -i\int d^3x f_p^*(\vec{x},-t')\Big[-\overleftarrow{\partial}_{0'} \langle\alpha; in|\, U(-t')\varphi(\vec{x},-t')U^{-1}(-t')U(-t)\,|0\rangle \\
&\qquad + \langle\alpha; in|\, U(-t')\dot{\varphi}(\vec{x},-t')U^{-1}(-t')U(-t)\,|0\rangle\Big] \\
&\quad -i\int' d^3x f_p^*(\vec{x},-t')\,\langle\alpha; in|\,\dot{U}(-t')\varphi(\vec{x},-t')U^{-1}(-t')U(-t)\,|0\rangle \\
&\quad -i\int d^3x f_p^*(\vec{x},-t')\,\langle\alpha; in|\,U(-t')\varphi(\vec{x},-t')\dot{U}^{-1}(-t')U(-t)\,|0\rangle \qquad (3.24)
\end{aligned}$$

We take now the $t = t' \to \infty$ limit. Then

$$\begin{aligned}
&\langle \alpha p; in|\,U(-t)\,|0\rangle = \sqrt{Z}\,\langle\alpha; in|\,U(-t)a_{in}(p)\,|0\rangle \\
&\quad -i\int d^3x f^*(\vec{x},-t)\left[\langle\alpha; in|\,\dot{U}(-t)\varphi(\vec{x},-t) + U(-t)\varphi(\vec{x},-t)\dot{U}^{-1}(-t)U(-t)\,|0\rangle\right] \quad (3.25)
\end{aligned}$$

Now the first term in Eq. (3.25) vanishes because $a_{in}(p)\,|0\rangle = 0$. The second term also vanishes because we have (we omit the arguments to simplify the notation),

$$\begin{aligned}
\dot{U}\varphi + U\varphi\dot{U}^{-1}U &= \dot{U}U^{-1}\varphi_{in}U + \varphi_{in}U\dot{U}^{-1}U \\
&= \dot{U}U^{-1}\varphi_{in}U - \varphi_{in}\dot{U}U^{-1}U \\
&= [\dot{U}U^{-1}, \varphi_{in}]U = -i[H_I, \varphi_{in}]U = 0 \qquad (3.26)
\end{aligned}$$

where we have used Eq. (3.7) and assumed that the interactions have no derivative[1]. We conclude then that,

$$\lim_{t\to\infty} \langle\alpha p; in|U(-t)|0\rangle = 0 \qquad (3.27)$$

for all states in that contain at least one particle. This means that,

$$\lim_{t\to\infty} U(-t)\,|0\rangle = \lambda_-\,|0\rangle \qquad (3.28)$$

In a similar way we could show that,

$$\lim_{t\to\infty} U(t)\,|0\rangle = \lambda_+\,|0\rangle \qquad (3.29)$$

[1]The study of theories with derivatives was not trivial before the quantization via path integral was introduced. As we will be viewing this method for gauge theories, we can avoid here the complications of the derivatives. The quantization via path integral is the only method that is available for non-abelian gauge theories as we will be discussing in chapter 6.

Returning now to the expression for the Green function, we can write,

$$G(x_1, \cdots x_n) = \lambda_- \lambda_+^* \langle 0| T \left(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \exp \left[-i \int_{-t}^{t} H_I'(t')dt' \right] \right) |0\rangle \tag{3.30}$$

The dependence in the operator U disappeared from the expectation value. To proceed, let us evaluate the constants $\lambda_\pm$, or more to the point, the combination $\lambda_- \lambda_+^*$ that appears in Eq. (3.30). We get (in the limit $\rightarrow \infty$),

$$\begin{aligned} \lambda_- \lambda_+^* &= \langle 0| U(-t) |0\rangle \langle 0| U^{-1}(t) |0\rangle \\ &= \langle 0| U(-t) U^{-1}(t) |0\rangle = \langle 0| U(-t, t) |0\rangle \\ &= \langle 0| T \left(\exp \left[+i \int_{-t}^{t} dt' H_I'(t') \right] \right) |0\rangle \\ &= \langle 0| T \left(\exp \left[-i \int_{-t}^{t} dt' H_I'(t') \right] \right) |0\rangle^{-1} \end{aligned} \tag{3.31}$$

Using this result we can write the Green function of Eq. (3.30) in the form,

$$G(x_1, \cdots, x_n) = \frac{\langle 0| T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \exp[-i \int_{-t}^{t} dt' H_I'(t')]) |0\rangle}{\langle 0| T(\exp[-i \int_{-t}^{t} dt' H_I'(t')) |0\rangle} \tag{3.32}$$

when $t \rightarrow \infty$. Before we write the final expression, we can now introduce the number $E_0(t)$. For that we recall that,

$$H_I' = H_I + E_0 \tag{3.33}$$

and noticing that E_0 is not an operator, we get a factor $\exp[-i \int_{-t}^{t} dt' E_0(t')]$ both in the numerator and denominator, canceling out in the final result. The final result can then be obtained from Eq. (3.32), just substituting H_I' by H_I. We get,

$$\begin{aligned} G(x_1 \cdots, x_n) &= \frac{\langle 0| T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \exp[-i \int_{-t}^{t} dt' H_I(t')]) |0\rangle}{\langle 0| T(\exp[-i \int_{-t}^{t} dt' H_I(t')) |0\rangle} \\ &= \frac{\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} d^4y_1 \cdots d^4y_m \langle 0| T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \mathcal{H}_I(y_1) \cdots \mathcal{H}_I(y_m) |0\rangle}{\sum_{m=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} d^4y_1 \cdots d^4y_m \langle 0| T(\mathcal{H}_I(y_1) \cdots \mathcal{H}_I(y_m)) |0\rangle} \end{aligned} \tag{3.34}$$

This equation is the fundamental result. The Green functions have been expressed in terms of the *in* fields whose algebra we know. It is therefore possible to reduce Eq. (3.34) to known quantities. In this reduction plays an important role the Wick's theorem, to which we now turn.

3.3 Wick's theorem

To evaluate the amplitudes that appear in Eq. (3.34) we have to move the annihilation operators to the right until they act on the vacuum. The final result from these manipulations can be stated in the form of a theorem, known as Wick's theorem, which reads,

$$T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n)) =$$

$$
\begin{aligned}
&= \ :\varphi_{in}(x_1)\cdots\varphi_{in}(x_n): +[\langle 0|\, T(\varphi_{in}(x_1)\varphi_{in}(x_2))\,|0\rangle :\varphi_{in}(x_3)\cdots\varphi_{in}(x_n): +\text{perm.}\\
&+ \ \langle 0|\, T(\varphi_{in}(x_1)\varphi_{in}(x_2))\,|0\rangle \langle 0|\, T(\varphi_{in}(x_3)\varphi_{in}(x_4))\,|0\rangle :\varphi_{in}(x_5)\cdots\varphi_{in}(x_n): +\text{perm.}\\
&+\cdots\\
&+\begin{cases}
\langle 0|\, T(\varphi_{in}(x_1)\varphi_{in}(x_2))\,|0\rangle \cdots \langle 0|\, T(\varphi_{in}(x_{n-1})\varphi_{in}(x_n))\,|0\rangle + \text{perm.} & \\
 & \text{n even}\\
\langle 0|\, T(\varphi_{in}(x_1)\varphi_{in}(x_2)\,|0\rangle \cdots \langle 0|\, T(\varphi_{in}(x_{n-2})\varphi_{in}(x_{n-1}))\,|0\rangle\, \varphi_{in}(x_n) + \text{perm.} & \\
 & \text{n odd}
\end{cases}
\end{aligned}
\tag{3.35}
$$

Proof:

The proof of the theorem is done by induction. For $n = 1$ it is certainly true (and trivial). Also for $n = 2$ we can shown that

$$T(\varphi_{in}(x_1)\varphi_{in}(x_2)) =: \varphi_{in}(x_1)\varphi_{in}(x_2) : +\text{c-number} \tag{3.36}$$

where the *c-number* comes from the commutations that are needed to move the annihilation operators to the right. To find this constant, we do not have to do any calculation, just to notice that

$$\langle 0| :\cdots: |0\rangle = 0 \tag{3.37}$$

Then

$$T(\varphi_{in}(x_1)\varphi_{in}(x_2)) =: \varphi_{in}(x_1)\varphi_{in}(x_2) : + \langle 0|\, T(\varphi_{in}(x_1)\varphi_{in}(x_2))\,|0\rangle \tag{3.38}$$

which is in agreement with Eq. (3.35).

Continuing with the induction, let us assume that Eq. (3.35) is valid for a given n. We have to show that it remains valid for $n+1$. Let us consider then $T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_{n+1}))$ and let us assume that t_{n+1} is the earliest time. Then

$$
\begin{aligned}
&T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_{n+1})) =\\
&\qquad = \ T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n))\varphi_{in}(x_{n+1})\\
&\qquad = \ :\varphi_{in}(x_1)\cdots\varphi_{in}(x_n): \varphi_{in}(x_{n+1})\\
&\qquad\quad + \sum_{\text{perm}} \langle 0|\, T(\varphi_{in}(x_1)\varphi_{in}(x_2))\,|0\rangle :\varphi_{in}(x_3)\cdots\varphi_{in}(x_n): \varphi_{in}(x_{n+1})\\
&\qquad\quad +\cdots
\end{aligned}
\tag{3.39}
$$

To write Eq. (3.39) in the form of Eq. (3.35) it is necessary to find the rule showing how to introduce $\varphi_{in}(x_{n+1})$ inside the normal product. For that, we introduce the notation,

$$\varphi_{in}(x) = \varphi_{in}^{(+)}(x) + \varphi_{in}^{(-)}(x) \tag{3.40}$$

where $\varphi_{in}^{(+)}(x)$ contains the annihilation operator and $\varphi_{in}^{(-)}(x)$ the creation operator. Then we can write,

$$:\varphi_{in}(x_1)\cdots\varphi_{in}(x_n):=\sum_{A,B}\prod_{i\in A}\varphi_{in}^{(-)}(x_i)\prod_{j\in B}\varphi_{in}^{(+)}(x_j) \tag{3.41}$$

where the sum runs over all the sets A, B that constitute partitions of the n indices. Then

$$\begin{aligned}
&:\varphi_{in}(x_1)\cdots\varphi_{in}(x_n):\varphi_{in}(x_{n+1}) = \\
&\quad= \sum_{A,B}\prod_{i\in A}\varphi_{in}^{(-)}(x_i)\prod_{j\in B}\varphi_{in}^{(+)}(x_j)[\varphi_{in}^{(+)}(x_{n+1})+\varphi_{in}^{(-)}(x_{n+1})] \\
&\quad= \sum_{A,B}\prod_{i\epsilon A}\varphi_{in}^{(-)}(x_i)\prod_{j\in B}\varphi_{in}^{(+)}(x_j)\varphi_{in}^{(+)}(x_{n+1}) \\
&\qquad+\sum_{A,B}\prod_{i\in A}\varphi_{in}^{(-)}(x_i)\varphi_{in}^{(-)}(x_{n+1})\prod_{j\in B}\varphi_{in}^{(+)}(x_j) \\
&\qquad+\sum_{A,B}\prod_{i\in A}\varphi_{in}^{(-)}(x_i)\sum_{k\in B}\prod_{j\in B j\neq k}\varphi_{in}^{(+)}(x_j)\left\langle 0\right|\varphi_{in}^{(+)}(x_k)\varphi_{in}^{(-)}(x_{n+1})\left|0\right\rangle
\end{aligned} \tag{3.42}$$

we can now write,

$$\begin{aligned}
\left\langle 0\right|\varphi_{in}^{(+)}(x_k)\varphi_{in}^{(-)}(x_{n+1})\left|0\right\rangle &= \left\langle 0\right|\varphi_{in}(x_k)\varphi_{in}(x_{n+1})\left|0\right\rangle \\
&= \left\langle 0\right|T(\varphi_{in}(x_k)\varphi_{in}(x_{n+1}))\left|0\right\rangle
\end{aligned} \tag{3.43}$$

where we have used the fact that t_{n+1} is the earliest time. We can then write Eq. (3.42) in the form,

$$\begin{aligned}
&:\varphi_{in}(x_1)\cdots\varphi_{in}(x_n):\varphi_{in}(x_{n+1}) =:\varphi_{in}(x_1)\cdots\varphi_{in}(x_{n+1}): \\
&\qquad+\sum_k :\varphi_{in}(x_1)\cdots\varphi_{in}(x_{k-1})\varphi_{in}(x_{k+1})\cdots\varphi_{in}(x_n):\left\langle 0\right|T(\varphi_{in}(x_k)\varphi_{in}(x_{n+1}))\left|0\right\rangle
\end{aligned} \tag{3.44}$$

With this result, Eq. (3.39) takes the general form of Eq. (3.35) for the $n+1$ case, ending the proof of the theorem. To fully understand the theorem, it is important to do in detail the case $n=4$, to see how things work. The importance of the Wick's theorem for the applications comes from the following two corollaries.

Corollary 1 : If n is odd, then $\left\langle 0\right|T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n))\left|0\right\rangle = 0$, as results trivially from Eqs. (3.35) and (3.37) and from,

$$\left\langle 0\right|\varphi_{in}(x)\left|0\right\rangle = 0 \tag{3.45}$$

Corollary 2: If n is even

$$\left\langle 0\right|T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n))\left|0\right\rangle =$$

$$= \sum_{\text{perm}} \delta_p \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)) |0\rangle \cdots \langle 0| T(\varphi_{in}(x_{n-1})\varphi_{in}(x_n)) |0\rangle \tag{3.46}$$

where δ_p is the sign of the permutation that is necessary to introduce in case of fermion fields. This result, that in practice is the most important one, also results from Eqs. (3.35), (3.37) and (3.45).

Therefore the vacuum expectation value of the time-ordered product of n operators that appear in the general formula, Eq. (3.34), are obtained considering all the vacuum expectation values of the fields taken two by two (*contractions*) in all possible ways. Now these *contractions* are nothing else than the Feynman propagators for free fields. For instance,

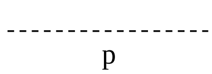
$$\begin{aligned}
&\langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4)) |0\rangle \\
&\quad= \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)) |0\rangle \langle 0| T(\varphi_{in}(x_3)\varphi_{in}(x_4) |0\rangle \\
&\qquad + \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_3)) |0\rangle \langle 0| T(\varphi_{in}(x_2)\varphi_{in}(x_4)) |0\rangle \\
&\qquad + \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_4)) |0\rangle \langle 0| T(\varphi_{in}(x_2)\varphi_{in}(x_3)) |0\rangle \\
&\quad= \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) \\
&\qquad + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3)
\end{aligned} \tag{3.47}$$

where

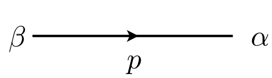
$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)} \tag{3.48}$$

is the Feynman propagator for the free field theory in the case of scalar fields.

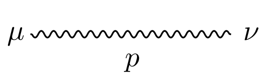
It is convenient to use a graphical (diagrammatic) representation for these propagators. We have in configuration space,



$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)} \tag{3.49}$$



$$S_F(x-y)_{\alpha\beta} = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{3.50}$$



$$D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\mu\nu}}{k^2 + i\epsilon} e^{-ik\cdot(x-y)} \tag{3.51}$$

respectively for scalar, spinor and photon (in the Feynman gauge) fields.

As the interaction Hamiltonian is normal ordered, there will be no contractions between the fields that appear in $\mathcal{H}_I$. The fields in $\mathcal{H}_I$ can only contract with fields outside. In this

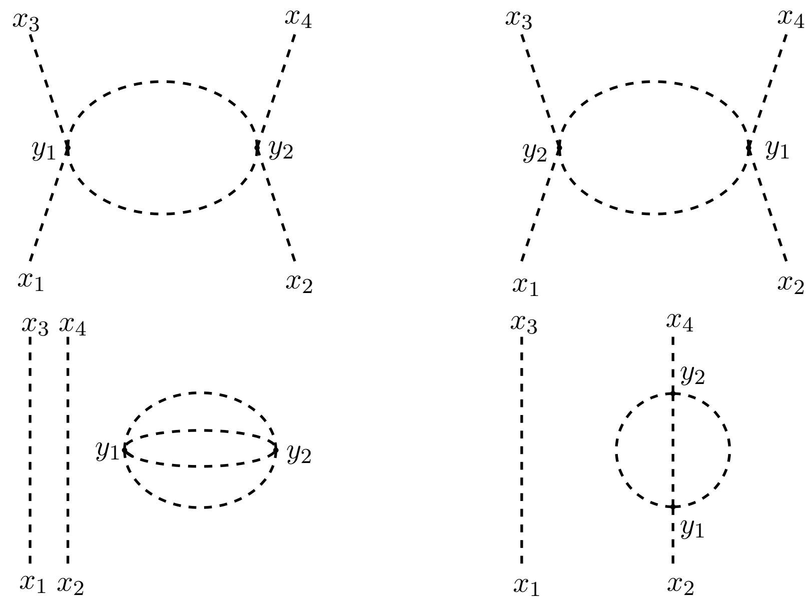


Figure 3.2: Some of the diagrams resulting from Eq (3.53).

way the contractions will connect the points corresponding to $\mathcal{H}_I$, the so-called vertices, to either external points or points in another $\mathcal{H}_I$, corresponding to another vertex. To illustrate this point let us consider the $\lambda\varphi^4$ theory where,

$$\mathcal{H}_I(x) = \frac{1}{4!}\lambda : \varphi_{in}^4(x) : \tag{3.52}$$

Then a contribution of order λ^2 to $G(x_1, x_2, x_3, x_4)$ comes from the term,

$$\frac{\lambda^2}{(4!)^2} \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4) : \varphi_{in}^4(y_1) :: \varphi_{in}^4(y_2) : |0\rangle \tag{3.53}$$

and leads to the diagrams in Fig. (3.2). In these diagrams, the interaction is represented by four lines coming from one point, y_1 or y_2. These lines are contractions between one field from one $\mathcal{H}_I$ with other field that might belong either to another $\mathcal{H}_I$, or be one of the external fields in $G(x_1 \cdots x_4)$. To obtain the Feynman rules we are left with a combinatorial problem. We are not going to find them here, as they are much easier to express in momentum space, as we will see in the following.

In Fig. (3.2) the diagrams a), b) and d) are called *connected* while the diagram c) is called *disconnected*. One diagram is *disconnected* when there is a part of the diagram that is not connected in any way to an external line. We will see in the following that these diagrams do not contribute to the Green functions. Diagram d) is *connected* but is also called *reducible* because it can be obtained by multiplication of simpler Green functions. As we will see only the *irreducible* diagrams are important.

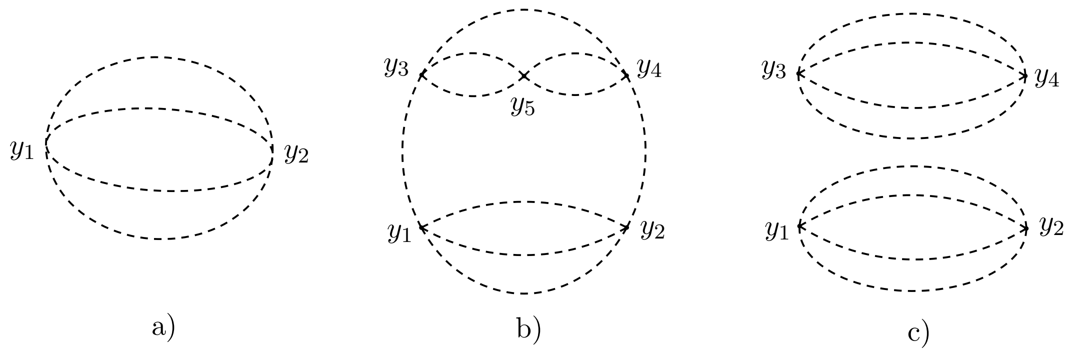


Figure 3.3: Some vacuum–vacuum amplitudes in Eq. (3.54).

3.4 Vacuum–Vacuum amplitudes

We have seen in the previous section examples of the numerator of Eq. (3.34). Let us now look at the denominator, the so-called vacuum-vacuum amplitudes. Continuing with the example of $\lambda\varphi^4$, some of the diagrams contributing for these amplitudes are shown in Fig. (3.3). The diagrams associated with the numerator of Eq. (3.34) can be separated into connected and disconnected parts. For all diagrams that have as connected part a contribution of order s in the interaction $\mathcal{H}_I$, the numerator of $G(x_1 \cdots x_n)$ takes the form,

$$\sum_{p=0}^{\infty} \frac{(-i)^p}{p!} \int d^4y_1 \cdots d^4y_p \, \langle 0| \, T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \mathcal{H}_I(y_1) \cdots \mathcal{H}_I(y_s)) \, |0\rangle_c$$
$$\times \frac{p!}{s!(p-s)!} \langle 0| \, T(\mathcal{H}_I(y_{s+1}) \cdots \mathcal{H}_I(y_p)) \, |0\rangle \tag{3.54}$$

where the subscript c indicates that only the connected parts are included. The combinatorial factor

$$\left(\begin{array}{c} p \\ \\ s \end{array} \right) = \frac{p!}{s!(p-s)!} \tag{3.55}$$

is the number of ways in which we can extract s terms $\mathcal{H}_I$ from a set of p terms. We write then Eq. (3.54) in the form $(r = p - s)$,

$$\sum_{s=0}^{\infty} \frac{(-i)^s}{s!} \int d^4y_1 \cdots d^4y_s \, \langle 0| \, T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \mathcal{H}_I(y_1) \cdots \mathcal{H}(y_s)) \, |0\rangle_c$$
$$\times \sum_{r=0}^{\infty} \frac{(-i)^r}{r!} \int d^4z_1 \cdots d^4z_r \, \langle 0| \, T(\mathcal{H}_I(z_1) \cdots \mathcal{H}_I(z_r)) \, |0\rangle \tag{3.56}$$

This equation has the form of a connected diagram of order s times an infinite series of vacuum-vacuum amplitudes, that cancels exactly against the denominator. This is true for all orders, and therefore we can write,

$$G(x_1, \cdots x_n) \quad = \quad \frac{\sum_i G_i(x_1 \cdots x_n)}{\sum_k D_k} = \frac{(\sum_i G_i^c(x_1, \cdots x_n))(\sum_k D_k)}{\sum_k D_k}$$

$$= \sum_i G_i^c(x_1 \cdots x_n) \tag{3.57}$$

where G_i^c are the connected diagrams and D_k the disconnected ones. This result means that we can simply ignore completely the disconnected diagrams and consider only the connected ones when evaluating the Green functions. These are simply the sum of all connected diagrams, simplifying enormously the structure of Eq. (3.34).

3.5 Feynman rules for $\lambda\varphi^4$

To understand how the Feynman rules appear, let us consider the case of a real scalar field with an interaction of the form,

$$\mathcal{H}_I = \frac{\lambda}{4!} : \varphi_{in}^4 := -\mathcal{L}_I \tag{3.58}$$

To be more precise we consider two particles in the initial and final state. Then the S matrix element is,

$$\begin{aligned} S_{fi} &= \langle p_1' p_2'; out | p_1 p_2; in \rangle \\ &= (i)^4 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{-ip_1\cdot x_1 - ip_2\cdot x_2 + ip_i' x_3 + ip_2' \cdot x_4} \\ & \qquad (\Box_{x_1} + m^2) \cdots (\Box_{x_4} + m^2) \langle 0 | T(\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)) | 0 \rangle \end{aligned} \tag{3.59}$$

For the Green function we use the expressions in Eqs. (3.34) and (3.57) and we obtain,

$$\begin{aligned} & G(x_1, x_2, x_3, x_4) = \sum_{p=0}^{\infty} \frac{(-i\lambda)^p}{p!} \int d^4z_1 \cdots d^4z_p \\ & \langle 0 | T(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4) : \frac{\varphi_{in}^4(z_1)}{4!} : \cdots : \frac{\varphi_{in}^4(z_p)}{4!} :) | 0 \rangle_c \end{aligned} \tag{3.60}$$

As the case $p = 0$ is trivial (there is no interaction) we begin by the $p = 1$ case.

- $p = 1$

Then the Green function is,

$$\begin{aligned} G(x_1, x_2, x_3, x_4) &= (-i\lambda) \int d^4z \, \langle 0 | T\left(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4) : \frac{\varphi_{in}^4(z)}{4!} :\right) | 0 \rangle \\ &= (-i\lambda) \frac{4!}{4!} \int d^4z \Delta_F(x_1 - z)\Delta_F(x_2 - z)\Delta_F(x_3 - z)\Delta_F(x_4 - z) \end{aligned} \tag{3.61}$$

to which corresponds, in the configuration space, the diagram of Fig. (3.4). To proceed,

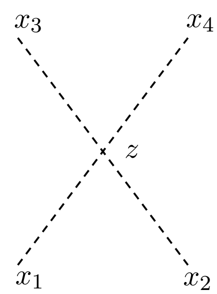


Figure 3.4: Vertex in the $\lambda\phi^4$ theory.

we introduce the Fourier transform of the propagators, that is,

$$\Delta_F(x_1 - z) = \int \frac{d^4 q_1}{(2\pi)^4} e^{-iq_1\cdot(x_1-z)} \Delta_F(q_1) \tag{3.62}$$

where

$$\Delta_F(q_1) = \frac{i}{q_1^2 - m^2} \tag{3.63}$$

then

$$\begin{aligned} G(x_1, \cdots x_4) &= (-i\lambda) \int d^4 z \frac{d^4 q_1}{(2\pi)^4} \cdots \frac{d^4 q_4}{(2\pi)^4}\ e^{-iq_1\cdot x_1 - iq_2 x_2 - iq_3 x_3 - iq_4 x_4 + i(q_1+q_2+q_3+q_4)\cdot z} \\ &\qquad \Delta_F(q_1)\Delta_F(q_2)\Delta_F(q_3)\Delta_F(q_4) \\ &= (-i\lambda) \int \frac{d^4 q_1}{(2\pi)^4} \cdots \frac{d^4 q_4}{(2\pi)^4}\ e^{-iq_1\cdot x_1 - iq_2\cdot x_2 - iq_3\cdot x_3 - iq_4\cdot x_4} \\ &\qquad (2\pi)^4 \delta^4(q_1 + q_2 + q_3 + q_4)\Delta_F(q_1)\Delta_F(q_2)\Delta_F(q_3)\Delta_F(q_4) \end{aligned} \tag{3.64}$$

If we now introduce the T matrix transition amplitude, defined by

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta(P_f - P_i)\ T_{fi} \tag{3.65}$$

we obtain

$$iT_{fi} = (-i\lambda) \tag{3.66}$$

for this amplitude we draw the Feynman diagram of Fig. (3.5), and we associate to the vertex the number $(-i\lambda)$.

• $p = 2$

Let us consider now a more complicated case, the evaluation of $G(x_1 \cdots x_4)$ in second order in the coupling λ. After this exercise we will be in position to be able to state the

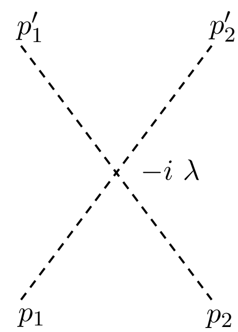
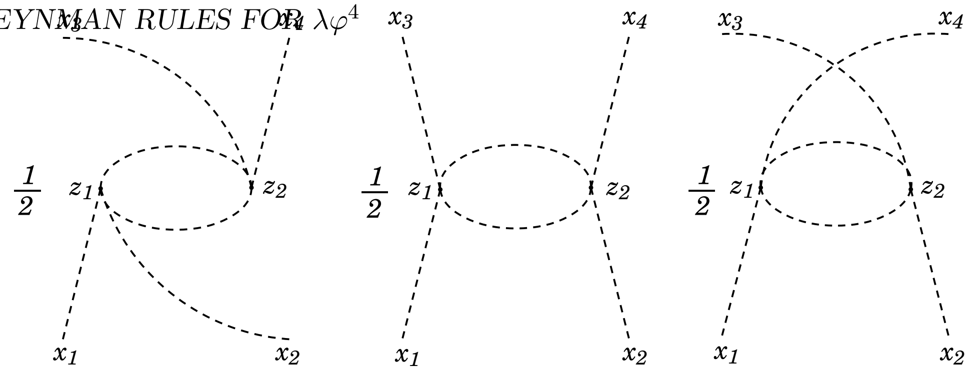


Figure 3.5: Vertex in momentum space.

Feynman rules in momentum space with all generality. From Eq. (3.60) we get in second order in λ,

$$
\begin{aligned}
& G(x_1, \cdots x_4) = \\
= \ & \frac{(-i\lambda)^2}{2!} \int d^4z_1 d^4z_2 \, \langle 0 | \, T\left(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4) : \frac{\varphi_{in}^4(z_1)}{4!} :: \frac{\varphi_{in}^4(z_2)}{4!} : \right) |0\rangle_c \\
= \ & \frac{(-i\lambda)^2}{2!} \int d^4z_1 d^4z_2 \left(\frac{4\times 3}{4!} \right) \times \left(\frac{4\times 3}{4!} \right) \times 2 \\
& \Big\{ \Delta_F(x_1-z_1)\Delta_F(x_2-z_1)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)\Delta_F(z_2-x_3)\Delta_F(z_2-x_4) \\
& +\Delta_F(x_1-z_1)\Delta_F(x_2-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-x_3)\Delta_F(z_2-x_4) \\
& +\Delta_F(x_1-z_1)\Delta_F(x_2-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-x_4)\Delta_F(z_2-x_3) \\
& +\Delta_F(x_1-z_2)\Delta_F(x_2-z_2)\Delta_F(z_2-z_1)\Delta_F(z_2-z_1)\Delta_F(z_1-x_3)\Delta_F(z_1-x_4) \\
& +\Delta_F(x_1-z_2)\Delta_F(x_2-z_1)\Delta_F(z_1-z_2)\Delta_F(z_2-z_1)\Delta_F(z_1-x_3)\Delta_F(z_2-x_4) \\
& +\Delta_F(x_1-z_2)\Delta_F(x_2-z_1)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-x_4)\Delta_F(z_1-x_3) \Big\} \\
= \ & \frac{(-i\lambda)^2}{2!} \int d^4z_1 d^4z_2 \\
& \Big\{
\end{aligned}
$$



$$
\left. + (z_1 \leftrightarrow z_2) \right\} \tag{3.67}
$$

Let us now go into momentum space, by introducing the Fourier transform of the propagators. We start by diagram a),

$$
\begin{aligned}
&G^{(a)}(x_1,x_2,x_3,x_4) = \\
&\quad= \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int d^4z_1 d^4z_2 \Delta_F(x_1-z_1)\Delta_F(x_2-z_1)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2) \\
&\qquad\qquad \Delta_F(z_2-x_3)\Delta_F(z_2-x_4) \\
&\quad= \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int d^4z_1 d^4z_2 \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\frac{d^4q_4}{(2\pi)^4}\frac{d^4q_5}{(2\pi)^4}\frac{d^4q_6}{(2\pi)^4} \\
&\qquad\qquad e^{i[(q_1\cdot x_1+q_2\cdot x_2-q_3\cdot x_3-q_4\cdot x_4)+z_1\cdot(q_5-q_1-q_2+q_6)+z_2\cdot(q_3+q_4-q_5-q_6)]} \\
&\qquad\qquad \Delta_F(q_1)\Delta_F(q_2)\Delta_F(q_3)\Delta_F(q_4)\Delta_F(q_5)\Delta_F(q_6) \\
&\quad= \frac{(-i\lambda)^2}{2!}\frac{1}{2}(2\pi)^4\int \frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_5}{(2\pi)^4}\delta^4(q_1+q_2-q_3-q_4)\ e^{i[q_1\cdot x_1+q_2\cdot x_2-q_2\cdot x_3-q_4\cdot x_4]} \\
&\qquad\qquad \Delta_F(q_1)\Delta_F(q_2)\Delta_F(q_3)\Delta_F(q_4)\Delta_F(q_5)\Delta_F(q_1+q_2-q_5)
\end{aligned} \tag{3.68}
$$

Now we insert the last equation into the reduction formula. We get

$$
\begin{aligned}
S_{fi}^{(a)} &= (i)^4\int d^4x_1\cdots d^4x_4 e^{-i[p_1\cdot x_1+p_2\cdot x_2-p_1'\cdot x_3-p_2'\cdot x_4]} \\
&\qquad (\Box_{x_1}+m^2)\cdots(\Box_{x_4}+m^2)G^{(a)}(x_1,\cdots,x_4)
\end{aligned} \tag{3.69}
$$

The only dependence of $G^{(a)}$ in the coordinates, $x_i (i=1,\cdots 4)$, is in the exponential, therefore,

$$
(\Box_{x_i}+m^2) \to (-q_i^2+m^2) \tag{3.70}
$$

and using

$$
(-q_i^2+m^2)\Delta_F(q_i) = -i \tag{3.71}
$$

we get

$$
S_{fi}^{(a)} = \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int d^4x_1\cdots d^4x_4\int\frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_5}{(2\pi)^4}(2\pi)^4\delta^4(q_1+q_2-q_3-q_4)
$$

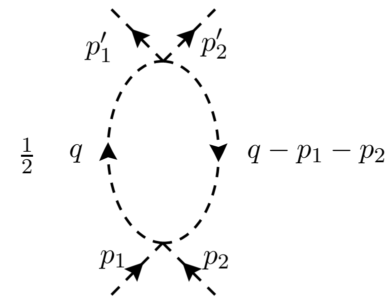


Figure 3.6:

$$
\begin{aligned}
& e^{-i[x_1\cdot(p_1-q_1)+x_2\cdot(p_2-q_2)-x_3\cdot(p_1'-q_3)-x_4\cdot(p_2'-q_4)]}\Delta_F(q_5)\Delta_F(q_1+q_2-q_5) \\
= & \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int\frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_5}{(2\pi)^4}(2\pi)^4\delta^4(q_1+q_2-q_3-q_4)(2\pi)^4\delta^4(p_1-q_1) \\
& (2\pi)^4\delta^4(p_2-q_2)(2\pi)^4\delta^4(p_1'-q_3)(2\pi)^4\delta^4(p_2'-q_4)\Delta_F(q_5)\Delta_F(q_1+q_2-q_5) \\
= & \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int\frac{d^4q_5}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\Delta_F(q_5)\Delta_F(p_1+p_2-q_5) \qquad (3.72)
\end{aligned}
$$

This expression can be written in the form

$$
S_{fi}^{(a)} = (2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\frac{(-i\lambda)^2}{2!}\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Delta_F(q)\Delta_F(p_1+p_2-q) \qquad (3.73)
$$

If we denote by $a')$ the diagram $a)$ with the interchange $z_1 \leftrightarrow z_2$ and redo the calculation we get exactly the same result as in Eq. (3.73). Therefore,

$$
S_{fi}^{(a+a')} = (2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')(-i\lambda)^2\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Delta_F(q)\Delta_F(p_1+p_2-q) \qquad (3.74)
$$

or in terms of the T_{fi} matrix,

$$
-iT_{fi}^{(a+a')} = (-i\lambda)^2\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Delta_F(q)\Delta_F(p_1+p_2-q) \qquad (3.75)
$$

To encode this result we draw the Feynman diagram of Fig. (3.6), that has the same topology as $a)$ and $a')$ but in momentum space. We find that in order to evaluate the $-iT$ matrix, we associate to each vertex a factor $(-i\lambda)$, to each internal line a propagator Δ_F and for each loop the integral $\int\frac{d^4q}{(2\pi)^4}$. Besides that we have 4-momentum conservation at each vertex. Finally there is a symmetry factor (see below) which takes the value $\frac{1}{2}$ for this diagram.

If we repeat the calculations for diagrams $b)+b')$ and $c)+c')$ it is easy to see that we get,

$$
-iT_{fi}^{(b+b')} = (-i\lambda)^2\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Delta_F(q)\Delta_F(q-p_1+p_1') \qquad (3.76)
$$

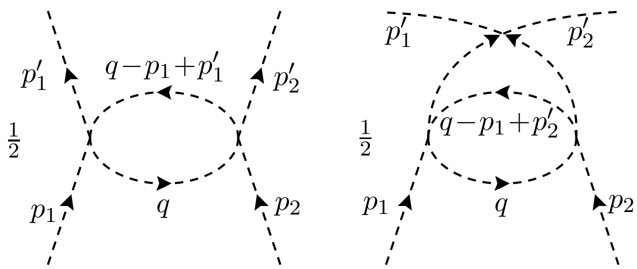


Figure 3.7:

and

$$-iT_{fi}^{(c+c')} = (-i\lambda)^2 \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_F(q) \Delta_F(q - p_1 + p_2') \tag{3.77}$$

to which correspond the diagrams of Fig. (3.7).

After this exercise we are in position to state the Feynman rules with all generality for the $\lambda\varphi^4$ theory. These are rules for the $-iT$ matrix, that is, after we factorize $(2\pi)^4\delta^4(\cdots)$. These are (for a process with n external legs):

1. Draw all topologically distinct diagrams with n external legs.
2. At each vertex multiply by the factor $(-i\lambda)$.
3. To each internal line associate a propagator $\Delta_F(q)$.
4. For each loop include an integral$\int \frac{d^4q}{(2\pi)^4}$. The direction of this momentum is irrelevant, but we have to respect 4-momentum conservation ate each vertex.
5. Multiply by the symmetry factor of the diagram. This is defined by,

$$S = \frac{\#\text{ of distinct ways of connecting the vertices to the external legs}}{\text{Permutations of each vertex} \times \text{Permutations of equal vertices}} \tag{3.78}$$

6. Add the contributions of all the topologically distinct diagrams. The result is the $-iT$ matrix amplitude that enters the formula for the cross section.

3.6 Feynman rules for QED

We now turn to the case of QED. Like $\lambda\phi^4$, it is a theory without derivatives and therefore,

$$\mathcal{L}_I = -\mathcal{H}_I = -e\ Q\overline{\psi}_{in}\gamma^\mu\psi_{in}A_\mu^{in} \tag{3.79}$$

where e is the absolute value of the electron charge, or the proton charge. For the electron the sign enters explicitly in $Q = -1$. This way of writing in Eq. (3.79), allows for obvious

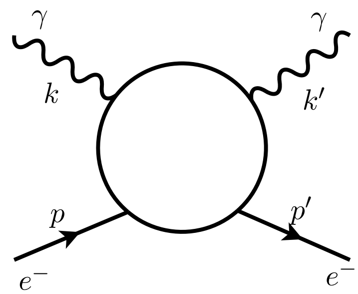


Figure 3.8: Kinematics for the Compton scattering.

generalizations for particles with other charges, like for instance the quarks. For QED we have then,

$$\mathcal{L}_I^{QED} = e\ \overline{\psi}_{in}\gamma^\mu\psi_{in}A_\mu^{in} \tag{3.80}$$

Due to the electric charge conservation, the Green functions that we have to deal with have an equal number of ψ and $\overline{\psi}$ fields. In general we have,

$$\begin{aligned} &G(x_1\cdots x_n x_{n+1}\cdots x_{2n}; y_1\cdots y_p) = \\ &\quad = \langle 0|\, T(\psi(x_1)\cdots\psi(x_n)\overline{\psi}(x_{n+1})\cdots\overline{\psi}(x_{2n})A^{\mu_1}(y_1)\cdots A^{\mu_p}(y_p))\,|0\rangle \end{aligned} \tag{3.81}$$

where, for simplicity, we omit the spinorial indices in the fermion fields. This equation is written in terms of the physical fields. Following a similar procedure to the scalar field case, we can obtain an expression for G in terms of the in fields. This will be,

$$\begin{aligned} G(x_1\cdots x_{2n}; y_1\cdots y_p) &= \frac{\langle 0|\, T\psi_{in}(x_1)\cdots\overline{\psi}_{in}(x_{2n})A_{in}^{\mu_1}(y_1)\cdots A_{in}^{\mu_p}(y_p)\ e^{[i\int d^4z\mathcal{L}_I(z)]}\,|0\rangle}{\langle 0|\, T\exp[i\int d^4z\mathcal{L}_I(z)]\,|0\rangle} \\ &= \langle 0|\, T\psi_{in}(x_1)\cdots\overline{\psi}_{in}(x_{2n})A_{in}^{\mu_1}(y_1)\cdots A_{in}^{\mu_p}(y_p)\ e^{[i\int d^4t\mathcal{L}_I(z)]}\,|0\rangle_c \end{aligned} \tag{3.82}$$

where the fields in $\mathcal{L}_I$ are normal ordered, and $\langle 0|\cdots|0\rangle_c$ means that we only consider the connected diagrams. To get the Feynman rules we will evaluate a few simple processes.

3.6.1 Compton scattering

Compton scattering corresponds to the following process,

$$e^- + \gamma \to e^- + \gamma \tag{3.83}$$

and we choose the kinematics in Fig. (3.8). The S matrix element to evaluate is therefore,

$$S_{fi} = \left\langle (p', s'), k'; out|(p, s), k; in\right\rangle \tag{3.84}$$

Using the LSZ reduction formula Eq. (2.103), and Eq. (2.111) we can write,

$$S_{fi} = \int d^4x d^4x' \int d^4y d^4y' e^{-i[p\cdot x + k\cdot y - p'\cdot x' - k'\cdot y']} \varepsilon^{\mu}(k) \varepsilon^{*\mu'}(k')$$
$$\overline{u}(p',s')_{\alpha'} (i\overrightarrow{\not\partial}_{x'} - m)_{\alpha'\beta'} \Box_y \Box_{y'} \langle 0| T(\psi_{\beta'}(x')\overline{\psi}_{\beta}(x) A_{\mu}(y) A_{\mu'}(y')) |0\rangle (-i\overleftarrow{\not\partial}_x - m)_{\beta\alpha} u_{\alpha}(p,s) \tag{3.85}$$

Our task is therefore to evaluate the Green function

$$G(x',x,y,y') \equiv \langle 0| T(\psi_{\beta'}(x')\overline{\psi}_{\beta}(x) A_{\mu}(y) A_{\mu'}(y')) |0\rangle \tag{3.86}$$

If we use Eq. (3.82) and the fact that the interaction has an odd number of fields, we find that the lowest contribution is quadratic in the interaction[2]. We get

$$\begin{aligned}
G(x,x',y,y') &= \\
&= \frac{(ie)^2}{2!} \int d^4z_1 d^4z_2 \langle 0| T(\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\beta}(x) A^{in}_{\mu}(y) A^{in}_{\mu'}(y') \\
&\qquad : \overline{\psi}_{in}(z_1)\gamma^{\sigma}\psi_{in}(z_1) A^{in}_{\sigma}(z_1) :: \overline{\psi}^{in}(z_2)\gamma^{\rho}\psi^{in}(z_2) A^{in}_{\rho}(z_2) :) |0\rangle \\
&= \frac{(ie)^2}{2!} (\gamma^{\sigma})_{\gamma\delta} \; (\gamma^{\rho})_{\gamma'\delta'} \int d^4z_1 d^4z_2 \langle 0| T(\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\beta}(x) A^{in}_{\mu}(y) A^{in}_{\mu'}(y') \\
&\qquad : \overline{\psi}^{in}_{\gamma}(z_1)\psi^{in}_{\delta}(z_1) A^{in}_{\sigma}(z_1) :: \overline{\psi}^{in}_{\gamma'}(z_2)\psi^{in}_{\delta'}(z_2) A^{in}_{\rho}(z_2) :) |0\rangle
\end{aligned} \tag{3.87}$$

Now we use Wick's theorem to write $\langle 0| T(\cdots) |0\rangle$ in terms of the propagators. We get,

$$\begin{aligned}
&\langle 0| T(\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\beta}(x) A^{in}_{\mu}(y) A^{in}_{\mu'}(y') : \overline{\psi}^{in}_{\gamma}(z_1)\psi^{in}_{\delta}(z_1) A^{in}_{\sigma}(z_1) :: \overline{\psi}^{in}_{\gamma'}(z_2)\psi^{in}_{\delta'}(z_2) A^{in}_{\rho}(z_2)) :) |0\rangle \\
&= \quad \langle 0| T\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\gamma}(z_1) |0\rangle \langle 0| T\psi^{in}_{\delta'}(z_2)\overline{\psi}^{in}_{\beta}(x) |0\rangle \langle 0| T\psi^{in}_{\delta}(z_1)\overline{\psi}_{\gamma'}(z_2) |0\rangle \\
&\qquad \langle 0| T(A^{in}_{\mu}(y) A^{in}_{\sigma}(z_1)) |0\rangle \langle 0| T A^{in}_{\mu'}(y') A^{in}_{\rho}(z_2) |0\rangle \\
&\quad + \langle 0| T\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\gamma}(z_1) |0\rangle \langle 0| T\psi^{in}_{\delta'}(z_2)\overline{\psi}^{in}_{\beta}(x) |0\rangle \langle 0| T\psi^{in}_{\delta}(z_1)\overline{\psi}_{\gamma'}(z_2) |0 \\
&\qquad \langle 0| T A^{in}_{\mu}(y) A^{in}_{\rho}(z_2) |0\rangle \langle 0| T A^{in}_{\mu'}(y') A^{in}_{\sigma}(z_1) |0\rangle \\
&\quad + \langle 0| T\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\gamma'}(z_2) |0\rangle \langle 0| T\psi^{in}_{\delta}(z_1)\overline{\psi}^{in}_{\beta}(x) |0\rangle \langle 0| T\psi^{in}_{\delta'}(z_2)\overline{\psi}^{in}_{\gamma}(z_1) |0\rangle \\
&\qquad \langle 0| T A^{in}_{\mu}(y) A^{in}_{\sigma}(z_1) |0\rangle \langle 0| T A^{in}_{\mu'}(y') A^{in}_{\rho}(z_2) |0\rangle \\
&\quad + \langle 0| T\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\gamma'}(z_2) |0\rangle \langle 0| T\psi^{in}_{\delta}(z_1)\overline{\psi}^{in}_{\beta}(x) |0\rangle \langle 0| T\psi^{in}_{\delta'}(z_2)\overline{\psi}^{in}_{\gamma}(z_1) |0\rangle \\
&\qquad \langle 0| T A^{in}_{\mu}(y) A^{in}_{\rho}(z_2) |0\rangle \langle 0| T A^{in}_{\mu'}(y') A^{in}_{\sigma}(z_1) |0\rangle \\
&= \quad S_{F\beta'\gamma}(x'-z_1) S_{F\delta'\beta}(z_2-x) S_{F\delta\gamma'}(z_1-z_2) D_{F\mu\sigma}(y-z_1) D_{F\mu'\rho}(y'-z_2)
\end{aligned}$$

[2]By Wick's theorem the expectation value of an odd number of fields vanishes.

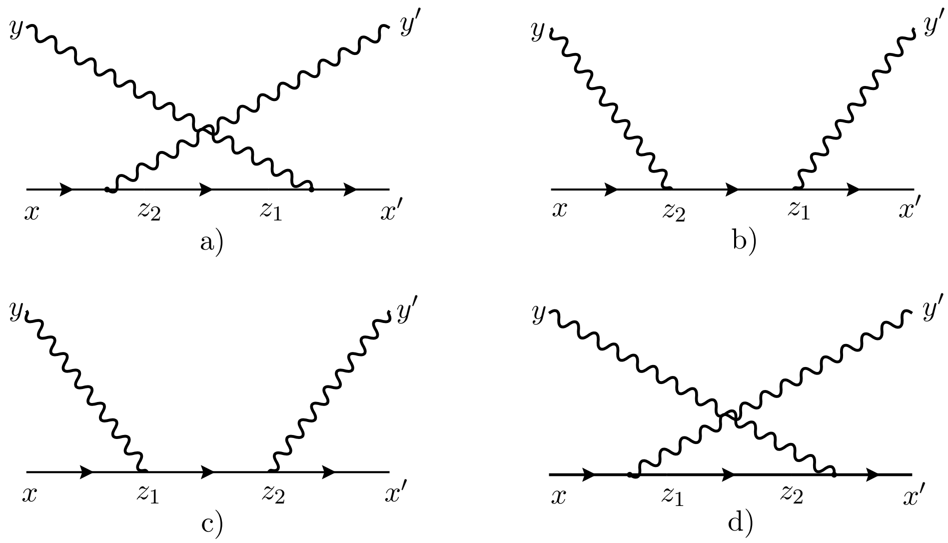


Figure 3.9: Diagrams for Compton scattering in configuration space.

$$
\begin{aligned}
&+S_{F\beta'\gamma}(x'-z_1)S_{F\delta'\beta}(z_2-x)S_{F\delta\gamma'}(z_1-z_2)D_{F\mu\rho}(y-z_2)D_{F\mu'\sigma}(y'-z_1)\\
&+S_{F\beta'\gamma'}(x'-z_2)S_{F\delta\beta}(z_1-x)S_{F\delta'\gamma}(z_2-z_1)D_{F\mu\sigma}(y-z_1)D_{F\mu'\rho}(y'-z_2)\\
&+S_{F\beta'\gamma'}(x'-z_2)S_{F\delta\beta}(z_1-x)S_{F\delta'\gamma}(z_2-z_1)D_{F\mu\rho}(y-z_2)D_{F\mu'\sigma}(y'-z_1) \qquad (3.88)
\end{aligned}
$$

To better understand Eq. (3.88) it is useful to draw the corresponding diagrams in configuration space. We show them in Fig. (3.9). From this figure it is clear that *b*) $\equiv$ *c*) and *a*) $\equiv$ *d*) because z_1 and z_2 are irrelevant labels. From this we get a factor of 2 that is going to cancel the $\frac{1}{2!}$ in Eq. (3.87)[3]. We have then only two distinct diagrams that we take as c) and d). Then, including already the factor of 2, we get for diagrama c)

$$
\begin{aligned}
G^{(c)}(x,x',y,y') \;=\;& (ie)^2(\gamma^\sigma)_{\gamma\delta}(\gamma^\rho)_{\gamma'\delta'}\int d^4z_1 d^4z_2\; S_{F\beta'\gamma'}(x'-z_2)S_{F\delta\beta}(z_1-x)\\
& S_{F\delta'\gamma}(z_2-z_1)D_{F\mu\sigma}(y-z_1)D_{F\mu'\rho}(y'-z_2) \qquad (3.89)
\end{aligned}
$$

To proceed we could, like in the case of $\lambda\varphi^4$, introduce the Fourier transform of the propagators. However, it is easier to get rid of the external legs using the results,

$$
\begin{aligned}
&(i\partial\!\!\!/_x - m)_{\alpha\lambda}S_{F\lambda\beta}(x-y) = i\delta_{\alpha\beta}\delta^4(x-y)\\
&S_{F\alpha\lambda}(x-y)(-i\overleftarrow{\partial\!\!\!/}_y - m)_{\lambda\beta} = i\delta_{\alpha\beta}\delta^4(x-y) \qquad (3.90)
\end{aligned}
$$

and

$$
\Box_x D_{F\mu\nu}(x-y) = ig_{\mu\nu}\delta^4(x-y) \qquad (3.91)
$$

[3]In fact this result is general, for n vertices we have $n!$ that cancels against the factor $\frac{1}{n!}$ from the expansion of the exponential.

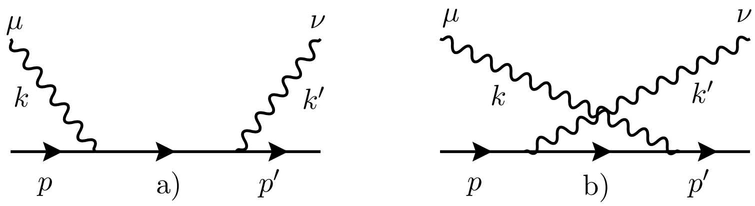


Figure 3.10: Diagrams for Compton scattering.

We get therefore,

$$
\begin{aligned}
S_{fi}^{(c)} &= (ie)^2 \int d^4x d^4x' d^4y d^4y' e^{-i(p\cdot x + k\cdot y - p'\cdot x' - k'\cdot y')} \varepsilon^{\mu}(k) g_{\mu\sigma} \varepsilon^{*\mu'}(k') g_{\mu'\rho} \\
&\qquad (\gamma^{\sigma})_{\gamma\delta} (\gamma^{\rho})_{\gamma'\delta'}\ \overline{u}(p',s')_{\alpha'} \delta_{\alpha'\gamma'} u_{\alpha}(p,s)\ \delta_{\delta\alpha} \\
&\qquad \int d^4z_1 d^4z_2 \delta^4(x'-z_2)\delta^4(x-z_1)\delta^4(y-z_1)\delta^4(y'-z_2)\ S_{F\delta'\gamma}(z_2-z_1) \\
&= (ie)^2 \int d^4z_1 d^4z_2 e^{-i(p\cdot z_1 + k\cdot z_1 - p'\cdot z_2 - k'\cdot z_2)} \varepsilon^{\mu}(k) \varepsilon^{*\mu'}(k') \\
&\qquad \overline{u}(p',s')_{\alpha'} (\gamma_{\mu'})_{\alpha'\delta'} S_{F\delta'\gamma}(z_2-z_1) (\gamma_{\mu})_{\gamma\alpha} u_{\alpha}(p,s)
\end{aligned}
\tag{3.92}
$$

Finally we use

$$
\begin{aligned}
S_F(z_2-z_1) &= \int \frac{d^4q}{(2\pi)^4} \frac{i(\not{q}+m)}{q^2-m^2+i\epsilon} e^{-iq\cdot(z_2-z_1)} \\
&\equiv \int \frac{d^4q}{(2\pi)^4} S_F(q) e^{-iq\cdot(z_2-z_1)}
\end{aligned}
\tag{3.93}
$$

to get

$$
\begin{aligned}
S_{fi}^{(c)} &= \int \frac{d^4q}{(2\pi)^4} d^4z_1 d^4z_2 e^{-iz_1\cdot(p+k-q) + iz_2\cdot(p'+k'-q)} \\
&\qquad \varepsilon^{\mu}(k) \varepsilon^{\mu'*}(k') \overline{u}(p',s')(ie\gamma_{\mu'}) S_F(q)(ie\gamma_{\mu}) u(p,s) \\
&= (2\pi)^4 \delta^{(4)}(p+k-p'-k) \cdot \\
&\qquad \varepsilon^{\mu}(k) \varepsilon^{\mu'*}(k') \overline{u}(p',s')(ie\gamma_{\mu'}) S_F(p+k)(ie\gamma_{\mu}) u(p,s)
\end{aligned}
\tag{3.94}
$$

Therefore, the T matrix transition amplitude is,

$$
-iT_{fi}^{(c)} = \varepsilon^{\mu}(k) \varepsilon^{\mu'*}(k') \overline{u}(p',s')(ie\gamma_{\mu'}) S_F(p+k)(ie\gamma_{\mu}) u(p,s) \tag{3.95}
$$

corresponding to the diagram on the left panel of Fig. (3.10). In Eq. (3.95) we factor out the quantity $(ie\gamma_\mu)$, because it will be clear that this quantity will be the Feynman rule for the vertex. The arrows in these diagrams correspond to the flow of electric charge. Notice that to an electron in the initial state we associate a spinor $u(p,s)$ and for an electron in

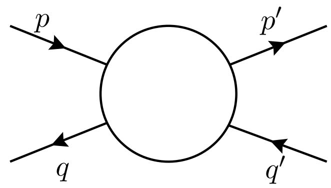


Figure 3.11: Diagram with positrons

the final state we associate the spinor $\overline{u}(p', s')$. Since the electron line as to be a c-number, we start writing the line in the reverse order of that of the arrows.

In a similar way for diagram d) we will get the diagram represented in Fig. (3.10), that corresponds to the following expression,

$$-iT_{fi}^{(d)} = \varepsilon^{\mu}(k)\varepsilon^{*\mu'}(k')\overline{u}(p', s')(ie\gamma_{\mu})S_F(p-k')(ie\gamma_{\mu'})u(p, s) \tag{3.96}$$

Looking at Eqs. (3.95) and (3.96) we are almost in a position to state the Feynman rules for QED. Before that we will look at a case where we have positrons.

3.6.2 Electron–positron elastic scattering (Bhabha scattering)

We will consider electron-positron elastic scattering, the so-called *Bhabha scattering*,

$$e^-(p) + e^+(q) \to e^-(p') + e^+(q') \tag{3.97}$$

This example will teach us two things. First, how positrons (that is the anti-particles) enter in the amplitudes. Secondly we will learn that, sometimes, due the anti-commutation rules of the fermions, we will get relative minus signs between different diagrams. We have,

$$S_{fi} = \left\langle (p', s'), (q', \overline{s'}); out \middle| (p, s), (q, \overline{s}); in \right\rangle \tag{3.98}$$

corresponding to the kinematics in Fig. (3.11). Notice that the arrows are in the direction of flow of charge of the electron, but the momenta do correspond to the real momenta of the particles or antiparticles in that frame: p entering and p' exiting for the electron, and q entering and q' exiting for the positron. In the following we will not show the spin dependence in order to simplify the notation. Then using Eq. (3.98) we write,

$$\begin{aligned} S_{fi} \;=\; & \int d^4x d^4y d^4x' d^4y' e^{-i[p\cdot x + q\cdot y - p'\cdot x' - q'\cdot y']} \\ & \qquad \overline{u}(p')_{\alpha}(i\overrightarrow{\partial\!\!\!/}_{x'} - m)_{\alpha\beta}\ \overline{v}_{\gamma}(q)(i\overrightarrow{\partial\!\!\!/}_{y} - m)_{\gamma\delta} \\ & \qquad \langle 0 | T\overline{\psi}_{\delta'}(y')\psi_{\beta}(x')\overline{\psi}_{\beta'}(x)\psi_{\delta}(y) | 0 \rangle \\ & \qquad (-i\overleftarrow{\partial\!\!\!/}_{x} - m)_{\beta'\alpha'}u_{\alpha'}(p)(-i\overleftarrow{\partial\!\!\!/}_{y'} - m)_{\delta'\gamma'}v_{\gamma'}(q') \end{aligned} \tag{3.99}$$

We have, therefore, to evaluate the Green function

$$G(y', x', x, y) \equiv \langle 0 | T\overline{\psi}_{\delta'}(y')\psi_{\beta}(x')\overline{\psi}_{\beta'}(x)\psi_{\delta}(y) | 0 \rangle \tag{3.100}$$

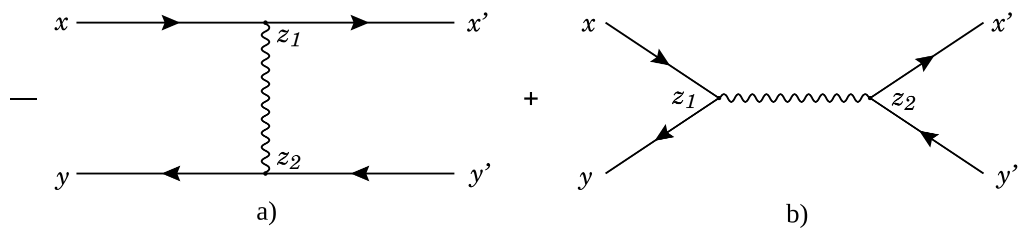


Figure 3.12:

The lowest order contribution is of second order in the coupling[4]. We have (to simplify we omit the label *in*),

$$\begin{aligned}
G(y',x',x,y) = &\frac{(ie)^2}{2}(\gamma^\mu)_{\epsilon\epsilon'}(\gamma^\nu)_{\varphi\varphi'}\int d^4z_1 d^4z_2 \\
&\langle 0|\, T\overline{\psi}_{\delta'}(y')\psi_\beta(x')\overline{\psi}_{\beta'}(x)\psi_\delta(y) :\overline{\psi}_\epsilon(z_1)\psi_{\epsilon'}(z_1)A_\mu(z_1)::\overline{\psi}_\varphi(z_2)\psi_{\varphi'}(z_2)A_\nu(z_2):|0\rangle \\
= &\frac{(ie)^2}{2}(\gamma^\mu)_{\varepsilon\varepsilon'}(\gamma^\nu)_{\varphi\varphi'}\int d^4z_1 d^4z_2 \\
&\Big[-S_{F\beta\epsilon}(x'-z_1)S_{F\epsilon'\beta'}(z_1-x)S_{F\delta\varphi}(y-z_2)S_{F\varphi'\delta'}(z_2-y')D_{F\mu\nu}(z_1-z_2) \\
&+S_{F\delta\epsilon}(y-z_1)S_{F\epsilon'\beta'}(z_1-x)S_{F\beta\varphi}(x'-z_2)S_{F\varphi'\delta'}(z_2-y')D_{F\mu\nu}(z_1-z_2) \\
&+(z_1\leftrightarrow z_2)\Big]
\end{aligned} \tag{3.101}$$

Once more the exchange $(z_1 \leftrightarrow z_2)$ compensates for the factor $\frac{1}{2!}$ and we have two diagrams with a relative minus sign, as it is shown in Fig. (3.12). Let us look at the contribution of diagram a),

$$\begin{aligned}
S_{fi}^{(a)} &= -\int d^4x d^4y d^4x' d^4y' d^4z_1 d^4z_2 (ie)^2(\gamma^\mu)_{\varepsilon\varepsilon'}(\gamma^\nu)_{\varphi\varphi'} e^{-i[p\cdot x+q\cdot y-p'\cdot x'-q'\cdot y']} \\
&\qquad \overline{u}(p')_\alpha(i\overrightarrow{\not\partial}'_x - m)_{\alpha\beta}\overline{v}_\gamma(q)(i\overrightarrow{\not\partial}_y - m)_{\gamma\delta} \\
&\qquad S_{F\beta\varepsilon}(x'-z_1)S_{F\varepsilon'\beta'}(z_1-x)S_{F\delta\varphi}(y-z_2)S_{F\varphi'\delta'}(z_2-y') \\
&\qquad (-i\overleftarrow{\not\partial}_x - m)_{\beta'\alpha'}u_{\alpha'}(p)(-i\overleftarrow{\not\partial}'_y - m)_{\delta'\gamma'}v_{\gamma'}(q')D_{F\mu\nu}(z_1-z_2) \\
&= -\int d^4z_1 d^4z_2 e^{-i[p\cdot z_1+q\cdot z_2-p'\cdot z_1-q'\cdot z_2]} \\
&\qquad \overline{u}(p')(ie\gamma^\mu)u(p)\overline{v}(q)(ie\gamma^\nu)v(q')D_{F\mu\nu}(z_1-z_2)
\end{aligned} \tag{3.102}$$

[4] There is, of course, a contribution without interaction, but that corresponds to disconnected terms in which we are not interested.

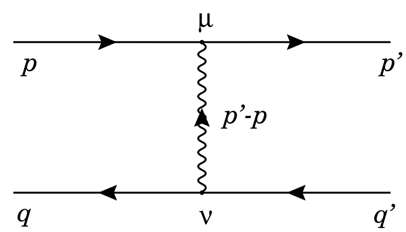


Figure 3.13:

Using now the Fourier transform of the photon propagator,

$$\begin{aligned} D_{F\mu\nu}(z_1 - z_2) &= \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon} e^{-ik\cdot(z_1-z_2)} \\ &\equiv \int \frac{d^4k}{(2\pi)^u} D_{F\mu\nu}(k) e^{-ik\cdot(z_1-z_2)} \end{aligned} \tag{3.103}$$

we get

$$\begin{aligned} S_{fi}^{(a)} &= -\overline{u}(p')(ie\gamma^\mu)u(\rho)\overline{v}(q)(ie\gamma^\nu)v(q') \\ &\qquad \int d^4z_1 d^4z_2 \frac{d^4k}{(2\pi)^4} D_{F\mu\nu}(k) e^{-iz_1\cdot(p-p'+k)} e^{-iz_2\cdot(q-q'-k)} \\ &= -(2\pi)^4\delta^4(p+q-p'-q')\overline{u}(p')(ie\gamma^\nu)u(p)\overline{v}(q)(ie\gamma^\mu)v(q')D_{F\mu\nu}(p'-p) \end{aligned} \tag{3.104}$$

and therefore the T matrix element is,

$$-iT_{fi}^{(a)} = -\overline{u}(p')(ie\gamma^\mu)u(p)D_{F\mu\nu}(p'-p)\overline{v}(q)(ie\gamma^\nu)v(q') \tag{3.105}$$

to which corresponds the Feynman diagram of Fig. (3.13).

In a similar way we would get

$$-iT_{fi}^{(b)} = \overline{v}(q)(ie\gamma^\mu)u(p)D_{F\mu\nu}(p+q)\overline{u}(p')(ie\gamma^\nu)v(q') \tag{3.106}$$

that corresponds to the diagram of Fig. (3.14). Which of the diagrams has the minus sign is irrelevant, because this is the lowest order diagram. It depends on the conventions determining how to build the *in* state that lead to Eq. (3.98). Only the relative sign is important. However, higher order terms have to respect the same conventions.

3.6.3 Fermion Loops

Before we summarize the Feynman rules for `QED` let us look at what happens with fermion loops. One such example is the second order correction to the photon propagator shown

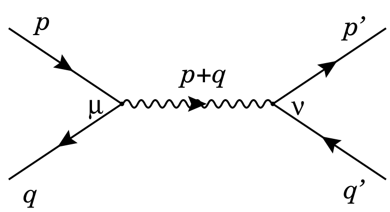


Figure 3.14:

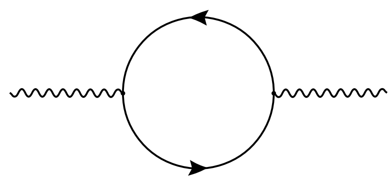


Figure 3.15: Vacuum polarization

in Fig. (3.15). First of all, the loop orientation it is only relevant if leads to topologically different diagrams. Therefore the diagrams of Fig. (3.16) are topologically equivalent and only one should be considered. However the diagrams in Fig. (3.17) are topologically distinct and both should be considered.

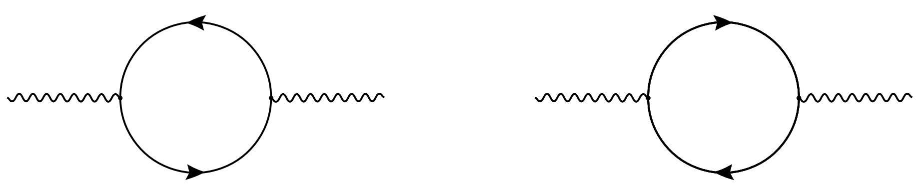


Figure 3.16: Topologically equivalent diagrams

The second aspect that is relevant is a possible sign coming from the anti-commutation of the fermion fields, that should affect some diagrams, and in particular the fermion loop. To understand this sign we should note that by definition of loop, the internal lines are not connected to external fermion lines, they should originate only in the interaction. Therefore they should come from terms of the form

$$\langle 0|\, T \cdots : \overline{\psi}(z_1) \not{A}(z_1) \psi(z_1) : \cdots : \overline{\psi}(z_n) \not{A}(z_n) \psi(z_n) : \cdots |0\rangle \ . \tag{3.107}$$

Now it is clear that in order to make the appropriate contractions of the fermion fields to bring them to the form of the Feynman propagator, $\langle 0|\, T \psi(z_1) \overline{\psi}(z_2)\, |0\rangle$, it is necessary

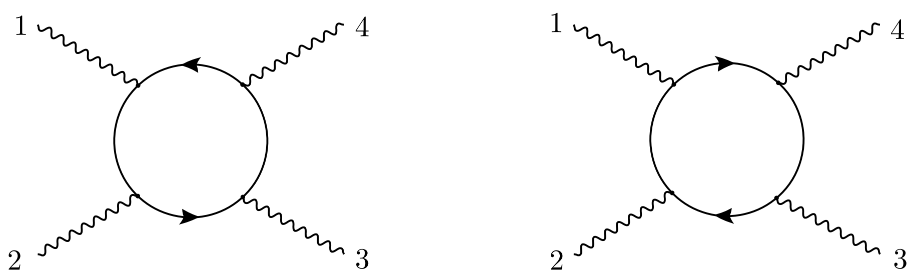


Figure 3.17: Topologically distinct diagrams

to make an odd number of permutations of the fermion fields, and therefore we get a $(-)$ sign for the loops. This sign is physically relevant because there is a lowest order diagram where the photons do not interact, corersponding to the free propagator. So the minus sign is defined in relation to this lowest order diagram and therefore it is not arbitrary (see the difference with respect to the discussion of the Bahbha scattering).

3.6.4 Feynman rules for QED

We are now in position to state the Feynman rules for `QED`

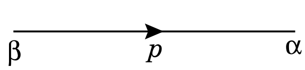
1. For a given process, draw all topologically distinct diagrams.

2. For each electron entering a diagram a factor $u(p,s)$. If it leaves the diagram a factor $\overline{u}(p,s)$.

3. For each positron leaving the diagram (final state) a factor $v(p,s)$. It it enters the diagram (initial state) then we have a factor $\overline{v}(p,s)$.

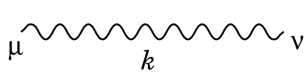
4. For each photon in the initial state we have the vector $\varepsilon^{\mu}(k)$ and in the final state $\varepsilon^{*\mu}(k)$.

5. For each internal fermionic line the propagator



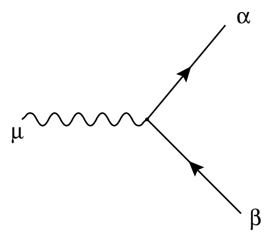
$$S_{F_{\alpha\beta}}(p) = i\frac{(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon} \tag{3.108}$$

6. For each virtual photon the propagator (Feynman gauge)



$$D_{F\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2} \tag{3.109}$$

7. For each vertex the factor



$$ie\gamma^\mu \tag{3.110}$$

8. For each internal momentum, not fixed by conservation of momenta, as in the case of loops, a factor

$$\int \frac{d^4q}{(2\pi)^4} \tag{3.111}$$

9. For each loop of fermions a -1 sign.

10. A factor of -1 between diagrams that differ by exchange of fermionic lines. In doubt, revert to Wick's theorem.

Comments

- In QED there are no symmetry factors, that is, they are always equal to 1.
- In our discussion we did not consider the Z factors that come in the reduction formulas, like in Eq. (2.67). This is true in lowest order in perturbation theory. They can be calculated also in perturbation theory. Their definition is (for instance for the electron),

$$\lim_{\not{p}\to m} S'_F(p) = Z_2 S_F(p) \tag{3.112}$$

where $S'_F(p)$ is the propagator of the theory with interactions. Then we can obtain, in perturbation theory,

$$Z_2 = 1 + O(\alpha) + \cdots \tag{3.113}$$

In higher orders it is necessary to correct the external lines with these $\sqrt{Z}$ factors.

3.7 General formalism for getting the Feynman rules

After showing how to obtain the Feynman rules for $\lambda\phi^4$ and QED, we are going to present here, without proof, a general method to obtain the Feynman rules of any theory, including the case when the interactions have derivatives, that we have excluded up to now, and that is very important for the Standard Model. This method can only be fully justified with the methods of Chapter 5. For simplicity we will consider only scalar fields.

The starting point is the action taken as a functional of the fields,

$$\Gamma_0[\varphi] \equiv \int d^4x \mathcal{L}[\varphi]\cdot \tag{3.114}$$

In fact, $\Gamma_0[\varphi]$ is the generating functional of the one particle irreducible Green functions in lowest order, as we will see in Chapter 5. The rules are as follows:

Propagators

1. Start by evaluating $\Gamma_0^{(2)}(x_i, x_j) \equiv \dfrac{\delta^2 \Gamma_0[\varphi]}{\delta\varphi(x_i)\delta\varphi(x_j)}$
2. Then evaluate the Fourier Transform (FT) to get $\Gamma_0^{(2)}(p_i, p_j)$ defined by the relation
$$(2\pi)^4\delta^4(p_i + p_j)\Gamma_0^{(2)}(p_i, p_j) \equiv \int d^4x_i d^4x_j e^{-i(p_i\cdot x_i + p_j\cdot x_j)}\Gamma_0^{(2)}(x_i, x_j) \tag{3.115}$$
where all the momenta are *incoming*.
3. The Feynman propagator is then
$$G^{(0)}_{Fij} = i[\Gamma_0^{(2)}(p_i, p_j)]^{-1} \ . \tag{3.116}$$
Do not forget that $p_i = -p_j$.

Vertices

1. Evaluate $\Gamma_0^{(n)}(x_1 \cdots x_n) = \dfrac{\delta^n \Gamma_0[\varphi]}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}$
2. Then take the Fourier Transform to obtain
$$\begin{aligned}(2\pi)^4\delta^4(p_1 + p_2 + \cdots + p_n)&\Gamma_0^{(n)}(p_1 \cdots p_n)\\ &\equiv \int d^4x_1 \cdots d^4x_n e^{-i(p_1\cdot x_1 + \cdots p_n\cdot x_n)}\Gamma_0^{(n)}(x_1 \cdots x_n)\end{aligned} \tag{3.117}$$
3. The vertex in momenta space is then given by the rule
$$i\Gamma_0^{(n)}(p_1, \cdots p_n) \tag{3.118}$$

Comments

- For fermionic fields it is necessary to take care with the order of the derivation. The convention that we take is

$$\frac{\delta^2}{\delta\psi_\alpha(x)\delta\overline{\psi}_\beta(y)}\left(\overline{\psi}(z)\Gamma\psi(z)\right) \equiv \Gamma_{\beta\alpha}\delta^4(z-x)\delta^4(z-y) \tag{3.119}$$

$\psi_\alpha(x)$ e $\psi_\beta(x)$ are here taken as classical anti-commuting fields (Grassmann variables, see Chapter 5).

- The functional derivatives are defined by

$$\frac{\delta\varphi_i(x)}{\delta\varphi_k(y)} \equiv \delta_{ik}\delta^4(x-y) \tag{3.120}$$

3.7.1 Example: scalar electrodynamics

The Lagrangian is

$$\mathcal{L} = (\partial_\mu - ieQA_\mu)\varphi^*(\partial^\mu + ieQA^\mu)\varphi - m\varphi^*\varphi + \mathcal{L}_{\text{Maxwell}} - \frac{\lambda}{4}(\varphi^*\varphi)^2 \tag{3.121}$$

Therefore

$$\mathcal{L}_{int} = -ieQ\varphi^*\overleftrightarrow{\partial}_\mu\varphi A^\mu + e^2Q^2\varphi^*\varphi A_\mu A^\mu \tag{3.122}$$

The propagators are the usual ones, let us consider only the vertices. There are two vertices. The cubic vertex is

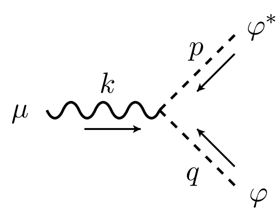


Figure 3.18: Cubic vertex in scalar QED.

$$\Gamma^{(3)}_\mu(x_1,x_2,x_3) = -ieQ\int d^4z\delta^4(z-x_1)(\overrightarrow{\partial}^z_\mu - \overleftarrow{\partial}^z_\mu)\delta^4(z-x_2)\delta^4(z-x_3) \tag{3.123}$$

therefore

$$\begin{aligned}(2\pi)^4\delta^4(p+k+q)\Gamma^{(3)}_\mu(p,q,k) \quad &\equiv \quad -ieQ\int d^4zd^4x_1d^4x_2d^4x_3e^{-i(x_1\cdot p+x_2\cdot q+x_3\cdot k)}\\ &\qquad \delta^4(z-x_1)\;(\overrightarrow{\partial}^z_\mu - \overleftarrow{\partial}^z_\mu)\delta^4(z-x_2)\delta^4(z-x_3)\end{aligned}$$

$$
\begin{aligned}
&= -ieQ \int d^4z d^4x_2 e^{-i[(p+k)\cdot z+q\cdot x_2]} \partial^z_\mu \delta^4(z-x_2) \\
&\quad +ieQ \int d^4z d^4x_1 e^{-i[p\cdot x_1+(q+k)\cdot z]} \partial^z_\mu \delta^4(z-x_1) \\
&= -ieQ(ip_\mu - iq_\mu)(2\pi)^4 \delta^4(p+q+k)
\end{aligned}
\tag{3.124}
$$

Therefore we obtain for this vertex

$$
i\Gamma_\mu(p,q,k) = ieQ\ (p_\mu - q_\mu) = -ieQ\ (q_\mu - p_\mu) \tag{3.125}
$$

The other vertex is

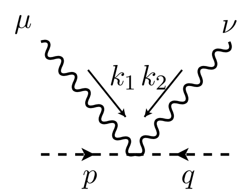


Figure 3.19: Quartic vertex in scalar QED (seagull).

We obtain,

$$
\Gamma^{(4)}_{\mu\nu}(x_1,x_2,x_3,x_4) = 2e^2Q^2\delta^4(x_1-x_2)\delta^4(x_1-x_3)\delta^4(x_1-x_4)g_{\mu\nu} \tag{3.126}
$$

and

$$
\Gamma^{(4)}_{\mu\nu}(p,q,k_1,k_2) = 2(eQ)^2 g_{\mu\nu} \tag{3.127}
$$

and we finally get for the Feynman rule.

$$
i2e^2Q^2\ g_{\mu\nu} \tag{3.128}
$$

Comment

- From the above results we can enunciate a simple rule for interactions that have derivatives of fields.

  Consider that we have one field in the Lagrangian that has a derivative, say $\partial_\mu\phi$. Then the rule is

$$
\partial_\mu\phi \to -i\ (\textit{incoming momentum})_\mu \tag{3.129}
$$

  In the end do not forget to multiply the result by i.

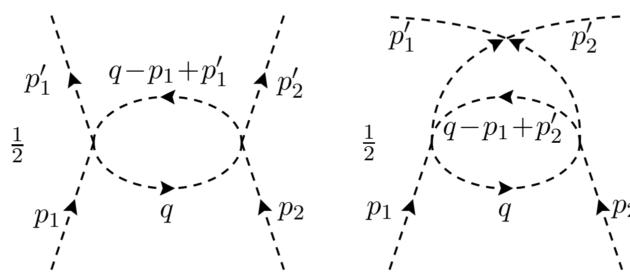
- As an example consider the following term in the Lagrangian for scalar electrodynamics
$$\mathcal{L} = ieQ\partial_\mu\varphi^*\varphi A^\mu + \cdots \tag{3.130}$$
If p is the incoming momentum of the line associated with the field φ^*, see Fig. 3.18, we have
$$\text{Vertex} = i \times (ieQ) \times (-ip_\mu) = i\, e\, Q\, p_\mu \tag{3.131}$$
in agreement with Eq. (3.125).

Problems for Chapter 3

3.1 Show explicitly that Wick's theorem is valid for the case of 4 fields, that is

$$T(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4)) =: \varphi_{in}(x_1)\cdots\varphi_{in}(x_4) : + \cdots \tag{3.132}$$

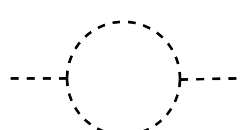
3.2 For the case of the $\lambda\varphi^4$ theory verify the Feynman rules for the diagrams



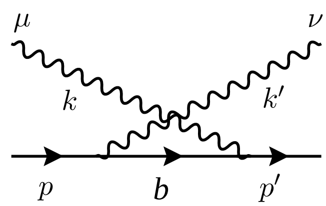
3.3 Consider a theory with the following interaction Lagrangian

$$\mathcal{L}_I = -\frac{\lambda}{3!}\varphi_{in}^3 \tag{3.133}$$

- a) Find the Feynman rules for this theory.
- b) Find the symmetry factor for the diagram



3.4 Verify that for Compton scattering the diagram



gives the result of Eq. (3.96).

3.5 Verify Eq. (3.106).

3.6 Show that in `QED` the symmetry factors are always 1.

3.7 Explicitly calculate the T matrix element for the process $e^+e^- \to \gamma\gamma$ and verify that is in agreement with the general rules.

3.8 Show that the amplitudes for $e^+e^- \to \gamma\gamma$ and $e^-\gamma \to e^-\gamma$ are related. How can one obtain one from the other?

Chapter 4

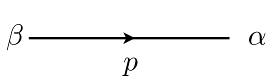
Radiative Corrections

4.1 QED Renormalization at one-loop

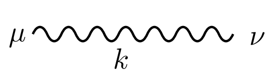
We will consider the theory described by the Lagrangian

$$\mathcal{L}_{\rm QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial\cdot A)^2 + \overline{\psi}(i\not{\partial} + e\not{A} - m)\psi \ . \tag{4.1}$$

The free propagators are

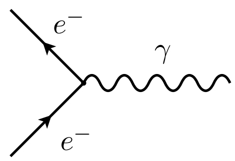


$$\left(\frac{i}{\not{p} - m + i\varepsilon}\right)_{\beta\alpha} \equiv S^0_{F\beta\alpha}(p) \tag{4.2}$$



$$\begin{aligned} & -i\left[\frac{g_{\mu\nu}}{k^2+i\varepsilon} - (1-\xi)\frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2}\right] \\ = \quad & -i\left\{\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2+i\varepsilon} + \xi\frac{k_\mu k_\nu}{k^4}\right\} \\ \equiv \quad & G^0_{F\mu\nu}(k) \end{aligned} \tag{4.3}$$

and the vertex



$$+\, ie(\gamma_\mu)_{\beta\alpha} \qquad e = |e| > 0 \tag{4.4}$$

We will now consider the one-loop corrections to the propagators and to the vertex. We will work in the Feynman gauge ($\xi = 1$).

4.1.1 Vacuum Polarization

In first order the contribution to the photon propagator is given by the diagram of Fig. 4.1 that we write in the form

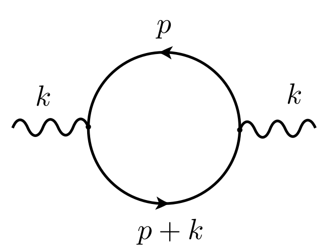


Figure 4.1: Vacuum polarization.

$$G^{(1)}_{\mu\nu}(k) \equiv G^0_{\mu\mu'}(k)\ i\,\Pi^{\mu'\nu'}(k) G^0_{\nu'\nu}(k) \tag{4.5}$$

where

$$\begin{aligned} i\,\Pi_{\mu\nu}(k) &= -(+ie)^2 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}[\gamma_\mu(\not{p}+m)\gamma_\nu(\not{p}+\not{k}+m)]}{(p^2-m^2+i\varepsilon)((p+k)^2-m^2+i\varepsilon)} \\ &= -4e^2 \int \frac{d^4p}{(2\pi)^4} \frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2+p\cdot k - m^2)}{(p^2-m^2+i\varepsilon)((p+k)^2-m^2+i\varepsilon)} \end{aligned} \tag{4.6}$$

Simple power counting indicates that this integral is quadratically divergent for large values of the internal loop momenta. In fact the divergence is milder, only logarithmic. The integral being divergent we have first to regularize it and then to define a renormalization procedure to cancel the infinities. For this purpose we will use the method of dimensional regularization. For a value of d small enough the integral converges. If we define $\epsilon = 4 - d$, in the end we will have a divergent result in the limit $\epsilon \to 0$. We get therefore[1]

$$\begin{aligned} i\,\Pi_{\mu\nu}(k,\epsilon) &= -4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)]}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)} \\ &= -4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)} \end{aligned} \tag{4.7}$$

where

$$N_{\mu\nu}(p,k) = 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2) \tag{4.8}$$

To evaluate this integral we first use the Feynman parameterization to rewrite the denominator as a single term. For that we use (see Appendix)

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} \tag{4.9}$$

to get

$$\begin{aligned} i\,\Pi_{\mu\nu}(k,\epsilon) &= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{[x(p+k)^2 - xm^2 + (1-x)(p^2 - m^2) + i\varepsilon]^2} \\ &= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{[p^2 + 2k\cdot px + xk^2 - m^2 + i\varepsilon]^2} \\ &= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{[(p+kx)^2 + k^2x(1-x) - m^2 + i\varepsilon]^2} \end{aligned} \tag{4.10}$$

For dimension d sufficiently small this integral converges and we can change variables

$$p \to p - kx \tag{4.11}$$

We then get

$$i\,\Pi_{\mu\nu}(k,\epsilon) = -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p-kx,k)}{[p^2 - C + i\epsilon]^2} \tag{4.12}$$

where

$$C = m^2 - k^2x(1-x) \tag{4.13}$$

$N_{\mu\nu}$ is a polynomial of second degree in the loop momenta as can be seen from Eq. (4.8). However as the denominator in Eq. (4.12) only depends on p^2 is it easy to show that

$$\begin{aligned} &\int \frac{d^dp}{(2\pi)^d}\ \frac{p^\mu}{[p^2 - C + i\epsilon]^2} = 0 \\ &\int \frac{d^dp}{(2\pi)^d}\ \frac{p^\mu p^\nu}{[p^2 - C + i\epsilon]^2} = \frac{1}{d}\,g^{\mu\nu} \int \frac{d^dp}{(2\pi)^d}\ \frac{p^2}{[p^2 - C + i\epsilon]^2} \end{aligned} \tag{4.14}$$

[1] Where μ is a parameter with dimensions of a mass that is introduced to ensure the correct dimensions of the coupling constant in dimension d, that is, $[e] = \frac{4-d}{2} = \frac{\epsilon}{2}$. We take then $e \to e\mu^{\frac{\epsilon}{2}}$. For more details see the Appendix.

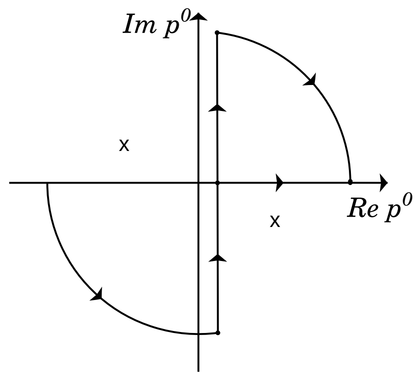


Figure 4.2: Wick rotation.

This means that we only have to calculate integrals of the form

$$\begin{aligned} I_{r,m} &= \int \frac{d^d p}{(2\pi)^d} \frac{(p^2)^r}{[p^2 - C + i\epsilon]^m} \\ &= \int \frac{d^{d-1} p}{(2\pi)^d} \int dp^0 \frac{(p^2)^r}{[p^2 - C + i\epsilon]^m} \end{aligned} \tag{4.15}$$

To make this integration we will use integration in the plane of the complex variable p^0 as described in Fig. 4.2. The deformation of the contour corresponds to the so called Wick rotation,

$$p^0 \to i p_E^0 \qquad ; \qquad \int_{-\infty}^{+\infty} \to i \int_{-\infty}^{+\infty} dp_E^0 \tag{4.16}$$

and $p^2 = (p^0)^2 - |\vec{p}|^2 = -(p_E^0)^2 - |\vec{p}|^2 \equiv -p_E^2$, where $p_E = (p_E^0, \vec{p})$ is an euclidean vector, that is

$$p_E^2 = (p_E^0)^2 + |\vec{p}|^2 \tag{4.17}$$

We can then write (see the Appendix for more details),

$$I_{r,m} = i(-1)^{r-m} \int \frac{d^d p_E}{(2\pi)^d} \frac{p_E^{2^r}}{\left[p_E^2 + C\right]^m} \tag{4.18}$$

where we do not need the $i\epsilon$ anymore because the denominator is positive definite[2] $(C > 0)$. To proceed with the evaluation of $I_{r,m}$ we write,

$$\int d^d p_E = \int d\overline{p}\; \overline{p}^{\;d-1} d\Omega_{d-1} \tag{4.19}$$

where $\overline{p} = \sqrt{(p_E^0)^2 + |\vec{p}|^2}$ is the length of of vector p_E in the euclidean space with d dimensions and $d\Omega_{d-1}$ is the solid angle that generalizes spherical coordinates. We can show (see Appendix) that

$$\int d\Omega_{d-1} = 2 \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{4.20}$$

[2] The case when $C < 0$ is obtained by analytical continuation of the final result.

The $\overline{p}$ integral is done using the result,

$$\int_0^\infty dx \ \frac{x^p}{(x^2+C)^m} = \frac{\Gamma\left(\frac{p+1}{2}\right) C^{\frac{1}{2}(p-2m+1)} \Gamma\left(-\frac{p}{2}+m-\frac{1}{2}\right)}{2\,\Gamma(m)} \tag{4.21}$$

and we finally get

$$I_{r,m} = iC^{r-m+\frac{d}{2}} \ \frac{(-1)^{r-m}}{(4\pi)^{\frac{d}{2}}} \ \frac{\Gamma(r+\frac{d}{2})}{\Gamma(\frac{d}{2})} \ \frac{\Gamma(m-r-\frac{d}{2})}{\Gamma(m)} \tag{4.22}$$

Note that the integral representation of $I_{r,m}$, Eq. (4.15) is only valid for $d < 2(m-r)$ to ensure the convergence of the integral when $\overline{p} \to \infty$. However the final form of Eq. (4.22) can be analytically continued for all the values of d except for those where the function $\Gamma(m-r-d/2)$ has poles, which are (see section C.6),

$$m - r - \frac{d}{2} \neq 0, -1, -2, \ldots \tag{4.23}$$

For the application to dimensional regularization it is convenient to write Eq. (4.22) after making the substitution $d = 4 - \epsilon$. We get

$$I_{r,m} = i\frac{(-1)^{r-m}}{(4\pi)^2} \ \left(\frac{4\pi}{C}\right)^{\frac{\epsilon}{2}} \ C^{2+r-m} \ \frac{\Gamma(2+r-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})} \ \frac{\Gamma(m-r-2+\frac{\epsilon}{2})}{\Gamma(m)} \tag{4.24}$$

that has poles for $m - r - 2 \leq 0$ (see section C.6).

We now go back to calculate $\Pi_{\mu\nu}$. First we notice that after the change of variables of Eq. (4.11) we get, neglecting terms that vanish due to Eq. (4.14),

$$N_{\mu\nu}(p-kx,k) = 2p_\mu p_\nu + 2x^2 k_\mu k_\nu - 2xk_\mu k_\nu - g_{\mu\nu}\left(p^2 + x^2k^2 - xk^2 - m^2\right) \tag{4.25}$$

and therefore

$$\begin{aligned} \mathcal{N}_{\mu\nu} &\equiv \mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \, \frac{N_{\mu\nu}(p-kx,k)}{\left[p^2 - C + i\epsilon\right]^2} \\ &= \left(\frac{2}{d} - 1\right) g_{\mu\nu}\mu^\epsilon I_{1,2} + \Big[-2x(1-x)k_\mu k_\nu + x(1-x)k^2 g_{\mu\nu} + g_{\mu\nu}m^2 \Big] \mu^\epsilon I_{0,2} \end{aligned} \tag{4.26}$$

Using now Eq. (4.24) we can write

$$\begin{aligned} \mu^\epsilon I_{0,2} &= \frac{i}{16\pi^2} \left(\frac{4\pi\mu^2}{C}\right)^{\frac{\epsilon}{2}} \frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(2)} \\ &= \frac{i}{16\pi^2} \left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon) \end{aligned} \tag{4.27}$$

where we have used the expansion of the Γ function, Eq. (C.78),

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \tag{4.28}$$

γ being the Euler constant and we have defined, Eq. (C.82),

$$\Delta_\epsilon = \frac{2}{\epsilon} - \gamma + \ln 4\pi \tag{4.29}$$

In a similar way

$$\begin{aligned}
\mu^\epsilon I_{1,2} &= -\frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\frac{\epsilon}{2}} C\,\frac{\Gamma(3-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})}\,\frac{\Gamma(-1+\frac{\epsilon}{2})}{\Gamma(2)} \\
&= \frac{i}{16\pi^2}C\left(1+2\Delta_\epsilon - 2\ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)
\end{aligned} \tag{4.30}$$

Due to the existence of a pole in $1/\epsilon$ in the previous equations we have to expand all quantities up to $\mathcal{O}(\epsilon)$. This means for instance, that

$$\frac{2}{d} - 1 = \frac{2}{4-\epsilon} - 1 = -\frac{1}{2} + \frac{1}{8}\epsilon + \mathcal{O}(\epsilon^2) \tag{4.31}$$

Substituting back into Eq. (4.26), and using Eq. (4.13), we obtain

$$\begin{aligned}
\mathcal{N}_{\mu\nu} &= g_{\mu\nu}\left[-\frac{1}{2}+\frac{1}{8}\epsilon+\mathcal{O}(\epsilon^2)\right]\left[\frac{i}{16\pi^2}C\left(1+2\Delta_\epsilon-2\ln\frac{C}{\mu^2}\right)+\mathcal{O}(\epsilon)\right] \\
&\quad + \Big[-2x(1-x)k_\mu k_\nu + x(1-x)k^2 g_{\mu\nu} + g_{\mu\nu}m^2\Big]\left[\frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)+\mathcal{O}(\epsilon)\right] \\
&= -\frac{i}{16\pi^2}k_\mu k_\nu\left[\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)2x(1-x)\right] \\
&\quad + \frac{i}{16\pi^2}g_{\mu\nu}k^2\left[\Delta_\epsilon\Big(x(1-x)+x(1-x)\Big) + \ln\frac{C}{\mu^2}\Big(-x(1-x)-x(1-x)\Big)\right. \\
&\qquad\qquad \left. + x(1-x)\left(\frac{1}{2}-\frac{1}{2}\right)\right] \\
&\quad + \frac{i}{16\pi^2}g_{\mu\nu}m^2\left[\Delta_\epsilon(-1+1) + \ln\frac{C}{\mu^2}(1-1) + (-\frac{1}{2}+\frac{1}{2})\right]
\end{aligned} \tag{4.32}$$

and finally

$$\mathcal{N}_{\mu\nu} = \frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)2x(1-x) \tag{4.33}$$

Now using Eq. (4.7) we get

$$\begin{aligned}
\Pi_{\mu\nu}(k) &= -4e^2\frac{1}{16\pi^2}\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\int_0^1 dx\; 2x(1-x)\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) \\
&= -\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\Pi(k^2,\epsilon)
\end{aligned} \tag{4.34}$$

where

$$\Pi(k^2,\epsilon) \equiv \frac{2\alpha}{\pi}\int_0^1 dx\; x(1-x)\left[\Delta_\epsilon - \ln\frac{m^2 - x(1-x)k^2}{\mu^2}\right] \tag{4.35}$$

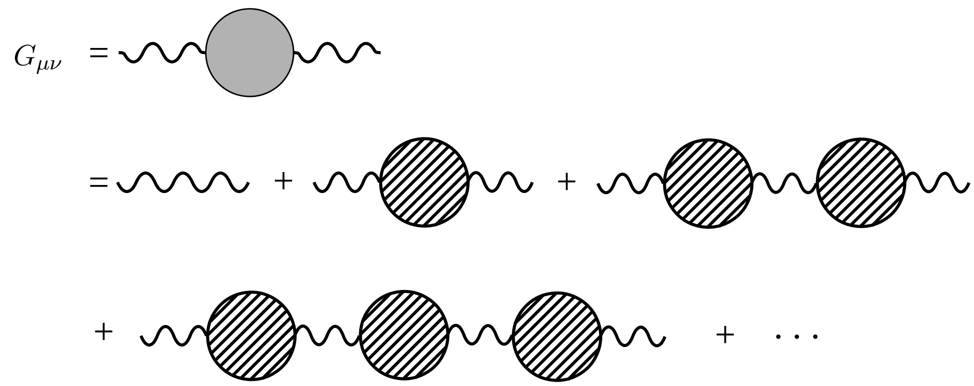
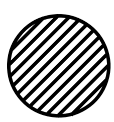


Figure 4.3: Full photon propagator.

This expression clearly diverges as $\epsilon \to 0$. Before we show how to renormalize it let us discuss the meaning of $\Pi_{\mu\nu}(k)$. The full photon propagator is given by the series represented in Fig. 4.3, where

 $\equiv i\,\Pi_{\mu\nu}(k) =$ sum of all one-particle irreducible (*proper*) diagrams to *all* orders (4.36)

In lowest order we have the contribution represented in Fig. 4.4, which is what we have

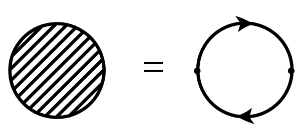


Figure 4.4: Lowest order contribution.

just calculated. To continue it is convenient to rewrite the free propagator of the photon (in an arbitrary gauge ξ) in the following form

$$\begin{aligned} iG^0_{\mu\nu} &= \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2} + \xi\frac{k_\mu k_\nu}{k^4} = P^T_{\mu\nu}\frac{1}{k^2} + \xi\frac{k_\mu k_\nu}{k^4} \\ &\equiv iG^{0T}_{\mu\nu} + iG^{0L}_{\mu\nu} \end{aligned} \tag{4.37}$$

where we have introduced the transversal projector $P^T_{\mu\nu}$ defined by

$$P^T_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \tag{4.38}$$

obviously satisfying the relations,

$$\begin{cases} k^\mu P^T_{\mu\nu} = 0 \\ \\ P^{T\,\nu}_{\mu} P^T_{\nu\rho} = P^T_{\mu\rho} \end{cases} \tag{4.39}$$

The full photon propagator can also in general be written separating its transversal an longitudinal parts

$$G_{\mu\nu} = G^T_{\mu\nu} + G^L_{\mu\nu} \tag{4.40}$$

where $G^T_{\mu\nu}$ satisfies

$$G^T_{\mu\nu} = P^T_{\mu\nu} G_{\mu\nu} \tag{4.41}$$

Eq. (4.34) means that, to first order, the vacuum polarization tensor is transversal, that is

$$i\,\Pi_{\mu\nu}(k) = -ik^2 P^T_{\mu\nu}\,\Pi(k) \tag{4.42}$$

This result is in fact valid to all orders of perturbation theory, a result that can be shown using the Ward-Takahashi identities. This means that the longitudinal part of the photon propagator is not renormalized,

$$G^L_{\mu\nu} = G^{0L}_{\mu\nu} \tag{4.43}$$

For the transversal part we obtain from Fig. 4.3,

$$\begin{aligned} iG^T_{\mu\nu} &= P^T_{\mu\nu}\frac{1}{k^2} + P^T_{\mu\mu'}\frac{1}{k^2}(-i)k^2 P^{T\mu'\nu'}\Pi(k^2)(-i)P^T_{\nu'\nu}\frac{1}{k^2} \\ &\quad + P^T_{\mu\rho}\frac{1}{k^2}(-i)k^2 P^{T\rho\lambda}\;\Pi(k^2)(-i)P^T_{\lambda\tau}\frac{1}{k^2}(-i)k^2 P^{T\tau\sigma}\;\Pi(k^2)(-i)P^T_{\sigma\nu}\frac{1}{k^2} + \cdots \\ &= P^T_{\mu\nu}\frac{1}{k^2}\left[1 - \Pi(k^2) + \Pi^2(k^2) + \cdots\right] \end{aligned} \tag{4.44}$$

which gives, after summing the geometric series,

$$iG^T_{\mu\nu} = P^T_{\mu\nu}\frac{1}{k^2\left[1+\Pi(k^2)\right]} \tag{4.45}$$

All that we have done up to this point is formal because the function $\Pi(k)$ diverges. The most satisfying way to solve this problem is the following. The initial lagrangian from which we started has been obtained from the classical theory and nothing tell us that it should be exactly the same in quantum theory. In fact, as we have just seen, the normalization of the wave functions is changed when we calculate *one-loop* corrections, and the same happens to the physical parameters of the theory, the charge and the mass. Therefore we can think that the correct lagrangian is obtained by adding corrections to the classical lagrangian, order by order in perturbation theory, so that we keep the definitions of charge and mass as well as the normalization of the wave functions. The terms that we add to the lagrangian are called *counterterms*[3]. The total lagrangian is then,

$$\mathcal{L}_{\text{total}} = \mathcal{L}(e, m, ...) + \Delta\mathcal{L} \tag{4.46}$$

Counterterms are defined from the normalization conditions that we impose on the fields and other parameters of the theory. In QED we have at our disposal the normalization of the electron and photon fields and of the two physical parameters, the electric charge and the electron mass. The normalization conditions are, to a large extent, arbitrary. It is however convenient to keep the expressions as close as possible to the free field case, that is, without radiative corrections. We define therefore the normalization of the photon field as,

$$\lim_{k\to 0} k^2 i G^{RT}_{\mu\nu} = 1 \cdot P^T_{\mu\nu} \tag{4.47}$$

where $G^{RT}_{\mu\nu}$ is the renormalized propagator (the transversal part) obtained from the lagrangian $\mathcal{L}_{\text{total}}$. The justification for this definition comes from the following argument. Consider the Coulomb scattering to all orders of perturbation theory. We have then the situation described in Fig. 4.5. Using the Ward-Takahashi identities one can show that

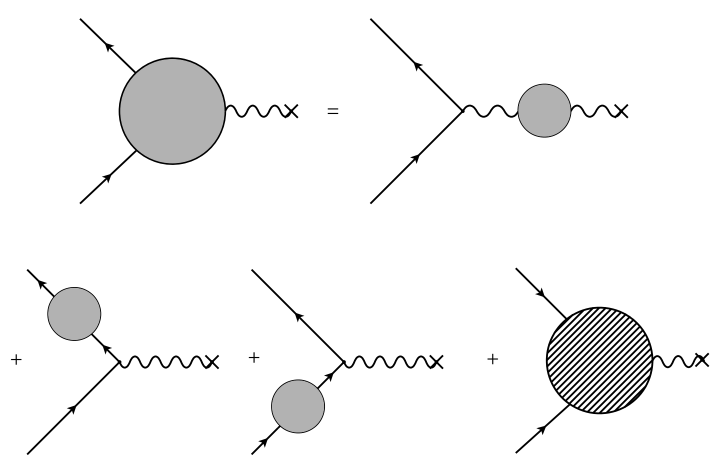


Figure 4.5: Corrections to Coulomb scattering.

the last three diagrams cancel in the limit $q = p' - p \to 0$. Then the normalization condition, Eq. (4.47), means that we have the situation described in Fig. 4.6, that is, the experimental value of the electric charge is determined in the limit $q \to 0$ of the Coulomb scattering.

[3] This interpretation in terms of quantum corrections makes sense. In fact we can show that an expansion in powers of the coupling constant can be interpreted as an expansion in $\hbar^L$, where L is the number of the loops in the expansion term.

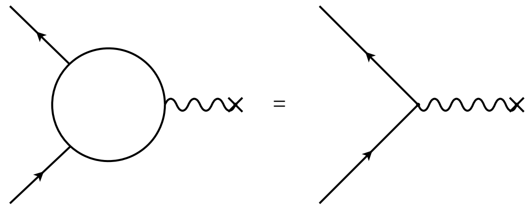
$\lim_{q\to 0}$ 

Figure 4.6: Definition of the electric charge.

The counterterm lagrangian has to have the same form as the classical lagrangian to respect the symmetries of the theory. For the photon field it is traditional to write

$$\Delta\mathcal{L} = -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\delta Z_3\, F_{\mu\nu}F^{\mu\nu} \tag{4.48}$$

corresponding to the following Feynman rule

$$-i\,\delta Z_3 k^2 \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \tag{4.49}$$

We have then

$$\begin{aligned} i\Pi_{\mu\nu} &= i\Pi^{loop}_{\mu\nu} - i\,\delta Z_3 k^2 \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \\ &= -i\,\left(\Pi(k,\epsilon) + \delta Z_3\right) k^2\, P^T_{\mu\nu} \end{aligned} \tag{4.50}$$

Therefore we should make the substitution

$$\Pi(k,\epsilon) \to \Pi(k,\epsilon) + \delta Z_3 \tag{4.51}$$

in the photon propagator. We obtain,

$$iG^T_{\mu\nu} = P^T_{\mu\nu}\,\frac{1}{k^2}\,\frac{1}{1+\Pi(k,\epsilon)+\delta Z_3} \tag{4.52}$$

The normalization condition, Eq. (4.47), implies

$$\Pi(0,\epsilon) + \delta Z_3 = 0 \tag{4.53}$$

from which one determines the constant δZ_3. We get

$$\begin{aligned} \delta Z_3 &= -\Pi(0,\epsilon) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left[\Delta_\epsilon - \ln\frac{m^2}{\mu^2}\right] \\ &= -\frac{\alpha}{3\pi}\left[\Delta_\epsilon - \ln\frac{m^2}{\mu^2}\right] \end{aligned} \tag{4.54}$$

The renormalized photon propagator can then be written as [4]

$$iG_{\mu\nu}(k) = \frac{P^T_{\mu\nu}}{k^2[1+\Pi(k,\epsilon)-\Pi(0,\epsilon)]} + i\,G^L_{\mu\nu} \tag{4.55}$$

The *finite* radiative corrections are given through the function

$$\begin{aligned}\Pi^R(k^2) &\equiv \Pi(k^2,\epsilon)-\Pi(0,\epsilon)\\ &= -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2-x(1-x)k^2}{m^2}\right]\\ &= -\frac{\alpha}{3\pi}\left\{\frac{1}{3}+2\left(1+\frac{2m^2}{k^2}\right)\left[\left(\frac{4m^2}{k^2}-1\right)^{1/2}\cot^{-1}\left(\frac{4m^2}{k^2}-1\right)^{1/2}-1\right]\right\}\end{aligned} \tag{4.56}$$

where the last equation is valid for $k^2 < 4m^2$. For values $k^2 > 4m^2$ the result for $\Pi^R(k^2)$ can be obtained from Eq. (4.56) by analytical continuation. Using ($k^2 > 4m^2$)

$$\cot^{-1} iz = i\left(-\tanh^{-1} z + \frac{i\pi}{2}\right) \tag{4.57}$$

and

$$\left(\frac{4m^2}{k^2}-1\right)^{1/2} \to i\sqrt{1-\frac{4m^2}{k^2}} \tag{4.58}$$

we get

$$\Pi^R(k^2) = -\frac{\alpha}{3\pi}\left\{\frac{1}{3}+2\left(1+\frac{2m^2}{k^2}\right)\left[-1+\sqrt{1-\frac{4m^2}{k^2}}\tanh^{-1}\left(1-\frac{4m^2}{k^2}\right)^{1/2}\right.\right. \tag{4.59}$$

$$\left.\left.-i\frac{\pi}{2}\sqrt{1-\frac{4m^2}{k^2}}\right]\right\} \tag{4.60}$$

The imaginary part of Π^R is given by

$$Im\ \Pi^R(k^2) = \frac{\alpha}{3}\left(1+\frac{2m^2}{k^2}\right)\sqrt{1-\frac{4m^2}{k^2}}\theta\left(1-\frac{4m^2}{k^2}\right) \tag{4.61}$$

and it is related to the pair production that can occur [5] for $k^2 > 4m^2$.

4.1.2 Self-energy of the electron

The electron full propagator is given by the diagrammatic series of Fig. 4.7, which can be written as,

[4] Notice that the photon mass is not renormalized, that is the pole of the photon propagator remains at $k^2 = 0$.

[5] For $k^2 > 4m^2$ there is the possibility of producing one pair e^+e^-. Therefore on top of a virtual process (vacuum polarization) there is a real process (pair production).

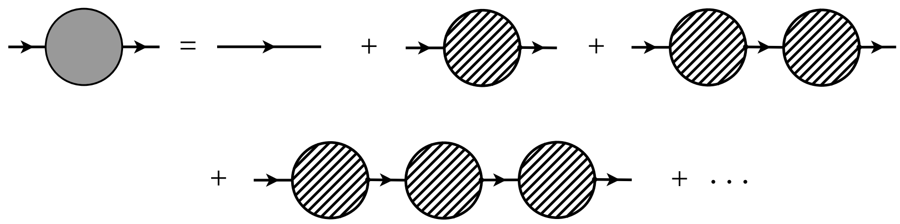
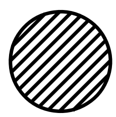


Figure 4.7: Full electron propagator

$$\begin{aligned} S(p) &= S^0(p) + S^0(p)\Big(-i\,\Sigma(p)\Big)S^0(p) + \cdots \\ &= S^0(p)\Big[1 - i\,\Sigma(p) S(p)\Big] \end{aligned} \tag{4.62}$$

where we have identified



$$\equiv -i\,\Sigma(p) \tag{4.63}$$

Multiplying on the left with $S_0^{-1}(p)$ and on the right with $S^{-1}(p)$ we get

$$S_0^{-1}(p) = S^{-1}(p) - i\,\Sigma(p) \tag{4.64}$$

which we can rewrite as

$$S^{-1}(p) = S_0^{-1}(p) + i\,\Sigma(p) \tag{4.65}$$

Using the expression for the free field propagator,

$$S_0(p) = \frac{i}{\not p - m} \Longrightarrow S_0^{-1}(p) = -i(\not p - m) \tag{4.66}$$

we can then write

$$\begin{aligned} S^{-1}(p) &= S_0^{-1}(p) + i\Sigma(p) \\ &= -i\Big[\not p - (m + \Sigma(p))\Big] \end{aligned} \tag{4.67}$$

We conclude that it is enough to calculate $\Sigma(p)$ to all orders of perturbation theory to obtain the full electron propagator. The name *self-energy* given to $\Sigma(p)$ comes from the fact that, as can be seen in Eq. (4.67), it comes as an additional (momentum dependent) contribution to the mass.

In lowest order there is only the diagram of Fig. 4.8 contributing to $\Sigma(p)$ and therefore we get,

$$-i\Sigma(p) = (+ie)^2 \int \frac{d^4k}{(2\pi)^4} (-i) \frac{g_{\mu\nu}}{k^2 - \lambda^2 + i\varepsilon} \gamma^\mu \frac{i}{\not p + \not k - m + i\varepsilon} \gamma^\nu \tag{4.68}$$

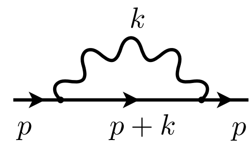


Figure 4.8: Self-energy of the electron

where we have chosen the Feynman gauge ($\xi = 1$) for the photon propagator and we have introduced a small mass for the photon λ, in order to control the infrared divergences (IR) that will appear when $k^2 \to 0$ (see below). Using dimensional regularization and the results of the Dirac algebra in dimension d,

$$\begin{aligned}
\gamma_\mu(\not{p}+\not{k})\gamma^\mu &= -(\not{p}+\not{k})\gamma_\mu\gamma^\mu + 2(\not{p}+\not{k}) = -(d-2)(\not{p}+\not{k}) \\
m\gamma_\mu\gamma^\mu &= m\,d
\end{aligned} \tag{4.69}$$

we get

$$\begin{aligned}
-i\,\Sigma(p) &= -\mu^\epsilon e^2 \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\lambda^2+i\varepsilon}\gamma_\mu\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2+i\varepsilon}\gamma^\mu \\
&= -\mu^\epsilon e^2 \int \frac{d^dk}{(2\pi)^d}\frac{-(d-2)(\not{p}+\not{k})+m\,d}{[k^2-\lambda^2+i\varepsilon]\,[(p+k)^2-m^2+i\varepsilon]} \\
&= -\mu^\epsilon e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{-(d-2)(\not{p}+\not{k})+m\,d}{[(k^2-\lambda^2)\,(1-x)+x(p+k)^2-xm^2+i\varepsilon]^2} \\
&= -\mu^\epsilon e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{-(d-2)(\not{p}+\not{k})+m\,d}{[(k+px)^2+p^2x(1-x)-\lambda^2(1-x)-xm^2+i\varepsilon]^2} \\
&= -\mu^\epsilon e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{-(d-2)\,[\not{p}(1-x)+\not{k}]+m\,d}{[k^2+p^2x(1-x)-\lambda^2(1-x)-xm^2+i\varepsilon]^2} \\
&= -\mu^\epsilon e^2 \int_0^1 dx\,\Big[-(d-2)\not{p}(1-x)+m\,d\Big]\,I_{0,2}
\end{aligned} \tag{4.70}$$

where[6]

$$I_{0,2} = \frac{i}{16\pi^2}\left[\Delta_\epsilon - \ln\left[-p^2x(1-x)+m^2x+\lambda^2(1-x)\right]\right] \tag{4.71}$$

The contribution from the *loop* in Fig. 4.8 to the electron self-energy $\Sigma(p)$ can then be written in the form,

$$\Sigma(p)^{loop} = A(p^2) + B(p^2)\,\not{p} \tag{4.72}$$

with

$$\begin{aligned}
A &= e^2\mu^\epsilon(4-\epsilon)m\frac{1}{16\pi^2}\int_0^1 dx\,\left[\Delta_\epsilon - \ln\left[-p^2x(1-x)+m^2x+\lambda^2(1-x)\right]\right] \\
B &= -e^2\mu^\epsilon(2-\epsilon)\frac{1}{16\pi^2}\int_0^1 dx\,(1-x)\,\Big[\Delta_\epsilon
\end{aligned}$$

[6] The linear term in k vanishes.

$$- \ln\left[-p^2 x(1-x) + m^2 x + \lambda^2 (1-x)\right]\Big] \tag{4.73}$$

Using now the expansions

$$\begin{aligned}
\mu^\epsilon (4-\epsilon) &= 4\left[1 + \epsilon\left(\ln\mu - \frac{1}{4}\right) + \mathcal{O}(\epsilon^2)\right] \\
\mu^\epsilon (4-\epsilon)\Delta_\epsilon &= 4\left[\Delta_\epsilon + 2\left(\ln\mu - \frac{1}{4}\right) + \mathcal{O}(\epsilon)\right] \\
\mu^\epsilon (2-\epsilon) &= 2\left[1 + \epsilon\left(\ln\mu - \frac{1}{2}\right) + \mathcal{O}(\epsilon^2)\right] \\
\mu^\epsilon (2-\epsilon)\Delta_\epsilon &= 2\left[\Delta_\epsilon + 2\left(\ln\mu - \frac{1}{2}\right) + \mathcal{O}(\epsilon)\right]
\end{aligned} \tag{4.74}$$

we can finally write,

$$A(p^2) = \frac{4\, e^2 m}{16\pi^2} \int_0^1 dx\, \left[\Delta_\epsilon - \frac{1}{2} - \ln\left[\frac{-p^2 x(1-x) + m^2 x + \lambda^2(1-x)}{\mu^2}\right]\right] \tag{4.75}$$

and

$$B(p^2) = -\frac{2\, e^2}{16\pi^2} \int_0^1 dx\, (1-x) \left[\Delta_\epsilon - 1 - \ln\left[\frac{-p^2 x(1-x) + m^2 x + \lambda^2(1-x)}{\mu^2}\right]\right] \tag{4.76}$$

To continue with the renormalization program we have to introduce the counterterm lagrangian and define the normalization conditions. We have

$$\Delta\mathcal{L} = i\,(Z_2 - 1)\,\overline{\psi}\gamma^\mu\partial_\mu\psi - (Z_2 - 1)\, m\,\overline{\psi}\psi + Z_2\delta m\,\overline{\psi}\psi + (Z_1 - 1)e\,\overline{\psi}\gamma^\mu\psi A_\mu \tag{4.77}$$

and therefore we get for the self-energy

$$- i\,\Sigma(p) = -i\,\Sigma^{loop}(p) + i\,\left(\not{p} - m\right)\delta Z_2 + i\,\delta m \tag{4.78}$$

Contrary to the case of the photon we see that we have two constants to determine. In the *on-shell* renormalization scheme that is normally used in QED the two constants are obtained by requiring that the pole of the propagator corresponds to the physical mass (hence the name of *on-shell* renormalization), and that the residue of the pole of the renormalized electron propagator has the same value as the free field propagator. This implies,

$$\begin{aligned}
&\Sigma(\not{p} = m) = 0 \quad\to\quad \delta m = \Sigma^{loop}(\not{p} = m) \\
&\left.\frac{\partial\Sigma}{\partial\not{p}}\right|_{\not{p}=m} = 0 \quad\to\quad \delta Z_2 = \left.\frac{\partial\Sigma^{loop}}{\partial\not{p}}\right|_{\not{p}=m}
\end{aligned} \tag{4.79}$$

We then get for δm,

$$\delta m \;=\; A(m^2) + m\, B(m^2)$$

$$
\begin{aligned}
&= \frac{2\,me^2}{16\pi^2}\int_0^1 dx\,\left\{\left[2\Delta_\epsilon - 1 - 2\ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right]\right.\\
&\qquad\qquad \left. -(1-x)\left[\Delta_\epsilon - 1 - \ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right]\right\}\\
&= \frac{2\,m\,e^2}{16\pi^2}\left[\frac{3}{2}\Delta_\epsilon - \frac{1}{2} - \int_0^1 dx\,(1+x)\ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right]\\
&= \frac{3\alpha m}{4\pi}\left[\Delta_\epsilon - \frac{1}{3} - \frac{2}{3}\int_0^1 dx\,(1+x)\ln\left(\frac{m^2x^2}{\mu^2}\right)\right] \qquad (4.80)
\end{aligned}
$$

where in the last step in Eq. (4.80) we have taken the limit $\lambda \to 0$ because the integral does not diverge in that limit[7]. In a similar way we get for δZ_2,

$$
\delta Z_2 = \left.\frac{\partial \Sigma^{loop}}{\partial \not{p}}\right|_{\not{p}=m} = \left.\frac{\partial A}{\partial \not{p}}\right|_{\not{p}=m} + B + m\left.\frac{\partial B}{\partial \not{p}}\right|_{\not{p}=m} \qquad (4.81)
$$

where

$$
\begin{aligned}
\left.\frac{\partial A}{\partial \not{p}}\right|_{\not{p}=m} &= \frac{4\,e^2\,m^2}{16\pi^2}\int_0^1 dx\,\frac{2(1-x)x}{-m^2x(1-x)+m^2x+\lambda^2(1-x)}\\
&= \frac{2\,\alpha\,m^2}{\pi}\int_0^1 dx\,\frac{(1-x)x}{m^2x^2+\lambda^2(1-x)}\\
B &= -\frac{\alpha}{2\pi}\int_0^1 dx\,(1-x)\left[\Delta_\epsilon - 1 - \ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right]\\
m\left.\frac{\partial B}{\partial \not{p}}\right|_{\not{p}=m} &= -\frac{\alpha}{2\pi}\,m^2\int_0^1 dx\,\frac{2x(1-x)^2}{m^2x^2+\lambda^2(1-x)} \qquad (4.82)
\end{aligned}
$$

Substituting Eq. (4.82) in Eq. (4.81) we get,

$$
\begin{aligned}
\delta Z_2 &= -\frac{\alpha}{2\pi}\left[\frac{1}{2}\Delta_\epsilon - \frac{1}{2} - \int_0^1 dx\,(1-x)\ln\left(\frac{m^2x^2}{\mu^2}\right) - 2\int_0^1 dx\,\frac{(1+x)(1-x)xm^2}{m^2x^2+\lambda^2(1-x)}\right]\\
&= \frac{\alpha}{4\pi}\left[-\Delta_\epsilon - 4 + \ln\frac{m^2}{\mu^2} - 2\ln\frac{\lambda^2}{m^2}\right] \qquad (4.83)
\end{aligned}
$$

where we have taken the $\lambda \to 0$ limit in all cases that was possible. It is clear that the final result in Eq. (4.83) diverges in that limit, therefore implying that Z_2 is IR divergent. This is not a problem for the theory because δZ_2 is not a physical parameter. We will see in section 4.4.2 that the IR diverges cancel for real processes. If we had taken a general gauge ($\xi \neq 1$) we would find out that δm would not be changed but that Z_2 would show a gauge dependence. Again, in physical processes this should cancel in the end.

4.1.3 The Vertex

The diagram contributing to the QED vertex at one-loop is the one shown in Fig. 4.9. In

[7] δm is not IR divergent.

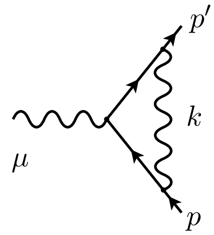


Figure 4.9: The QED vertex.

the Feynman gauge ($\xi = 1$) this gives a contribution,

$$\begin{aligned} ie\,\mu^{\epsilon/2}\Lambda_\mu^{loop}(p',p) &= (ie\,\mu^{\epsilon/2})^3 \int \frac{d^dk}{(2\pi)^d}(-i)\frac{g_{\rho\sigma}}{k^2-\lambda^2+i\varepsilon} \\ &\qquad \gamma^\sigma \frac{i[(\not{p}'+\not{k})+m]}{(p'+k)^2-m^2+i\varepsilon}\gamma_\mu \frac{i[(\not{p}+\not{k})+m]}{(p+k)^2-m^2+i\varepsilon}\gamma^\rho \end{aligned} \tag{4.84}$$

where Λ_μ is related to the full vertex Γ_μ through the relation

$$\begin{aligned} i\Gamma_\mu &= ie\,(\gamma_\mu + \Lambda_\mu^{loop} + \gamma_\mu \delta Z_1) \\ &= ie\,\left(\gamma_\mu + \Lambda_\mu^R\right) \end{aligned} \tag{4.85}$$

The integral that defines $\Lambda_\mu^{loop}(p',p)$ is divergent. As before we expect to solve this problem by regularizing the integral, introducing counterterms and normalization conditions. The counterterm has the same form as the vertex and is already included in Eq. (4.85). The normalization constant is determined by requiring that in the limit $q = p' - p \to 0$ the vertex reproduces the tree level vertex because this is what is consistent with the definition of the electric charge in the $q \to 0$ limit of the Coulomb scattering. Also this should be defined for on-shell electrons. We have therefore that the normalization condition gives,

$$\overline{u}(p)\left(\Lambda_\mu^{loop} + \gamma_\mu\delta Z_1\right)u(p)\Big|_{\not{p}=m} = 0 \tag{4.86}$$

If we are interested only in calculating δZ_1 and in showing that the divergences can be removed with the normalization condition then the problem is simpler. It can be done in two ways.

1st method

We use the fact that δZ_1 is to be calculated on-shell and for $p = p'$. Then

$$i\Lambda_\mu^{loop}(p,p) = e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\lambda^2+i\varepsilon}\gamma_\rho \frac{1}{\not{p}+\not{k}-m+i\varepsilon}\gamma_\mu \frac{1}{\not{p}+\not{k}-m+i\varepsilon}\gamma^\rho \tag{4.87}$$

However we have

$$\frac{1}{\not p+\not k-m+i\varepsilon}\gamma_\mu\frac{1}{\not p+\not k-m+i\varepsilon}=-\frac{\partial}{\partial p^\mu}\frac{1}{\not p+\not k-m+i\varepsilon} \tag{4.88}$$

and therefore

$$\begin{aligned} i\Lambda_\mu^{loop}(p,p) &= -e^2\mu^\epsilon\frac{\partial}{\partial p^\mu}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\lambda^2+i\varepsilon}\gamma_\rho\frac{\not p+\not k+m}{(p+k)^2-m^2+i\varepsilon}\gamma^\rho & (4.89)\\ &= -i\frac{\partial}{\partial p^\mu}\Sigma^{loop}(p) & (4.90)\end{aligned}$$

We conclude then, that $\Lambda_\mu^{loop}(p,p)$ is related to the self-energy of the electron[8],

$$\Lambda_\mu^{loop}(p,p)=-\frac{\partial}{\partial p^\mu}\Sigma^{loop} \tag{4.91}$$

On-shell we have

$$\Lambda_\mu^{loop}(p,p)\bigg|_{\not p=m} = -\frac{\partial\Sigma^{loop}}{\partial p^\mu}\bigg|_{\not p=m}=-\delta Z_2\gamma_\mu \tag{4.92}$$

and the normalization condition, Eq. (4.86), gives

$$\delta Z_1=\delta Z_2 \tag{4.93}$$

As we have already calculated δZ_2 in Eq. (4.83), then δZ_1 is determined.

2nd method

In this second method we do not rely in the Ward identity but just calculate the integrals for the vertex in Eq. (4.84). For the moment we do not put $p'=p$ but we will assume that the vertex form factors are to be evaluated for on-shell spinors. Then we have

$$\begin{aligned} i\,\overline{u}(p')\Lambda_\mu^{loop}u(p) &= e^2\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\,\frac{\overline{u}(p)\gamma_\rho\left[(\not p'+\not k+m)\right]\gamma_\mu\left[(\not p+\not k+m)\right]\gamma^\rho u(p)}{D_0D_1D_2}\\ &= e^2\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\,\frac{\mathcal{N}_\mu}{D_0D_1D_2} & (4.94)\end{aligned}$$

where

$$\mathcal{N}_\mu = \overline{u}(p)\Big[(-2+d)k^2\gamma_\mu+4p\cdot p'\gamma_\mu+4(p+p')\cdot k\,\gamma_\mu+4m\,k_\mu$$

[8]This result is one of the forms of the Ward-Takahashi identity.

$$- 4\not{k}\,(p+p')_\mu + 2(2-d)\not{k}\not{k} k_\mu \Big] u(p) \tag{4.95}$$

$$\begin{aligned} D_0 &= k^2 - \lambda^2 + i\epsilon & (4.96)\\ D_1 &= (k+p')^2 - m^2 + i\epsilon & (4.97)\\ D_2 &= (k+p)^2 - m^2 + i\epsilon & (4.98)\end{aligned}$$

Now we use the results of section C.7.3 to do the momentum integrals. We have for our case

$$\begin{aligned} r_1^\mu &= p'^\mu \quad ; \quad r_2^\mu = p^\mu & (4.99)\\ P^\mu &= x_1 p'^\mu + x_2 p^\mu & (4.100)\\ C &= (x_1+x_2)^2 m^2 - x_1 x_2\, q^2 + (1-x_1-x_2)\lambda^2 & (4.101)\end{aligned}$$

where

$$q = p' - p\ . \tag{4.102}$$

We get,

$$\begin{aligned} i\,\overline{u}(p')\Lambda_\mu^{loop} u(p) &= i\,\frac{\alpha}{4\pi}\Gamma(3)\int_0^1 dx_1 \int_0^{1-x_1} dx_2\ \frac{1}{2C}\\ &\Big\{\overline{u}(p')\gamma_\mu u(p)\Big[-(-2+d)(x_1^2 m^2 + x_2^2 m^2 + 2x_1x_2 p'\cdot p) - 4p'\cdot p\\ &+4(p+p')\cdot(x_1p'+x_2p) + \frac{(2-d)^2}{2}\,C\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)\Big]\\ &+\overline{u}(p)u(p)m\ \Big[4(x_1p'+x_2p)_\mu - 4(p'+p)_\mu(x_1+x_2)\\ &-2(2-d)(x_1+x_2)(x_1p'+x_2p)_\mu\Big]\Big\} & (4.103)\\ &= i\,\overline{u}(p)\left[G(q^2)\,\gamma_\mu + H(q^2)\,(p+p')_\mu\right]u(p) & (4.104)\end{aligned}$$

where we have defined[9],

$$G(q^2) \equiv \frac{\alpha}{4\pi}\Bigg[\Delta_\epsilon - 2 - 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2\ \ln\frac{(x_1+x_2)^2m^2 - x_1x_2q^2 + (1-x_1-x_2)\lambda^2}{\mu^2}$$

[9] To obtain Eq. (4.109) one has to show that the symmetry of the integrals in $x_1 \leftrightarrow x_2$ implies that the coefficient of p is equal to the coefficient of p'. To see this define

$$H = \int_0^1 dx_1 \int_0^{1-x_1} f(x_1,x_2) \tag{4.105}$$

Then use

$$f(x_1,x_2) = \frac{1}{2}\left[f(x_1,x_2)+f(x_2,x_1)\right]\frac{1}{2}\left[f(x_1,x_2)-f(x_2,x_1)\right] \tag{4.106}$$

to show that

$$H = \int_0^1 dx_1 \int_0^{1-x_1} \frac{1}{2}\left(f(x_1,x_2)+f(x_2,x_1)\right)\ . \tag{4.107}$$

Also notice that you can put $d=4$ in this term because H is not divergent.

$$+\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left(\frac{-2(x_1+x_2)^2 m^2 - x_1 x_2 q^2 - 4m^2 + 2q^2}{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2} + \frac{2(x_1+x_2)(4m^2-q^2)}{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2} \right) \Bigg] \tag{4.108}$$

$$H(q^2) \equiv \frac{\alpha}{4\pi} \left[\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \, \frac{-2m\,(x_1+x_2) + 2m\,(x_1+x_2)^2}{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2} \right] \tag{4.109}$$

Now, using the definition of Eq. (4.85), we get for the renormalized vertex,

$$\overline{u}(p')\Lambda^R_\mu(p',p)u(p) = \overline{u}(p') \left[\left(G(q^2) + \delta Z_1 \right) \gamma_\mu + H(q^2)\,(p+p')_\mu \right] u(p) \tag{4.110}$$

As δZ_1 is calculated in the limit of $q = p' - p \to 0$ it is convenient to use the Gordon identity to get rid of the $(p'+p)^\mu$ term. We have

$$\overline{u}(p')\left(p'+p\right)_\mu u(p) = \overline{u}(p') \left[2m\gamma_\mu - i\sigma_{\mu\nu}\, q^\nu \right] u(p) \tag{4.111}$$

and therefore,

$$\begin{aligned} \overline{u}(p')\Lambda^R_\mu(p',p)u(p) &= \overline{u}(p') \left[\left(G(q^2) + 2mH(q^2) + \delta Z_1 \right) \gamma_\mu - i\, H(q^2)\, \sigma_{\mu\nu}\, q^\nu \right] u(p) \\ &= \overline{u}(p') \left[\gamma_\mu F_1(q^2) + \frac{i}{2m} \sigma_{\mu\nu} q^\nu F_2(q^2) \right] u(p) \end{aligned} \tag{4.112}$$

where we have introduced the usual notation for the vertex form factors,

$$F_1(q^2) \equiv G(q^2) + 2mH(q^2) + \delta Z_1 \tag{4.113}$$

$$F_2(q^2) \equiv -2mH(q^2) \tag{4.114}$$

The normalization condition of Eq. (4.86) implies $F_1(0) = 0$, that is,

$$\delta Z_1 = -G(0) - 2m\, H(0) \tag{4.115}$$

We have therefore to calculate $G(0)$ and $H(0)$. In this limit the integrals of Eqs. (4.108) and (4.109) are much simpler. We get (we change variables $x_1 + x_2 \to y$),

$$\begin{aligned} G(0) &= \frac{\alpha}{4\pi} \left[\Delta_\epsilon - 2 - 2 \int_0^1 dx_1 \int_{x_1}^1 dy \ln \frac{y^2 m^2 + (1-y)\lambda^2}{\mu^2} \right. \\ &\qquad \left. + \int_0^1 dx_1 \int_{x_1}^1 dy \, \frac{-2y^2 m^2 - 4m^2 + 8ym^2}{y^2 m^2 + (1-y)\lambda^2} \right] \end{aligned} \tag{4.116}$$

$$H(0) = \frac{\alpha}{4\pi} \int_0^1 dx_1 \int_{x_1}^1 dy \, \frac{-2m\, y + 2m\, y^2}{y^2 m^2 + (1-y)\lambda^2} \tag{4.117}$$

Now using

$$\int_0^1 dx_1 \int_{x_1}^1 dy \ln \frac{y^2 m^2 + (1-y)\lambda^2}{\mu^2} = \frac{1}{2}\left(\ln \frac{m^2}{\mu^2} - 1 \right) \tag{4.118}$$

$$\int_0^1 dx_1 \int_{x_1}^1 dy \, \frac{-2y^2m^2 - 4m^2 + 8ym^2}{y^2m^2 + (1-y)\lambda^2} = 7 + 2\ln\frac{\lambda^2}{m^2} \tag{4.119}$$

$$\int_0^1 dx_1 \int_{x_1}^1 dy \, \frac{-2m\,y + 2m\,y^2}{y^2m^2 + (1-y)\lambda^2} = -\frac{1}{m} \tag{4.120}$$

(where we took the limit $\lambda \to 0$ if possible) we get,

$$G(0) = \frac{\alpha}{4\pi}\left[\Delta_\epsilon + 6 - \ln\frac{m^2}{\mu^2} + 2\ln\frac{\lambda^2}{m^2}\right] \tag{4.121}$$

$$H(0) = -\frac{\alpha}{4\pi}\frac{1}{m} \tag{4.122}$$

Substituting the previous expressions in Eq. (4.115) we get finally,

$$\delta Z_1 = \frac{\alpha}{4\pi}\left[-\Delta_\epsilon - 4 + \ln\frac{m^2}{\mu^2} - 2\ln\frac{\lambda^2}{m^2}\right] \tag{4.123}$$

in agreement with Eq. (4.83) and Eq. (4.93). The general form of the form factors $F_i(q^2)$, for $q^2 \neq 0$, is quite complicated. We give here only the result for $q^2 < 0$ (in section C.10.3 we will give a general formula for numerical evaluation of these functions),

$$\begin{aligned} F_1(q^2) &= \frac{\alpha}{4\pi}\left\{\left(2\ln\frac{\lambda^2}{m^2} + 4\right)(\theta\coth\theta - 1) - \theta\,\tanh\frac{\theta}{2} - 8\coth\theta\int_0^{\theta/2}\beta\tanh\beta d\beta\right\} \\ F_2(q^2) &= \frac{\alpha}{2\pi}\frac{\theta}{\sinh\theta} \end{aligned} \tag{4.124}$$

where

$$\sinh^2\frac{\theta}{2} = -\frac{q^2}{4m^2}. \tag{4.125}$$

In the limit of zero transferred momenta ($q = p' - p = 0$) we get

$$\begin{cases} F_1(0) = 0 \\ F_2(0) = \dfrac{\alpha}{2\pi} \end{cases} \tag{4.126}$$

a result that we will use in section 4.4.1 while discussing the anomalous magnetic moment of the electron.

4.2 Ward-Takahashi identities in QED

In the study of the QED vertex, in one of the methods, we used the Ward identity[10]

$$\Lambda_\mu(p,p) = -\frac{\partial}{\partial p^\mu}\Sigma(p) \tag{4.127}$$

[10] The sign in Eq. (4.127) is connected with our identification in Eq. (4.63) of the one-particle irreducible contribution to the fermion propagator as $-i\Sigma(p)$.

We are going to derive here the general form for these identities. The following discussion is formal in the sense that the various Green functions are divergent. We have to prove that we can find a regularization scheme that preserves the identities. This happens when one uses a regularization that preserves the gauge invariance of the theory. Examples are dimensional regularization and the Pauli-Villars regularization.

Ward identities are a consequence of the gauge invariance of the theory, as will be fully discussed in chapters 5 and 6. Here we are only going to use the fact that there is a conserved current[11],

$$\begin{aligned} j_\mu &= -e\overline{\psi}\gamma_\mu\psi \\ \partial_\mu j^\mu &= 0 \end{aligned} \tag{4.128}$$

We are interested in calculating the quantity

$$\partial_x^\mu \langle 0|\, T j_\mu(x)\psi(x_1)\overline{\psi}(y_1)\cdots\psi(x_n)\overline{\psi}(y_n)A_{\nu_1}(z_1)\cdots A_{\nu_p}(z_p)\,|0\rangle \tag{4.129}$$

This quantity does not vanish, despite the fact that $\partial^\mu j_\mu = 0$. This happens because in the time ordered product we have θ functions that depend on the coordinate x^0. For instance, for the field $\psi(x_i)$ we should have a contribution of the form,

$$\begin{aligned} &\partial_x^0\big[\theta(x^0-x_i^0)j_0(x)\psi(x_i)+\theta(x_i^0-x^0)\psi(x_i)j_0(x)\big] \\ &\quad= \;\; \delta(x^0-x_i^0)j_0(x)\psi(x_i)-\delta(x^0-x_i^0)\psi(x_i)j_0(x) \\ &\quad= \;\; \big[j_0(x),\psi(x_i)\big]\delta(x^0-x_i^0) \end{aligned} \tag{4.130}$$

In this way we get ($\widehat{\ }$ means that we omit that term from the sum),

$$\begin{aligned} &\partial_x^\mu \langle 0|\, T j_\mu(x)\psi(x_1)\cdots\overline{\psi}(y_n)A_{\nu_1}(z_1)\cdots A_{\nu_p}(z_p)\,|0\rangle \\ &=\sum_{i=1}^n \langle 0|\, T\left\{ [j_0(x),\psi(x_i)]\,\delta(x^0-x_i^0)\overline{\psi}(y_i)\right. \\ &\quad\left. +\psi(x_i)\left[j_0(x),\overline{\psi}(y_i)\right]\delta(x^0-y_i^0)\right\}\psi(x_1)\overline{\psi}(y_1)\cdots\widehat{\psi(x_i)\overline{\psi}(y_i)}\cdots A_{\nu_p}(z_p)\,|0\rangle \\ &\quad+\sum_{j=1}^p \langle 0|\, T\psi(x_1)\cdot\overline{\psi}(y_n)A_{\nu_p}(z_1)\cdots[j_0(x),A_{\nu_j}(z_j)]\delta(x^0-z_j^0)\cdots A_{\nu_p}(z_p)\,|0\rangle \end{aligned} \tag{4.131}$$

Using now the equal time commutation relations,

$$\begin{aligned} &\big[j_0(x),\psi(x')\big]\delta(x^0-x'^0)=e\psi(x)\delta^4(x-x') \\ &\big[j_0(x),\overline{\psi}(x')\big]\delta(x^0-x'^0)=-e\overline{\psi}(x)\delta^4(x-x') \\ &\big[j_0(x),A_\mu(x')\big]\delta(x^0-x'^0)=0 \end{aligned} \tag{4.132}$$

that express that $\psi,\overline{\psi}$ and A_μ create *quanta* with charge $Q=\int d^3x j^0(x)$ equal to $e,-e$ and zero, respectively, we get,

$$\partial_x^\mu \langle 0|\, T j_\mu(x)\psi(x_1)\cdots\overline{\psi}(y_n)A_{\nu_1}(z_1)\cdots A_{\nu_p}(z_p)\,|0\rangle$$

[11] We follow here Ref.[2], but use our identification of the electric charge as $e=|e|$, so we have modified the signs to be consistent.

$$= -e \langle 0| T\psi(y_1)\cdots A_{\nu_p}(z_p)|0\rangle \sum_{i=1}^{n}\left[\delta^4(x-y_i)-\delta^4(x-x_i)\right] \tag{4.133}$$

Taking different values for n and p we get different relations among the Green functions of the theory. We will consider in the following, two important cases.

4.2.1 Transversality of the photon propagator $n=0$, $p=1$

The Green function $\langle 0| Tj_\mu(x)A_\nu(y)|0\rangle$ corresponds to the Feynman diagram of Fig. 4.10, and it is related with the full photon propagator shown in Fig. 4.11, by the diagrammatic

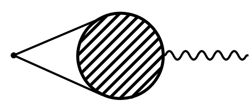


Figure 4.10: Green function $\langle 0| Tj_\mu(x)A_\nu(y)|0\rangle$.

relation shown in Fig. 4.12 known as the Dyson-Schwinger equation for `QED`. It can be written as

$$G_{\mu\nu}(x-y) = G^0_{\mu\nu}(x-y) - i\int d^4x' G^0_{\mu\rho}(x-x')\langle 0| Tj^\rho(x')A_\nu(y)|0\rangle \tag{4.134}$$

We apply now the derivative ∂^μ_x to get,

$$\partial^\mu_x G_{\mu\nu}(x-y) = \partial^\mu_x G^0_{\mu\nu}(x-y) - i\int d^4x' \partial^\mu_x G^0_{\mu\rho}(x-x')\langle 0| Tj^\rho(x')A_\mu(y)|0\rangle \tag{4.135}$$

The free photon propagator is given by,

$$G^0_{\mu\rho}(x-x') = \int \frac{d^4p}{(2\pi)^4} e^{-i(x-x')\cdot p} G^0_{\mu\rho}(p) \tag{4.136}$$

where

$$G^0_{\mu\nu}(p) = -i\left[\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2} + \xi\frac{p_\mu p_\nu}{p^4}\right] . \tag{4.137}$$

Therefore

$$\partial^\mu_x G^0_{\mu\rho}(x-x') \quad = \quad \int \frac{d^4p}{(2\pi)^4} e^{-i(x-x')\cdot p}(-ip^\mu) G^0_{\mu\rho}(p)$$

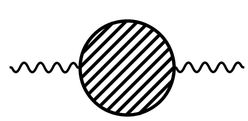


Figure 4.11: Photon propagator

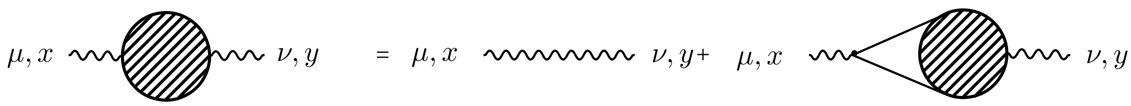


Figure 4.12: Dyson-Schwinger equation.

$$
\begin{aligned}
&= \int \frac{d^4p}{(2\pi)^4} e^{-i(x-x')\cdot p}(-ip_\rho)F(p^2) \\
&= -\partial_\rho^{x'} \int \frac{d^4p}{(2\pi)^4} e^{-i(x-x')\cdot p}F(p^2) \\
&= -\partial_\rho^{x'} \widetilde{F}(x-x') \qquad (4.138)
\end{aligned}
$$

and we get

$$
\begin{aligned}
\partial_x^\mu G_{\mu\nu}(x-y) &= \partial_x^\mu G^0_{\mu\nu}(x-y) + i\int d^4x' \partial^\rho_{x'} \widetilde{F}(x-x') \langle 0| T j_\rho(x') A_\nu(y) |0\rangle \\
&= \partial_x^\mu G^0_{\mu\nu}(x-y) - i\int d^4x' \widetilde{F}(x-x') \partial_\rho^{x'} \langle 0| T j^\rho(x') A_\nu(y) |0\rangle \\
&= \partial_x^\mu G^0_{\mu\nu}(x-y) \qquad (4.139)
\end{aligned}
$$

where we have made an integration by parts and used the Ward-Takashashi identity for $n = 0$, $p = 1$. We have then

$$
\partial_x^\mu G_{\mu\nu}(x-y) = \partial_x^\mu G^0_{\mu\nu}(x-y) \qquad (4.140)
$$

which in momenta space implies

$$
p^\mu G_{\mu\nu}(p) = p^\mu G^0_{\mu\nu}(p) \qquad (4.141)
$$

This means that the longitudinal part of the photon propagator is not renormalized, or in other words, that the self-energy of the photon (vacuum polarization) is transverse. In fact

$$
p^\mu G^0_{\mu\nu}(p) = -i\xi \frac{p_\nu}{p^2} \qquad (4.142)
$$

or

$$
p^\mu = -i\xi \frac{p_\nu}{p^2} G^{-1}_{\nu\mu}(p) = -\xi \frac{p_\nu}{p^2} \Gamma_{\nu\mu}(p) \qquad (4.143)
$$

But, in agreement with our conventions, we have

$$
\Gamma_{\nu\mu}(p) = -(g_{\nu\mu}p^2 - p_\nu p_\mu) - \frac{1}{\xi} p_\nu p_\mu + \Pi_{\nu\mu}(p^2) \qquad (4.144)
$$

and therefore

$$
-\xi \frac{p_\nu}{p^2} \Gamma_{\nu\mu}(p) = p_\mu - \frac{1}{\xi}\frac{1}{p^2} p^\nu \Pi_{\nu\mu}(p^2) = p_\mu \qquad (4.145)
$$

which gives

$$
p^\nu \Pi_{\nu\mu}(p^2) = 0 \qquad (4.146)
$$

that is, the self-energy is transverse.

4.2.2 Identity for the vertex $n = 1$, $p = 0$

We are now interested in the Green function,

$$\langle 0|\, T j_\mu(x)\psi_\beta(x_1)\overline{\psi}_\alpha(y_1)\,|0\rangle \tag{4.147}$$

to which corresponds the diagram of Fig. 4.13. This Green function can be related with

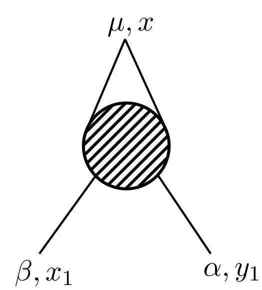


Figure 4.13: Green function $\langle 0|\, T j_\mu(x)\psi_\beta(x_1)\overline{\psi}_\alpha(y_1)\,|0\rangle$.

the vertex $\langle 0|\, T A_\mu(x)\psi_\beta(x_1)\overline{\psi}_\alpha(y_1)\,|0\rangle$ corresponding to the diagram of Fig. 4.14, through

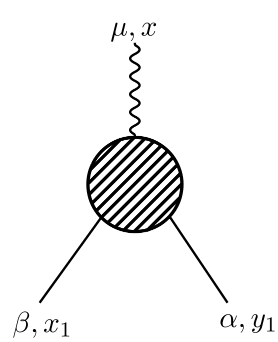
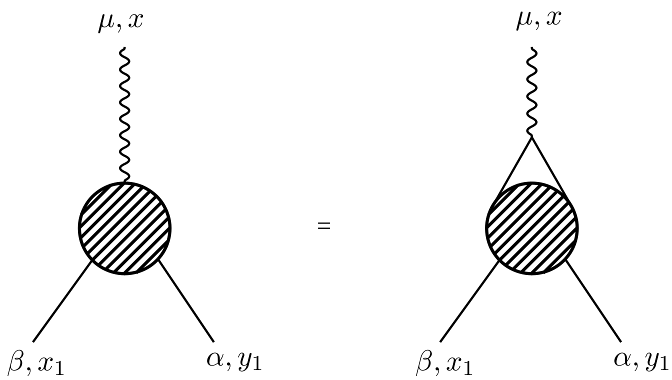


Figure 4.14: Full vertex.

the following diagrammatic equation,



(4.148)

that we can write as,

$$\langle 0| TA_\mu(x)\psi_\beta(x_1)\overline{\psi}_\alpha(y_1)|0\rangle = -i\int d^4x' G^0_{\mu\nu}(x-x')\langle 0| Tj^\nu(x')\psi_\beta(x_1)\overline{\psi}_\alpha(y_1)|0\rangle \quad (4.149)$$

Taking the Fourier transform,

$$\begin{aligned}&\int d^4x d^4x_1 d^4y_1 e^{i(p'\cdot x_1 - p\cdot y_1 - q\cdot x)}\langle 0| TA_\mu(x)\psi_\beta(x_1)\overline{\psi}_\alpha(y_1)|0\rangle \\ &\qquad = -iG^0_{\mu\nu}(q)\int d^4x d^4x_1 d^4y_1 e^{i(p'\cdot x_1 - p\cdot y_1 - q\cdot x)}\langle 0| Tj^\nu(x)\psi_\beta(x_1)\overline{\psi}_\alpha(y_1)|0\rangle \quad (4.150)\end{aligned}$$

where the direction of the momenta are shown in Fig. 4.15, and the momentum transfered

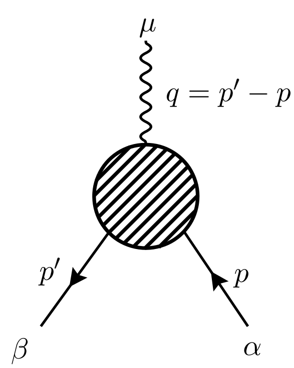


Figure 4.15: Definition of the momenta in the vertex.

is $q = p' - p$.

On the other side, using the definition of Γ_μ, we have,

$$\int d^4x d^4x_1 d^4y_1 e^{i(p'\cdot x_1 - p\cdot y_1 - q\cdot x)}\langle 0| TA_\mu(x)\psi_\beta(x_1)\overline{\psi}_\alpha(y_1)|0\rangle$$

$$= \quad (2\pi)^4\delta^4(p'-p-q)G_{\mu\nu}(q)\big[S(p')i\Gamma_\nu(p',p)S(p)\big]_{\beta\alpha} \tag{4.151}$$

Therefore we get,

$$\begin{aligned}
&(2\pi)^4\delta(p'-p-q)G_{\mu\nu}(q)S(p')i\Gamma^\nu(p',p)S(p)\\
&\quad = \quad -iG^0_{\mu\nu}(q)\int d^4xd^4x_1d^4y_1e^{i(p'\cdot x_1-p\cdot y_1-q\cdot x)}\ \langle 0|\,Tj^\nu(x)\psi(x_1)\overline{\psi}(y_1)\,|0\rangle
\end{aligned} \tag{4.152}$$

Multiplying by q^μ and using the result,

$$q^\mu G_{\mu\nu}(q) = q^\mu G^0_{\mu\nu}(q) = -i\xi\frac{q_\nu}{q^2} \tag{4.153}$$

we can then write (using the Ward identity for $n=1$, $p=0$)

$$\begin{aligned}
&(2\pi)^4\delta(p'-q-p)S(p')q^\nu\Gamma_\nu(p',p)S(p)\\
&\quad = \ i\int d^4xd^4x_1d^4y_1\partial^\nu_x\,\langle 0|\,Tj_\nu(x)\psi(x_1)\overline{\psi}(y_1)\,|0\rangle\, e^{i(p'\cdot x_1-p\cdot y_1-q\cdot x)}\\
&\quad = \ -i\,e\int d^4_xd^4x_1d^4y_1e^{i(p'\cdot x_1-p\cdot y_1-q\cdot x)}\,\langle 0|\,T\psi(x_1)\overline{\psi}(y_1)\,|0\rangle\,\big[\delta(x-y_1)-\delta(x-x_1)\big]\\
&\quad = \ -i\,e(2\pi)^4\delta(p'-p-q)\big[S(p')-S(p)\big]
\end{aligned} \tag{4.154}$$

or

$$q^\nu\Gamma_\nu(p',p) = -i\,e\left[S^{-1}(p)-S^{-1}(p')\right] \tag{4.155}$$

As $q^\nu = (p'-p)^\nu$ we get in the limit $p'=p$,

$$\begin{aligned}
\Gamma_\nu(p,p) \quad &= \quad ie\frac{\partial S^{-1}}{\partial p^\nu}\\
&= \quad e\left(\gamma_\nu - \frac{\partial\Sigma}{\partial p^\nu}\right)
\end{aligned} \tag{4.156}$$

Using $\Gamma_\nu = e(\gamma_\nu+\Lambda_\nu)$ we finally get the Ward identity in the form used before,

$$\Lambda_\nu(p,p) = -\frac{\partial\Sigma}{\partial p^\nu}\ . \tag{4.157}$$

4.3 Counterterms and power counting

All that we have shown in the previous sections can be interpreted as follows. The initial Lagrangian $\mathcal{L}(e,m,\cdots)$ has been obtained from a correspondence between classical and quantum theory. It is then natural that the initial Lagrangian has to be modified by quantum corrections. The total Lagrangian is then given by,

$$\mathcal{L}_{\rm total} = \mathcal{L}(e,m,\cdots) + \Delta\mathcal{L} \tag{4.158}$$

and

$$\Delta\mathcal{L} = \Delta\mathcal{L}^{(1)} + \Delta\mathcal{L}^{[2]} + \cdots \tag{4.159}$$

where $\Delta\mathcal{L}^{[i]}$ is the $i^{\text{th}} -$ *loops* correction. This also correspond to order $\hbar^i$ as counting in terms of loops is equivalent to counting in terms of $\hbar$[12]. This interpretation is quite attractive because in the limit $\hbar \to 0$ the total Lagrangian reduces to the classical one. With the Lagrangian $\mathcal{L}_{tot}$ we can then obtain finite results, although $\mathcal{L}_{tot}$ is divergent because of the counter-terms in $\Delta\mathcal{L}$.

With this language the results up to the first order in $\hbar$ can be written as,

$$\begin{aligned}
\mathcal{L}(e, m, \cdots) &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda^2}{2}A^\mu A_\mu - \frac{1}{2\xi}(\partial\cdot A)^2 \\
&\quad + i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi - e\overline{\psi}A\!\!\!/\psi \qquad (4.160)\\
\Delta\mathcal{L}^{(1)} &= -\frac{1}{4}(Z_3-1)F_{\mu\nu}F^{\mu\nu} + (Z_2-1)(i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi) \\
&\quad + Z_2\delta m\overline{\psi}\psi - e(Z_1-1)\overline{\psi}A\!\!\!/\psi \qquad (4.161)
\end{aligned}$$

The Lagrangian

$$\begin{aligned}
\mathcal{L}_{\text{total}} &= -\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} + \frac{\lambda^2}{2}A_\mu A^\mu - \frac{1}{2\xi}(\partial\cdot A)^2 \\
&\quad + Z_2(i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi + \delta m\overline{\psi}\psi) \\
&\quad - eZ_1\overline{\psi}A\!\!\!/\psi \qquad (4.162)
\end{aligned}$$

will give the renormalized Green's functions up the the order $\hbar$.

In fact, we have only shown that the two-point and three-point Green's functions (self-energies and vertex) were finite. It is important to verify that all the Green's functions, with an arbitrary number of external legs are finite, as we have already used all our freedom in the renormalization of those Green's functions. This leads us to the so-called *power counting*.

Let us consider a Feynman diagram G, with L *loops*, I_B bosonic and I_F fermionic internal lines. If there are vertices with derivatives, δ_v is the number of derivatives in that vertex. We define then the *superficial degree of divergence* of the diagram (note that $L = I_B + I_F + 1 - V$) by,

$$\begin{aligned}
\omega(G) &= 4L + \sum_v \delta_v - I_F - 2I_B \\
&= 4 + 3I_F + 2I_B + \sum_v (\delta_v - 4) \qquad (4.163)
\end{aligned}$$

For large values of the momenta the diagram will be divergent as

$$\Lambda^\omega(G) \quad \text{if} \quad \omega(G) > 0 \tag{4.164}$$

[12] $\hbar^{E-1+L} = \hbar^{\frac{E}{2}+\frac{V}{2}}$. We have the following relations $L = I - V + 1$; $3V = E + 2I$ (this only for `QED`).

and as

$$\ln\Lambda \quad \text{if} \quad \omega(G)=0 \tag{4.165}$$

where Λ is a *cutoff*. The origin of the different terms can be seen in the following correspondence,

$$\begin{aligned}
&\int \frac{d^4q}{(2\pi)^4}\ \text{(for each loop)} && \rightarrow \quad 4L\\
&\partial_\mu \Leftrightarrow k_\mu && \rightarrow \quad \delta_v\\
&\frac{i}{\not{q}-m} && \rightarrow \quad -I_F\\
&\frac{i}{q^2-m^2} && \rightarrow \quad -2I_B
\end{aligned} \tag{4.166}$$

The expression for $\omega(G)$ is more useful when expressed in terms of the number of external legs and of the dimensionality of the vertices of the theory. Let ω_v be the dimension, in terms of mass, of the vertex v, that is,

$$\omega_v = \delta_v + \#\text{campos bosónicos} \ + \ \frac{3}{2}\#\text{campos fermiónicos} \tag{4.167}$$

Then, if we denote by $f_v(b_v)$ the number of fermionic (bosonic) internal lines that join at the vertex v, we can write,

$$\sum_v \omega_v = \sum_v (\delta_v + \frac{3}{2}f_v + b_v) + \frac{3}{2}E_F + E_B \tag{4.168}$$

where $E_F(E_B)$ are the total number of *external* fermionic (bosonic) lines of the diagram. As we have,

$$\begin{aligned}
I_F &= \frac{1}{2}\sum_v f_v\\
I_B &= \frac{1}{2}\sum_v b_v
\end{aligned} \tag{4.169}$$

we get

$$\sum_v \omega_v = \sum_v \delta_v + 3I_F + 2I_B + \frac{3}{2}E_F + E_B \tag{4.170}$$

Substituting in the expression for $\omega(G)$ we get finally,

$$\begin{aligned}
\omega(G) =& 4 - \frac{3}{2}E_F - E_B + \sum_v(\omega_v - 4)\\
=& 4 - \frac{3}{2}E_F - E_B - \sum_v [g_v]
\end{aligned} \tag{4.171}$$

where $[g_v]$ denotes the dimension in terms of mass of the coupling constant of vertex v, satisfying,

$$\omega_v + [g_v] = 4\ . \tag{4.172}$$

From the previous expression for the superficial degree of divergence, Eq. (4.171), we can then classify theories in three classes,

i) Non-renormalizable Theories

They have at least one vertex with $\omega_v > 4$ (or $[g_v] < 0$). The superficial degree of divergence increases with the number of vertices, that is, with the order of perturbation theory. For an order high enough all the Green functions will diverge.

ii) Renormalizable Theories

All the vertices have $\omega_v \leq 4$ and at least one has $\omega_v = 4$. If all vertices have $\omega_v = 4$ then

$$\omega(G) = 4 - \frac{3}{2}E_F - E_B \tag{4.173}$$

and all the diagrams contributing to a given Green function have the same degree of divergence. Only a *finite* number of Green functions are divergent.

iii) Super-Renormalizable Theories

All the vertices have $\omega_v < 4$. Only a finite number of diagrams are divergent.[13]

Coming back to our question of knowing which are the divergent diagrams in QED, we can now summarize the situation in Table 4.1. All the other diagrams are superficially

E_F	E_B	$\omega(G)$	Effective degree of divergence
0	2	2	0 (Current Conservation (CC))
0	3		0 (Furry's Theorem)
0	4	0	Convergent (CC)
2	0	1	0 (Current Conservation)
2	1	0	0

Table 4.1: Superficial and effective degree of divergence in QED.

convergent. We have therefore a situation where there are only a finite number of divergent diagrams, exactly the ones that we considered before. This analysis shows that, up to order $\hbar$, the Lagrangian

$$\begin{aligned}
\mathcal{L}_{\text{total}} &= -\frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\lambda^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial\cdot A)^2 \\
&\quad + Z_2(i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi + \delta m\overline{\psi}\psi) \\
&\quad - eZ_1\overline{\psi}A\!\!\!/\psi
\end{aligned} \tag{4.174}$$

gives Green functions that are finite and renormalized with an arbitrary number of external legs. It remains to be shown that this Lagrangian is still valid up an arbitrary order in

[13] The effective degree of divergence it is sometimes smaller than the superficial degree because of symmetries of the theory. This is what happens for gauge theories like QED (see Table 4.1).

$\hbar$, with the only modification that the renormalization constants Z_1, Z_2, Z_3 e δm are now given by power series,

$$Z_1 = Z_1^{(1)} + Z_1^{(2)} + \cdots \tag{4.175}$$

The previous Lagrangian, Eq. (4.174), allows for another interpretation that it is also useful. The fields $A, \overline{\psi}$ and ψ are the renormalized fields that give the residues equal to 1 for the poles of the propagators and the constants e, m are the physical electric charge and mass of the electron. Let us define the non-renormalized fields $\psi_0, \overline{\psi}_0$ and A_0 and the *bare* (cutoff dependent) μ_0^2, m_0 through the definitions,

$$\begin{aligned}
&\psi_0 = \sqrt{Z_2}\ \psi && m_0 = m - \delta m \\
&\overline{\psi}_0 = \sqrt{Z_2}\ \overline{\psi} && \lambda_0^2 = Z_3^{-1}\lambda^2 \\
&A_0 = \sqrt{Z_3}\ A && e_0 = Z_1 Z_2^{-1}\sqrt{Z_3^{-1}}e = \tfrac{1}{\sqrt{Z_3}}e \\
&\xi_0 = Z_3 \xi
\end{aligned} \tag{4.176}$$

Then the Lagrangian written in terms of the bare quantities is identical to the original Lagrangian[14]

$$\begin{aligned}
\mathcal{L} \ &= \ -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + \frac{1}{2}\lambda_0^2\, A_{0\mu}A_0^{\mu} - \frac{1}{2\xi_0}(\partial\cdot A_0)^2 \\
&\quad + i(\overline{\psi}_0 \not{\partial}\psi_0 - m_0\overline{\psi}_0\psi_0) - e_0\overline{\psi}_0 \not{A}_0\psi_0
\end{aligned} \tag{4.177}$$

Finally we notice that the bare Green functions are related to the renormalized ones by

$$\begin{aligned}
&G_0^{n,\ell}(p_1, \cdots p_{2n}, k_1, \cdots k_\ell, \mu_0, m_0, \ell_0, \xi_0, \Lambda) \\
&\qquad = Z_2^n(\Lambda) Z_3^{\ell/2} G_R^{n,\ell}(p_1, \cdots p_{2n}, k_1 \cdots k_\ell, \mu, m, e, \xi)
\end{aligned} \tag{4.178}$$

where $p_1 \cdots p_{2n}$ ($k_1 \cdots k_\ell$) are the fermion (boson) momenta. We will come back to these relations in the study of the renormalization group, in chapter 7.

4.4 Finite contributions from RC to physical processes

4.4.1 Anomalous electron magnetic moment

We will show here, for the case of the electron anomalous moment, how the finite part of the radiative corrections can be compared with experiment, given credibility to the renormalization program. In fact we will just consider the first order, while to compare with the present experimental limit one has to go to fourth order in QED and to include also the weak and QCD corrections. The electron magnetic moment is given by

[14] The terms $\frac{\lambda^2}{2}A^2 = \frac{\lambda_0^2}{2}A_0^2$ and $\frac{1}{2\xi}(\partial\cdot A)^2 = \frac{1}{2\xi_0}(\partial\cdot A_0)^2$ are not renormalized. This a consequence of the Ward-Takashashi identities for QED. The Ward identity $Z_1 = Z_2$ is crucial for the equality $e_0 A_0 = eA$ giving a meaning to the electric charge independently of the renormalization scheme.

$$\vec{\mu} = \frac{e}{2m} g \frac{\vec{\sigma}}{2} \tag{4.179}$$

where $e = -|e|$ for the electron. One of the biggest achievements of the Dirac equation was precisely to predict the value $g = 2$. Experimentally we know that g is close to, but not exactly, 2. It is usual to define this difference as the anomalous magnetic moment. More precisely,

$$g = 2(1 + a) \tag{4.180}$$

or

$$a = \frac{g}{2} - 1 \tag{4.181}$$

Our task is to calculate a from the radiative corrections that we have computed in the previous sections. To do that let us start to show how a value $a \neq 0$ will appear in non relativistic quantum mechanics. Schrödinger's equation for a charged particle in an exterior field is,

$$i\frac{\partial \varphi}{\partial t} = \left[\frac{(\vec{p} - e\vec{A})^2}{2m} + e\phi - \frac{e}{2m}(1 + a)\vec{\sigma} \cdot \vec{B} \right] \varphi \tag{4.182}$$

Now we consider that the external field is a magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. Then keeping only terms first order in e we get

$$\begin{aligned} H &= \frac{p^2}{2m} - e\frac{\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p}}{2m} - \frac{e}{2m}(1 + a)\vec{\sigma} \cdot \vec{\nabla} \times \vec{A} \\ &\equiv H_0 + H_{int} \end{aligned} \tag{4.183}$$

With this interaction Hamiltonian we calculate the transition amplitude between two electron states of momenta p and p'. We get

$$\begin{aligned} \langle p' | H_{int} | p \rangle &= -\frac{e}{2m} \int \frac{d^3x}{(2\pi)^3} \chi^\dagger e^{-i\vec{p}' \cdot \vec{x}} \left[\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p} + (1 + a)\vec{\sigma} \times \vec{\nabla} \cdot \vec{A} \right] e^{i\vec{p} \cdot x} \chi \\ &= -\frac{e}{2m} \int \frac{d^3x}{(2\pi)^3} \chi^\dagger \left[(\vec{p}' + \vec{p}) \cdot \vec{A} + i(1 + a)\sigma^i \epsilon^{ijk} q^j A^k \right] e^{-i\vec{q} \cdot x} \chi \\ &= -\frac{e}{2m} \chi^\dagger \left[(p' + p)^k + i(1 + a)\sigma^i \epsilon^{ijk} q^j \right] A^k(q) \chi \end{aligned} \tag{4.184}$$

This is the result that we want to compare with the non relativistic limit of the renormalized vertex. The amplitude is given by,

$$\begin{aligned} A &= e\overline{u}(p')(\gamma_\mu + \Lambda^R_\mu) u(p) A^\mu(q) \\ &= e\overline{u}(p') \left[\gamma_\mu (1 + F_1(q^2)) + \frac{i}{2m} \sigma_{\mu\nu} q^\nu F_2(q^2) \right] u(p) A^\mu(q) \\ &= \frac{e}{2m} \overline{u}(p') \Big\{ (p' + p)_\mu \left[1 + F_1(q^2) \right] + i\sigma_{\mu\nu} q^\nu \left[1 + F_1(q^2) + F_2(q^2) \right] \Big\} u(p) A^\mu(q) \end{aligned} \tag{4.185}$$

where we have used Gordon's identity. For an external magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ and in the limit $q^2 \to 0$ this expression reduces to

$$
\begin{aligned}
A &= \frac{e}{2m}\overline{u}(p') \left\{(p'+p)_k\left[1+F_1(0)\right] + i\sigma_{kj}q^j\left[1+F_1(0)+F_2(0)\right]\right\} u(p)A^k(q) \\
&= \frac{e}{2m}\overline{u}(p') \left[-(p'+p)^k + i\Sigma^i\epsilon^{kij}q^j\left(1+\frac{\alpha}{2\pi}\right)\right] u(p)A^k(q)
\end{aligned}
\tag{4.186}
$$

where we have used the results of Eq. (4.126),

$$
\begin{cases}
F_1(0) = 0 \\
F_2(0) = \dfrac{\alpha}{2\pi}
\end{cases}
\tag{4.187}
$$

Using the explicit form for the spinors u

$$
u(p) = \begin{pmatrix} \chi \\ \dfrac{\vec{\sigma}\cdot(\vec{p}-e\vec{A})}{2m}\,\chi \end{pmatrix}
\tag{4.188}
$$

we can write in the non relativistic limit,

$$
A = -\frac{e}{2m}\chi^\dagger \left[(p'+p)^k + i\left(1+\frac{\alpha}{2\pi}\right)\sigma^i\epsilon^{ijk}q^j\right]\chi A^k
\tag{4.189}
$$

which after comparing with Eq. (4.184) leads to

$$
a^e_{th} = \frac{\alpha}{2\pi}
\tag{4.190}
$$

This result obtained for the first time by Schwinger and experimentally confirmed, was very important in the acceptance of the renormalization program of Feynman, Dyson and Schwinger for QED.

4.4.2 Cancellation of IR divergences in Coulomb scattering

In this section we will show how the IR divergences cancel in physical processes. We will take as an example the Coulomb scattering from a fixed nucleus. This is better done if we start from first principles. Coulomb scattering corresponds to the diagram of Fig. 4.16, which gives the following matrix element for the S matrix,

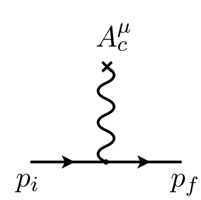


Figure 4.16: Lowest order diagram to Coulomb scattering.

$$S_{fi} = iZe^2(2\pi)\delta(E_i - E_f)\frac{1}{|\vec{q}|^2}\ \overline{u}(p_f)\gamma^0 u(p_i) \tag{4.191}$$

We will now study the radiative corrections to this result in lowest order in perturbation theory. Due to the IR divergences it is convenient to introduce a mass λ for the photon. For a classical field, as we are considering, this means a screening. If we take,

$$A_c^0(x) = Ze\frac{e^{-\lambda|\vec{x}|}}{4\pi|\vec{x}|} \tag{4.192}$$

the Fourier transform will be,

$$A_c^0(q) = Ze\frac{1}{|\vec{q}|^2 + \lambda^2} \tag{4.193}$$

that shows that the screening is equivalent to a mass for the photon. With these modifications we have,

$$S_{fi} = iZe^2(2\pi)\delta(E_f - E_i)\ \frac{i}{|\vec{q}|^2 + \lambda^2}\ \overline{u}(p_f)\gamma^0 u(p_i) \tag{4.194}$$

We are interested in calculating the corrections up to order e^3 in the amplitude. To this contribute[15] the diagrams of Fig. 4.17. Diagram 1 is of order e^2 while diagrams $2, 3, 4$

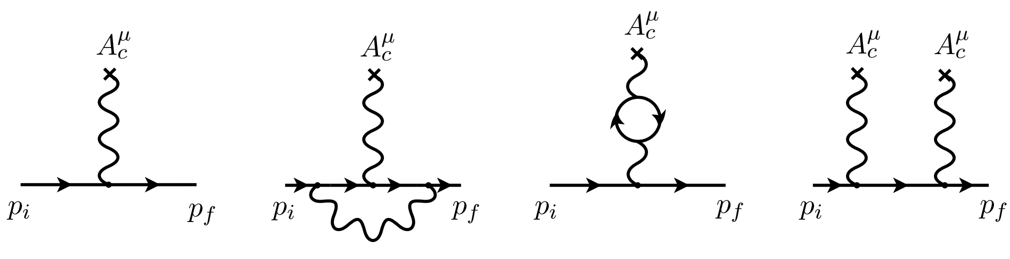


Figure 4.17: Coulomb scattering up to second order.

are of order e^4. Therefore the interference between 1 and $(2 + 3 + 4)$ is of order α^3 and should be added to the result of the bremsstrahlung in a Coulomb field. The contribution from $1 + 2 + 3$ can be easily obtained by noticing that

$$eA_c^\mu\gamma_\mu \to eA_c^\mu(\gamma_\mu + \Lambda_\mu^R + \Pi_{\mu\nu}^R G^{\nu\rho}\gamma_\rho) \tag{4.195}$$

where Λ_μ^R e $\Pi_{\mu\nu}^R$ have been calculated before. We get

$$S_{fi}^{(1+2+3)} \quad = \quad iZe^2(2\pi)\delta(E_i - E_f)\frac{1}{|\vec{q}|^2 + \lambda^2}\overline{u}(p_f)\gamma^0 \left\{1 + \frac{\alpha}{\pi}\left[-\frac{1}{2}\varphi \tanh\varphi\right.\right.$$

[15]We do not have to consider the self-energies of the external legs of the electron because they are on-shell.

$$\left(1+\ln\frac{\lambda}{m}\right)(2\varphi\coth 2\varphi-1)-2\coth 2\varphi\int_0^\varphi\beta\tanh\beta d\beta$$
$$+\left(1-\frac{\coth^2\varphi}{\beta}\right)(\varphi\coth\varphi-1)+\frac{1}{9}\Bigg]-\frac{\not{q}}{2m}\frac{\alpha}{\pi}\frac{\varphi}{\sinh 2\varphi}\Bigg\}u(p_i) \tag{4.196}$$

where

$$\frac{|\vec{q}|^2}{4m}=\sinh^2\varphi\,. \tag{4.197}$$

Finally the fourth diagram gives

$$\begin{aligned} S_{fi}^{(4)} &= (iZe)^2(e)^2\int\frac{d^4k}{(2\pi)^4}\overline{u}(p_f)\left[\frac{2\pi\delta(E_f-k^0)}{(p_f-k)^2-\lambda^2}\gamma^0\frac{i}{\not{k}-m+i\varepsilon}\gamma^0\frac{2\pi\delta(k^0-E_i)}{(k-p_i)^2-\lambda^2}\right] \\ &= -2i\frac{Z^2\alpha^2}{\pi}2\pi\delta(E_f-E_i)\;\overline{u}(p_f)\left[m(I_1-I_2)+\gamma^0E_i(I_1+I_2)\right]u(p_i) \end{aligned} \tag{4.198}$$

with

$$I_1=\int d^3k\frac{1}{[(\vec{p}_f-\vec{k})^2+\lambda^2][(\vec{p}_i-\vec{k})^2+\lambda^2][(\vec{p})^2-(\vec{k})^2+i\varepsilon]} \tag{4.199}$$

and

$$\frac{1}{2}(\vec{p}_i+\vec{p}_f)I_2\equiv\int d^3k\frac{\vec{k}}{[(\vec{p}_f-\vec{k})^2+\lambda^2][(\vec{p}_i-\vec{k})^2+\lambda^2][(\vec{p})^2-(\vec{k})^2+i\varepsilon]}\,. \tag{4.200}$$

In the limit $\lambda\to 0$ it can be shown that

$$I_1 = \frac{\pi^2}{2ip^3\sin^2\theta/2}\ln\left(\frac{2p\sin(\theta/2)}{\lambda}\right) \tag{4.201}$$

$$I_2 = \frac{\pi^2}{2p^3\cos^2\theta/2}\left\{\frac{\pi}{2}\left[1-\frac{1}{\sin\theta/2}\right]-i\left[\frac{1}{\sin^2\theta/2}\ln\left(\frac{2p\sin\theta/2}{\lambda}\right)+\ln\frac{\lambda}{2p}\right]\right\} \tag{4.202}$$

With these expressions we get for the cross section

$$\frac{d\sigma}{d\Omega}=\frac{Z^2\alpha^2}{|\vec{q}|^2}\frac{1}{2}\sum_{pol}|\overline{u}(p_f)\Gamma u(p_i)|^2 \tag{4.203}$$

where

$$\Gamma = \gamma^0(1+A) + \gamma^0 \frac{\not{q}}{2m} B + C \tag{4.204}$$

and

$$\begin{aligned} A &= \frac{\alpha}{\pi}\left[\left(1+\ln\frac{\lambda}{m}\right)(2\varphi\coth 2\varphi - 1) - 2\coth 2\varphi \int_0^\varphi d\beta\beta\tanh\beta - \frac{\varphi}{2}\tanh\varphi \right. \\ &\left. + \left(1 - \frac{1}{3}\coth^2\varphi\right)(\varphi\coth\varphi - 1) + \frac{1}{9}\right] - \frac{Z\alpha}{2\pi^2}|\vec{q}|^2 E(I_1 + I_2) \end{aligned} \tag{4.205}$$

$$B = -\frac{\alpha}{\pi}\frac{\varphi}{\sinh 2\varphi} \tag{4.206}$$

$$C = -\frac{Z\alpha}{2\pi^2} m|\vec{q}|^2 (I_1 - I_2) \tag{4.207}$$

Therefore

$$\begin{aligned} \frac{1}{4}\sum_{pol} |\overline{u}(p_f) p u(p_i)|^2 &= \frac{1}{4}\mathrm{Tr}[\Gamma(\not{p}_i + m)\overline{\Gamma}(\not{p}_f + m)] \\ &= 2E^2(1 - \beta^2\sin^2\theta/2) + 2E^2 2B\beta^2\sin^2\frac{\theta}{2} \\ &+ 2E^2 2ReA\left(1 - \beta^2\sin^2\frac{\theta}{2}\right) + 2ReC(2mE) + O(\alpha^2) \end{aligned} \tag{4.208}$$

Notice that A, B e C are of order α. Therefore the final result is, up to order α^3:

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}} &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left\{1 + \frac{2\alpha}{\pi}\left[\left(1+\ln\frac{\lambda}{m}\right)(2\varphi\coth\varphi - 1) - \frac{\varphi}{2}\tanh\varphi \right.\right. \\ & -2\coth 2\varphi\int_0^\varphi d\beta\beta\tanh\beta + \left(-\frac{\coth^2\varphi}{3}\right)(\varphi\coth\varphi - 1) + \frac{1}{9} \\ & \left.\left. -\frac{\varphi}{\sinh 2\varphi}\frac{B^2\sin^2\theta/2}{1-\beta^2\sin^2\theta/2}\right] + Z\alpha\pi\frac{\beta\sin\frac{\theta}{2}[1-\sin\theta/2]}{1-\beta^2\sin^2\theta/2}\right\} \end{aligned} \tag{4.209}$$

As we had said before the result is IR divergent in the limit $\lambda \to 0$. This divergence is not physical and can be removed in the following way. The detectors have an energy threshold, below which they can not detect. Therefore in the limit $\omega \to 0$ bremsstrahlung in a Coulomb field and Coulomb scattering can not be distinguished. This means that we have to add both results. If we consider an energy interval ΔE with $\lambda \leq \Delta E \leq E$ we get

$$\left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \int_{\omega\leq\Delta E} \frac{d^3k}{2\omega(2\pi)^3} e^2 \left[\frac{2p_i\cdot p_f}{k_i\cdot p_i k\cdot p_f} - \frac{m^2}{(k\cdot p\cdot)^2} - \frac{m^2}{(k\cdot p_f)^2}\right] \tag{4.210}$$

Giving a mass to the photon (that is $\omega = (|\vec{k}|^2 + \lambda^2)^{1/2}$) the integral can be done with the result,

$$\left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \frac{2\alpha}{\pi} \left\{(2\varphi \coth 2\varphi - 1) \ln \frac{2\Delta E}{\lambda} + \frac{1}{2\beta} \ln \frac{1+\beta}{1-\beta} \right.$$
$$\left. -\frac{1}{2} \cosh 2\varphi \frac{1-\beta^2}{\beta \sin\theta/2} \int_{\cos\theta/2}^{1} d\xi \frac{1}{(1-\beta^2\xi^2)[\xi - \cos^2\theta/2]^{1/2}} \ln \frac{1+\beta\xi}{1-\beta\xi} \right\} \tag{4.211}$$

The inclusive cross section can now be written as

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\Delta E) &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}} + \left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR} \\ &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} (1 - \delta_R + \delta_B) \end{aligned} \tag{4.212}$$

where δ_R and δ_B are complicated expressions that depend on the resolution of the detector ΔE but do not depend on λ that can be finally put to zero. One can show that in QED all the IR divergences can be treated in a similar way. One should note that the final effect of the bremsstrahlung is finite and can be important.

Chapter 5

Functional Methods

5.1 Introduction

In this chapter, called *Functional Methods*, we are going to present the path integral quantization. For systems that are not described by gauge theories this method may seem unnecessary, as the canonical quantization works without problems. However, for non-abelian gauge theories, as we shall see in the next chapter, this is the only known method. Besides this fundamental point, the quantization done using functional methods and the path integral formalism is very elegant and allows us to obtain the results much faster, even for the cases where the canonical quantization works. Examples of this are the Ward-Takahashi identities and the Dyson-Schwinger equations, as we will discuss at the end of the chapter.

We are going to assume that the reader is familiar with the path-integral quantization for systems of N particles in non-relativistic quantum mechanics. Therefore only a brief summary of the results will be given. A more detailed account is given in Appendix A. The step from the quantization of a system with N particles to the quantization of a field theory will be done heuristically. A more rigorous treatment will be given in Appendix B.

Before we start, let us clarify some questions related with the notation. Let us assume that we have real scalar field $\phi^a(x)$ where $a = 1, ...N$. In the following we will encounter expressions of the type,

$$I_1 = \int d^4x\, \phi^a(x)\phi^a(x) \tag{5.1}$$

or

$$I_2 = \int d^4x d^4y\, \phi^a(x) M^{ab}(x,y)\phi^b(y) \tag{5.2}$$

where $M^{ab}(x,y)$ is normally a differential operator. According to the rules for functional derivation, we have,

$$\frac{\delta I_1}{\delta \phi^b(y)} = 2\phi^b(y) \tag{5.3}$$

where we used the result

$$\frac{\delta \phi^a(x)}{\delta \phi^b(y)} = \delta^{ab}\delta^4(x-y) \tag{5.4}$$

If we keep all the indices in the previous expressions (and in some much more complicated that we will encounter soon), we will get a very complicated situation with respect to the notation. Therefore it will be useful to make use of a more compact notation. To this end we identify,

$$\phi_i \Longleftrightarrow \phi^a(x) \tag{5.5}$$

that is, the index i will represent *both* the discrete index a as well as the continuous x,

$$i \Longleftrightarrow \{a, x\} \tag{5.6}$$

In the case that the fields have further indices we will assume that i will always represent them collectively. We also use the Einstein convention meaning a sum for discrete indices and an integration for continuous indices. With these conventions Eq. (5.1) and Eq. (5.4) can be written as

$$\begin{aligned} &I_1 = \phi_i \phi_i \qquad I_2 = \phi_i M_{ij} \phi_j \\ &\frac{\delta I_1}{\delta \phi_j} = 2\phi_j \qquad \frac{\delta \phi_i}{\delta \phi_j} = \delta_{ij} \end{aligned} \tag{5.7}$$

In the following we will use these conventions, returning to the more usual notation when convenient or in case of a possible confusion.

5.2 Generating functional for Green's functions

5.2.1 Green's functions

The basic objects in Quantum Field Theory are the so-called Green functions. To avoid unnecessary complications we are going to use mostly the example of the scalar field. The generalizations are however quite straightforward. The Green function of order n is given by

$$G^{(n)}(x_1, \ldots, x_n) \equiv \langle 0 | T\phi(x_1) \cdots \phi(x_n) | 0 \rangle \ . \tag{5.8}$$

The Green functions defined in the previous equation are, sometimes called *complete* to distinguish from the Green functions *connected*, *truncated* or *one particle irreducible* that we now are going to define.

5.2.2 Connected Green's functions

We call *connected* Green functions those that in which none of the external lines goes through the diagram without interacting. As an example, in Fig. 5.1 we represent a disconnected contribution to $G^4(x_1, \ldots, x_4)$, while in Fig. 5.2 we have a connected contribution to the same Green function.

Therefore the connected Green functions are obtained summing over all the connected diagrams. The disconnected Green functions, corresponding to disconnected diagrams, can be obtained from connected Green functions of lower order, therefore the relevant quantities are the connected Green functions $G_c^n(x_1, \ldots, x_n)$. It is clear that we have

$$G_c^n(x_1, \ldots, x_n) = G^n(x_1, \ldots, x_n) - \text{disconnected terms}, \tag{5.9}$$

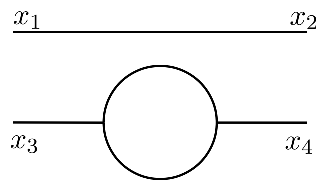


Figure 5.1: Disconnected contribution to $G^4(x_1,\ldots,x_4)$.

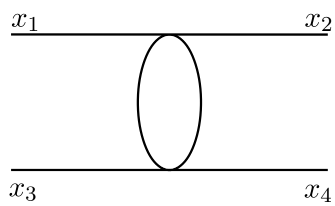


Figure 5.2: Connected contribution to $G^4(x_1,\ldots,x_4)$.

and

$$G_c^2(x_1,x_2) = G^2(x_1,x_2) \ . \tag{5.10}$$

Conventionally we represent $G_c^n(x_1,\ldots,x_n)$ by the diagram of Fig. 5.3,

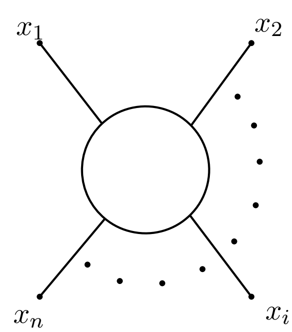


Figure 5.3: Graphical representation for $G_c^n(x_1,\ldots,x_n)$.

Sometimes it is important to consider the Green functions in momentum space. We define then $G_c^n(p_1,\ldots,p_n)$ through the relation (Fourier Transform)

$$\begin{aligned}
&(2\pi)^4\delta^4(p_1+p_2+\cdots p_n)\ G_c^n(p_1,\ldots,p_n)\\
&\equiv \int d^4x_1\cdots d^4x_n e^{-i(p_1\cdot x_1+\cdots+p_n\cdot x_n)}\ G_c^n(x_1,\ldots,x_n)\ ,
\end{aligned} \tag{5.11}$$

where all momenta, $p_1,\ldots,p_n$, are incoming, as shown in Fig. 5.4. Notice that in the definition we have factored out the delta function that ensures the conservation of 4-momentum. With these conventions $G^2(p,-p)\equiv G^2(p)$ is the full propagator represented in Fig. 5.5.

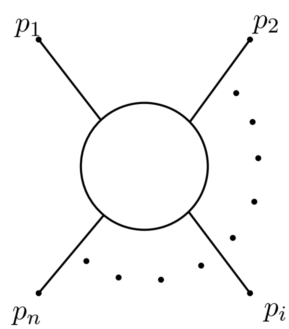


Figure 5.4: Graphical representation for $G_c^n(p_1, \ldots, p_n)$

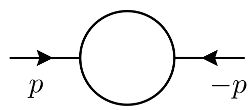


Figure 5.5: Full propagator $G^2(p)$.

5.2.3 Truncated Green's functions

For $n > 2$ one defines the *truncated* Green functions through the relation,

$$G^n_{\rm trunc}(p_1, \ldots, p_n) = \prod_{k=1}^n \left[G^2(p_k)\right]^{-1} G^n_c(p_1, \ldots, p_n) \tag{5.12}$$

that is, we multiply each external line by the inverse of the full propagator corresponding to that line. These are the functions that play a fundamental role in the Theory, as these the ones that are related to the S matrix elements. In fact the LSZ reduction formula for scalars fields is

$$\begin{aligned}\langle p_1, \ldots, p_n \text{ out}|q_1, \ldots, q_\ell \text{ in}\rangle &= \langle p_1, \ldots, p_n \text{ in}|S|q_1, \ldots, q_\ell \text{ in}\rangle = \text{disconnected terms}\\ &+ \left(iZ^{-1/2}\right)^{n+\ell} \int d^4y_1 \cdots d^4x_\ell \; \exp\left[i\left(\sum_{k=1}^n p_k \cdot y_k - \sum_{k=1}^\ell q_k \cdot x_k\right)\right]\\ &\times(\Box_{y_1} + m^2)\cdots(\Box_{x_\ell} + m^2)\,\langle 0|T\phi(y_1)\cdots\phi(x_\ell)|0\rangle_c \end{aligned}\tag{5.13}$$

which gives

$$\begin{aligned}\langle p_1, \ldots, p_n \text{ out}|q_1, \ldots, q_\ell \text{ in}\rangle &= \langle p_1, \ldots, p_n \text{ in}|S|q_1, \ldots, q_\ell \text{ in}\rangle = \text{disconnected terms}\\ &+Z^{-(n+\ell)/2}\,(2\pi)^4\delta\left(\sum p_i - \sum q_j\right)\; G^{n+\ell}_{\rm trunc}(-p_1, \ldots, -p_n, q_1, \ldots, q_\ell)\end{aligned}\tag{5.14}$$

5.2.4 Irreducible diagrams

We saw in 5.14 that the S elements related with the cross sections are expressed in terms of truncated diagrams. Among the truncated Green functions there is an important subset.

These are the Green functions that correspond to the subset of *one particle irreducible* or *proper* that are the truncated diagrams that remain connected when one cuts any arbitrary internal line. For instance the diagram of Fig. 5.6[1] is truncated but it is not proper, while the diagram of Fig. 5.7 is proper (in $\lambda\phi^3$).

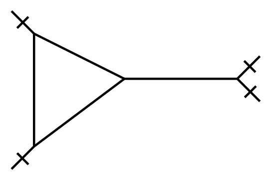


Figure 5.6: Example of a truncated diagram that is not proper.

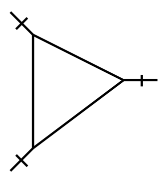


Figure 5.7: Example of a proper diagram.

The reason why the non-irreducible truncated diagrams are not important is because they can always be expressed in terms of irreducible diagrams of lower order (remember the self energy series that we saw in the last lecture). It is convenient to introduce a notation for the irreducible Green Functions (sum of all the irreducible diagrams for a given number of exterior legs) where the factor i was introduced by convenience. In Fig. 5.8 the external legs are drawn to make the figure more clear. They are in fact truncated. It is also convenient to define a notation for the truncated diagrams of order n. This is shown in Fig. 5.9 or, in another way, in Fig. 5.10. We can define similar diagrams in momentum space.

5.3 Generating functionals for Green's functions

The *Generating Functional* (FG) for the Green functions plays a very important role in Quantum Field Theory. In fact starting with it, by taking appropriate functional derivatives, one can obtain all the Green functions. Therefore they can treat simultaneously an infinite number of Green functions. The FG of the full Green functions is given by,

$$Z(J) \equiv \left\langle 0|Te^{iJ_i\phi_i}|0\right\rangle \tag{5.15}$$

[1] The bars indicate the the external lines are cuted.

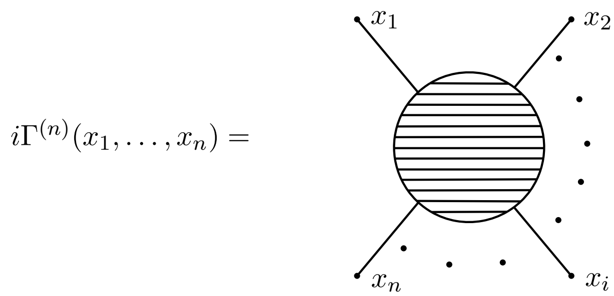


Figure 5.8: Irreducible Green functions.

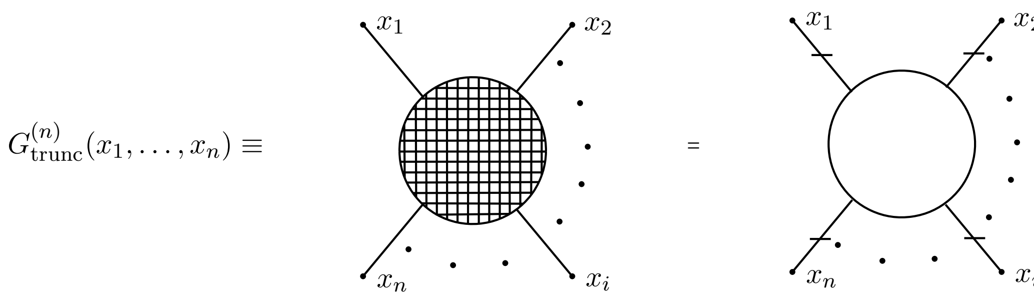


Figure 5.9: Graphical representation of truncated Green functions.

where we are using the compact notation explained before.

$$J_i\phi_i \equiv \int d^4x\ J(x)\phi(x)\ . \tag{5.16}$$

$Z(J)$ generates all the Green functions, because if we expand the exponential in 5.15 we get

$$\begin{aligned} Z(J) &= \textstyle\sum_{n=0}^{\infty} \frac{i^n}{n!} J_{i_1}\cdots J_{i_n}\,\langle 0|T\phi_{i_1}\cdots\phi_{i_n}|0\rangle \\ &= \textstyle\sum_{n=0}^{\infty} \frac{i^n}{n!} J_{i_1}\cdots J_{i_n} G^n_{i_1\cdots i_n} \end{aligned} \tag{5.17}$$

The Green functions are then obtained taking derivatives

$$G^n_{i_1\cdots i_n} = \left.\frac{\delta^n Z}{i\delta J_{i_1}\cdots i\delta J_{i_n}}\right|_{J_i=0} \tag{5.18}$$

The generating functional of the connected Green functions is defined by the relation,

$$Z(J) = e^{iW(J)} \tag{5.19}$$

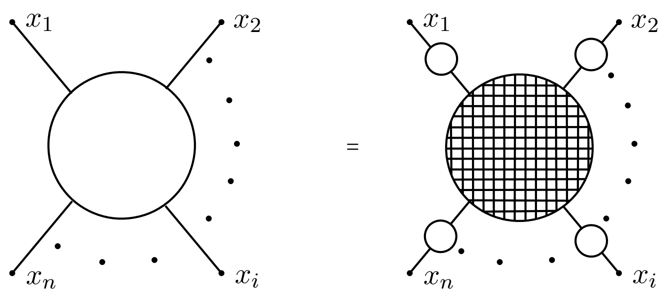


Figure 5.10: Another representation of truncated Green functions.

or

$$W(J) = -i \ln Z(J) \ . \tag{5.20}$$

The connected Green functions are then obtained by taking functional derivatives

$$G^n_{c\ i_1 \cdots i_n} = i \left. \frac{\delta^n W}{i\delta J_{i_1} \cdots i\delta J_{i_n}} \right|_{J_i=0} \tag{5.21}$$

Before we actually prove this statement, let us define the last important generating functional, the one that generates the irreducible Green functions. This is defined through the Legendre transformation of $W(J)$, that is

$$\Gamma(\phi) \equiv W(J) - J_i \phi_i \tag{5.22}$$

where

$$\begin{cases} \phi_i & \equiv \dfrac{\delta W(J)}{\delta J_i} \\ J_i & = -\dfrac{\delta \Gamma(\phi)}{\delta \phi_i} \end{cases} \tag{5.23}$$

The irreducible Green functions are then obtained through

$$\Gamma^n_{i_1 \cdots i_n} = \left. \frac{\delta^n \Gamma(\phi)}{\delta \phi_{i_1} \cdots \delta \phi_{i_n}} \right|_{\phi=0} . \tag{5.24}$$

Having given the definitions we have now to show that $W(J)$ and $\Gamma(\phi)$ do generate the connected and irreducible Green functions, respectively. Let us start with $W(J)$. The proof is done calculating $G^n_{c\ i_1 \cdots i_n}$. We are going only to do only the $n = 2$, $n = 3$ and $n = 4$ cases. The generalizations are immediate.

$n = 2$

$$G^2_{c\ i_1 i_2} = i \left. \frac{\delta^2 W}{i\delta J_{i_1} i\delta J_{i_2}} \right|_{J_i=0} = \left. \frac{\delta^2 \ln Z}{i\delta J_{i_1} i\delta J_{i_2}} \right|_{J_i=0} = \left. \frac{\delta}{i\delta J_{i_1}} \frac{1}{Z} \frac{\delta Z}{i\delta J_{i_2}} \right|_{J_i=0}$$

$$
\begin{aligned}
&= \frac{1}{Z}\left.\frac{\delta^2 Z}{i\delta J_{i_1} i\delta J_{i_2}}\right|_{J_i=0} - \frac{1}{Z}\frac{\delta Z}{i\delta J_{i_1}}\frac{1}{Z}\left.\frac{\delta Z}{i\delta J_{i_2}}\right|_{J_i=0} \\
&= \left.\frac{\delta^2 Z}{i\delta J_{i_1} i\delta J_{i_2}}\right|_{J_i=0}
\end{aligned}
\tag{5.25}
$$

or

$$G^2_{c\ i_1 i_2} = G^2_{i_1 i_2} \tag{5.26}$$

To obtain this result we have used,

$$
\begin{aligned}
Z(0) &= 1 \qquad \text{The vacuum is normalized} \\
\frac{\delta Z}{i\delta J_i} &= \langle 0|T\phi_i|0\rangle = 0 \qquad \text{There is no symmetry breaking}
\end{aligned}
\tag{5.27}
$$

$n = 3$

$$
\begin{aligned}
G^3_{c\ i_1 i_2 i_3} = & \left[\frac{1}{Z}\frac{\delta^3 Z}{i\delta J_{i_1} i\delta J_{i_2} i\delta J_{i_3}} - \frac{1}{Z}\frac{\delta^2 Z}{i\delta J_{i_1} i\delta J_{i_2}}\frac{1}{Z}\frac{\delta Z}{i\delta J_{i_3}} \right. \\
& - \frac{1}{Z}\frac{\delta^2 Z}{i\delta J_{i_2} i\delta J_{i_3}}\frac{1}{Z}\frac{\delta Z}{i\delta J_{i_1}} - \frac{1}{Z}\frac{\delta^2 Z}{i\delta J_{i_1} i\delta J_{i_3}}\frac{1}{Z}\frac{\delta Z}{i\delta J_{i_2}} \\
& \left. +2\,\frac{1}{Z}\frac{\delta Z}{i\delta J_{i_1}}\frac{1}{Z}\frac{\delta Z}{i\delta J_{i_2}}\frac{1}{Z}\frac{\delta Z}{i\delta J_{i_3}} \right]_{|J_i=0}
\end{aligned}
\tag{5.28}
$$

therefore

$$G^3_{i_1 i_2 i_3} = G^3_{c\ i_1 i_2 i_3} \tag{5.29}$$

The case $n = 4$ is left as an exercise (see Problem x.x). The extension to $n > 4$ is straightforward. We have therefore showed that $W(J)$ generates all the connected Green functions. Let us now show that $\Gamma(\phi)$ is the generating functional for the irreducible Green functions. For that we need two auxiliary results. The first one is based in the relation

$$\frac{\delta J_i}{\delta J_k} = \delta_{ik} \tag{5.30}$$

This relation it is obvious. However one can obtain starting with it another important relation. In fact

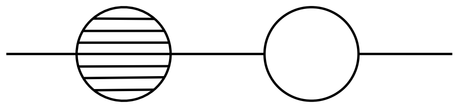
$$\frac{\delta J_i}{\delta J_k} = \frac{\delta J_i}{\delta \phi_\ell}\frac{\delta \phi_\ell}{\delta J_k} = -\frac{\delta^2 \Gamma}{\delta\phi_i \delta\phi_\ell}\frac{\delta^2 W}{\delta J_\ell \delta J_k} = -i\Gamma_{i\ell} G_{\ell k} \tag{5.31}$$

or

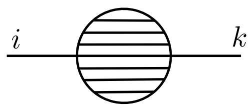
$$\Gamma_{i\ell} G_{\ell k} = i\delta_{ik} \tag{5.32}$$

This fundamental relation expresses the fact that Γ^2 is the inverse of the propagator (except for the i that comes from conventions). It is useful to write it in a diagrammatic form shown in 5.11. Notice that

$$i\Gamma^{(2)}_{ik} \equiv \tag{5.33}$$



$= \quad -1$

Figur

tation of Eq.5.32.

which explains the disappearance of the i. A second result is related to the following functional derivative,

$$\frac{\delta}{i\delta J_i} \tag{5.34}$$

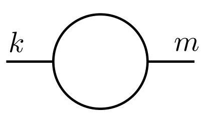
We want to derive in order to J_i quantities that depend on J_i indirectly through ϕ_k. We get

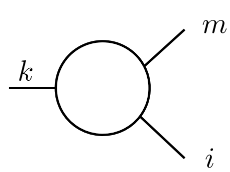
$$\frac{\delta}{i\delta J_i} = \frac{\delta\phi_k}{i\delta J_i}\frac{\delta}{\delta\phi_k} = \frac{\delta^2 W}{\delta J_k i\delta J_i}\frac{\delta}{\delta\phi_k} = G^{(2)}_{ik}\frac{\delta}{\delta\phi_k} \tag{5.35}$$

and therefore

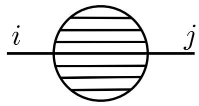
$$\frac{\delta}{i\delta J_i} = G_{ik}\frac{\delta}{\delta\phi_k} \tag{5.36}$$

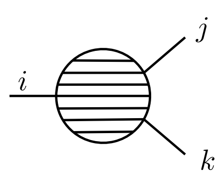
Equations 5.32 and 5.36 allow us to obtain all the relation between irreducible and connected Green functions. This analysis it is easier in a diagrammatic form, if we note the following identities,

$\frac{\delta}{i\delta J_i}$ 

$=$ 

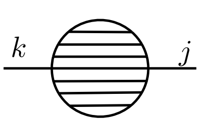
(5.37)

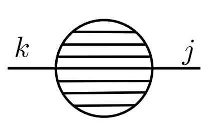
$\frac{\delta}{\delta\phi_k}$ 

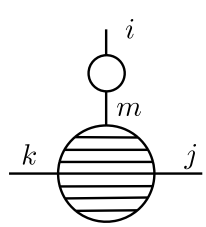
$=$ 

(5.38)

and

$\frac{\delta}{i\delta J_i}$ 

$= G_{im}\frac{\delta}{\delta\phi_m}$ 

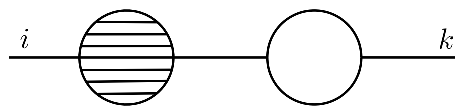
$=$ 

(5.39)

where we have used 5.36 to establish 5.39. In all these relations it is understood that at the end (after taking the functional derivatives) we do $J = 0$ e $\phi = 0$ at the right places We will use now these relations to relate the connected and irreducible Green functions for $n = 3$ and $n = 4$.

$n = 3$

The starting point is the equation 5.32. We apply $\frac{\delta}{i\delta J_\ell}$ to 5.32 and we get

$$\frac{\delta}{i\delta J_\ell}$$ 

$= 0$ (5.40)

Using equations 5.37 and 5.39 we get the diagram of Figure 5.12.

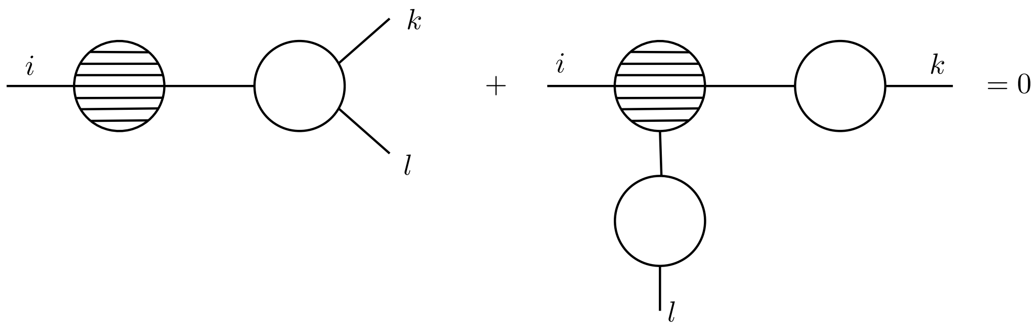
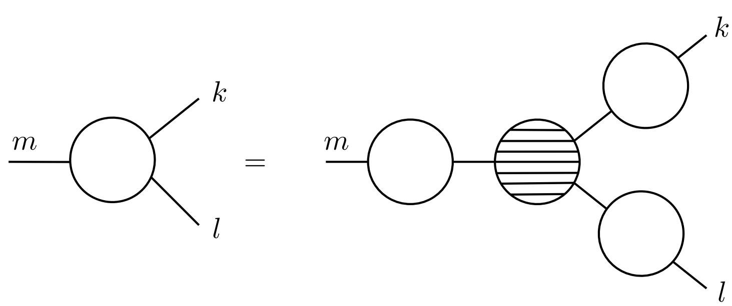


Figure 5.12: Graphical result of Eq.5.40.

Multiplying on the left by $G^{(2)}_{mi}$ and using 5.32 we get

$i\Gamma^{(3)}_{mkl} \equiv$ 

(5.41)

This shows that $\Gamma^{(3)}_{mkl}$ is in fact the irreducible Green function with 3 external legs, as for 3 external legs the irreducible and truncated Green functions coincide. To show that we have really irreducible functions and not only truncated ones, one has to go to $n = 4$ as the difference starts there.

$n = 4$

We start from equation 5.41 and take the functional derivative with respect to $\frac{\delta}{i\delta J_n}$. Using the previous methods we get the equation represented in Figure 5.13. If we use 5.41 to

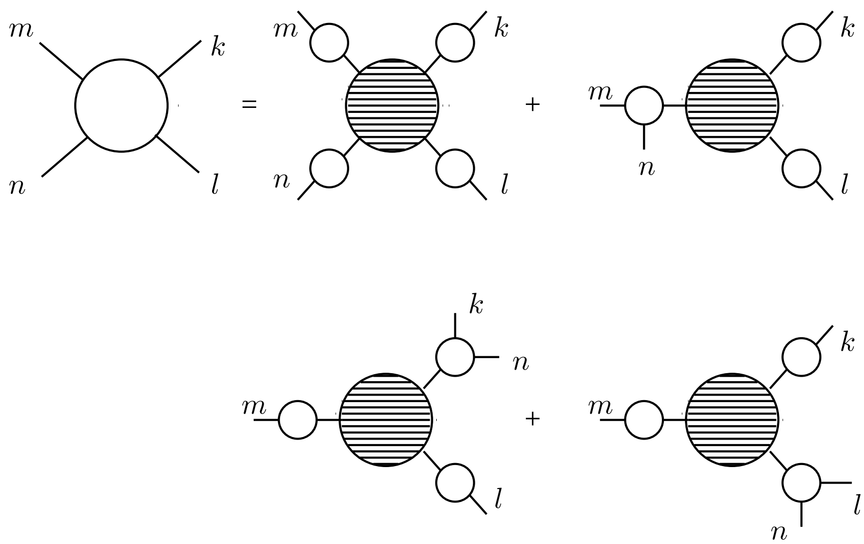


Figure 5.13: Graphical representation of G_{mkln} in terms of Γ_{mkln}.

express $G^{(3)}_{kml}$ in terms of $\Gamma^{(3)}_{kml}$ we obtain the diagrammatic equation of Figure 5.14 that ends the proof.

$n > 4$

It is now trivial to continue the process for $n > 4$. For a given n we start from the relation for $n - 1$ and we apply Eqs. 5.37 and 5.39. These results show that the irreducible Green functions are the important ones, all the other can be obtained from them. This is an important result as it reduces enormously the number of Feynman diagrams to be evaluated.

5.4 Feynman rules

The formalism of functional generators allows us to obtain the Feynman rules of any theory with all the correct conventions. We have already shown how to get the Feynman rules in section 3.7. There we used the result that in lowest order (tree level) we have

$$\Gamma_{\text{tree}}(\phi) = \int d^4x \mathcal{L}[\phi] \equiv \Gamma_0(\phi) \tag{5.42}$$

Here we are going just to show this result. For the interaction terms ($n > 2$) this is clear. For instance for $n = 3$ we have

$$i\Gamma^{(3)} = G^{(3)}_{\text{tree}} \tag{5.43}$$

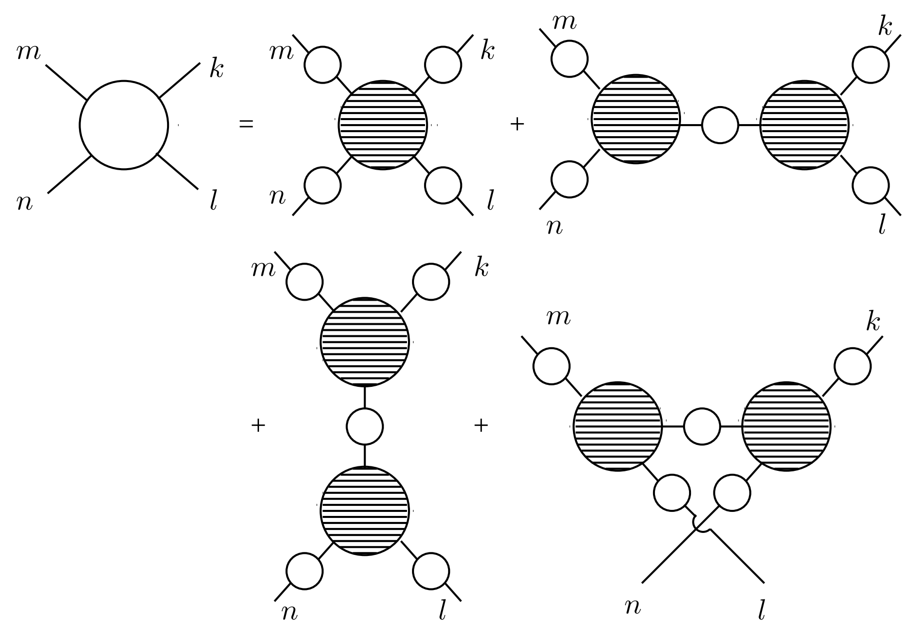


Figure 5.14: Graphical representation of G_{mkln} in terms of Γ_{mkln}.

while for $n = 4$ we get

$$i\Gamma^{(4)} = G^{(4)} - \text{ irreducible parts} \tag{5.44}$$

and it is obvious that $i\Gamma^{(4)}_{\text{tree}}$ generates the vertices.

The i factor is in agreement with the usual conventions for the Feynman rules as it comes from the term $\exp\left(i\int d^4x\mathcal{L}_{int}\right)$ in the calculation of the Green functions. There is however an important detail. In our definiton of $\Gamma^{(n)}(p_1, p_2, \ldots)$ we have factored out $(2\pi)^4\delta^4(p_1 + p_2 + \cdots)$. Before doing the inverse Fourier transform one has to put it back. For instance for the quadratic terms we have, Eq. 5.32,

$$\Gamma^{(2)}_{\text{tree}}(p) = p^2 - m^2 \tag{5.45}$$

therefore, doing the inverse Fourier transform,

$$\Gamma^{(2)}_{\text{tree}}(x, y) = \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}e^{i(p_1\cdot x+p_2\cdot y)}\ (p_1^2 - m^2) \times (2\pi)^4\delta^4(p_1 + p_2) \tag{5.46}$$

and

$$\begin{aligned}
&\frac{1}{2}\int d^4xd^4y\,\phi(x)\Gamma^{(2)}_{\text{tree}}(x, y)\phi(y) = \\
&= \frac{1}{2}\int d^4xd^4y\,\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\,e^{i(p_1\cdot x+p_2\cdot y)}(2\pi)^4\delta(p_1 + p_2)(p_1^2 - m^2)\phi(x)\phi(y) \\
&= \frac{1}{2}\int d^4x\,\phi(x)(-\Box - m^2)\phi(x) = \frac{1}{2}\int d^4x\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right)
\end{aligned} \tag{5.47}$$

which shows that Γ_{tree} is in fact the action. In getting to Eq. (5.47) we have done an integration by parts and discarded, usual, the boundary term. We refer the reader to Section 3.7 for the actual recipes on how to determine the Feynman rules of any theory.

5.5 Path integral for generating functionals

5.5.1 Quantum mechanics of n degrees of freedom

Let us start reviewing the results for the path integral in systems with 1 degree of freedom. In appendix A a more detailed explanation is given for the results that we are going to use. The fundamental result is the following transition amplitude,

$$\langle q';t'|q;t\rangle = N\int \mathcal{D}(q)e^{i\int_t^{t'} dtL(q,\dot{q})} = N\int \mathcal{D}(q)e^{iS} \tag{5.48}$$

where N is a normalization factor and $\mathcal{D}(q)$ is a symbolic form for representing the measure of integration. This is in fact a complicated limit (see appendix A). Another important result concerns the time ordered matrix elements of operators. Let

$$O(t_1,\ldots,t_n) = T[O_1^H(t_1)O_2^H(t_2)...O_n^H(t_n)] \tag{5.49}$$

such that

$$t' \geq (t_1,t_2,\ldots,t_n) \geq t \tag{5.50}$$

Then

$$\langle q';t'|\, O(t_1,\ldots,t_n)\,|q;t\rangle = N\int \mathcal{D}(q)O_1(q(t_1))\cdots O_n(q(t_n))e^{iS} \tag{5.51}$$

where we have assumed that the operators O_i are diagonal in the coordinate space. For the generalization to Quantum Field Theory the important objects are not the transition amplitudes but the generating functionals. Consider, for instance, the Green function

$$G(t_1,t_2) \equiv \langle 0|\, T(Q^H(t_1)Q^H(t_2))\,|0\rangle \tag{5.52}$$

where $|0\rangle$ is the ground state and $Q^H(t)$ is the coordinate operator in the Heisenberg representation. To write Eq. (5.52) using the path integral, we introduce a complete set of states and we write

$$\begin{aligned} G(t_1,t_2) &= \int dq\; dq'\,\langle 0|q';t'\rangle\,\langle q';t'|\,T(Q^H(t_1)Q^H(t_2))\,|q;t\rangle\,\langle q;t|0\rangle \\ &= \int dqdq'\phi_0(q',t')\phi_0^*(q,t)\int \mathcal{D}(q)q(t_1)q(t_2)e^{i\int_t^{t'} Ld\tau} \end{aligned} \tag{5.53}$$

where

$$\phi_0(q,t) = \langle 0|q;t\rangle = \phi_0(q)e^{-iE_0t} \tag{5.54}$$

The appearance in this expression of the wave functions of the fundamental state makes the expression not very useful. We can remove them in the following way. Consider the matrix element

$$\begin{aligned} &\langle q';t'|\, O(t_1,t_2)\,|q;t\rangle \\ &= \int dQ\; dQ'\,\langle q';t'|Q';T'\rangle\,\langle Q';T'|\,O(t_1,t_2)\,|Q;T\rangle\,\langle Q;T|q;t\rangle \end{aligned} \tag{5.55}$$

where

$$O(t_1,t_2) = T(Q^H(t_1)Q^H(t_2))$$

$$t' \geq T' \geq (t_1, t_2) \geq T \geq t \tag{5.56}$$

Let $|n\rangle$ be the eigenstates of energy E_n with wave function $\phi_n(q)$ that is

$$\begin{aligned} H\,|n\rangle &= E_n\,|n\rangle \\ \langle q|n\rangle &= \phi_n^*(q) \end{aligned} \tag{5.57}$$

Then

$$\begin{aligned} \langle q';t'|Q';T'\rangle &= \langle q'|\, e^{-iH(t'-T')}\,|Q'\rangle \\ &= \sum_n \langle q'|n\rangle \langle n|\, e^{-iH(t'-T')}\,|Q'\rangle \\ &= \sum_n \phi_n^*(q')\phi_n(Q')e^{-iE_n(t'-T')} \end{aligned} \tag{5.58}$$

We consider now the limit $t' \to -i\infty$. We obtain then

$$\lim_{t' \to -i\infty} \langle q';t'|Q';T'\rangle = \phi_0^*(q')\phi_0(Q')e^{-E_0|t'|}e^{iE_0T'} \tag{5.59}$$

In a similar way

$$\lim_{t \to i\infty} \langle Q;T|q;t\rangle = \phi_0(q)\phi_0^*(Q)e^{-E_0|t|}e^{-iE_0T} \tag{5.60}$$

Applying these limits to Eq. (5.55) we get

$$\begin{aligned} &\lim_{t' \to -i\infty} \lim_{t \to i\infty} \langle q';t'|\, O(t_1,t_2)\,|q;t\rangle \\ &\quad = \int dQdQ'\phi_0^*(q')\phi_0(Q')e^{-E_0|t'|}e^{iE_0T'} \\ &\qquad\quad \langle Q';T'|\, O(t_1,t_2)\,|Q;T\rangle\, \phi_0(q)\phi_0^*(Q)e^{-E_0|t|}e^{-iE_0T} \\ &\quad = \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|} \\ &\qquad \int dQdQ'\phi_0(Q',T')\phi_0^*(Q,T)\,\langle Q';T'|\, O(t_1,t_2)\,|Q;T\rangle \end{aligned} \tag{5.61}$$

Using the definition of Green function in Eq.5.53 we obtain the important result

$$\lim_{t' \to -i\infty} \lim_{t \to i\infty} \langle q';t'|\, O(t_1,t_2)\,|q;t\rangle = \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|}G(t_1,t_2) \tag{5.62}$$

On the other hand

$$\lim_{t' \to -i\infty} \lim_{t \to i\infty} \langle q';t'|q;t\rangle = \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{E_0|t|} \tag{5.63}$$

Therefore we can finally write

$$G(t_1,t_2) = \lim_{t' \to -i\infty} \lim_{t \to i\infty} \left[\frac{\langle q';t'|\, T(Q^H(t_1)Q^H(t_2))\,|q;t\rangle}{\langle q';t'|q;t\rangle} \right] \tag{5.64}$$

Using 5.51 we can finally write $G(t_1,t_2)$ in terms of a path integral, as

$$G(t_1,t_2) = \lim_{t'\to -i\infty} \lim_{t\to i\infty} \frac{1}{\langle q';t'|q;t\rangle} \int \mathcal{D}(q) q(t_1) q(t_2) e^{i\int_t^{t'} L d\tau} \tag{5.65}$$

This result is easily generalized to Green functions with *n-points*

$$\begin{aligned} G(t_1,\ldots,t_n) &= \langle 0| T(q(t_1)\cdots q(t_n)) |0\rangle \\ &= \lim_{t'\to -i\infty} \lim_{t\to i\infty} \frac{1}{\langle q';t'|q;t\rangle} \int \mathcal{D}(q) q(t_1)\cdots q(t_n) e^{i\int_t^{t'} L d\tau} \end{aligned} \tag{5.66}$$

We can now see that all the Green functions can be obtained from the generating functional

$$Z[J] = \lim_{t'\to -i\infty} \lim_{t\to i\infty} \frac{1}{\langle q';t'|q;t\rangle} \int \mathcal{D}(q) e^{i\int_t^{t'} [L(q,\dot{q})+Jq]d\tau} \tag{5.67}$$

by functional derivation

$$G(t_1,\ldots,t_n) = \left.\frac{\delta^n Z[J]}{i\delta J(t_1)\cdots i\delta J(t_n)}\right|_{J=0} \tag{5.68}$$

The expression 5.67 for the generating functional shows that its the transition amplitude between the ground state at time t and the ground state at time t', in the presence of an exterior source and with normalization such that $Z[J=0]=1$

$$Z[J] = \langle 0|0\rangle_J \tag{5.69}$$

For a system with n degrees of freedom, we have the generalization of 5.67

$$Z[J_1,\ldots,J_n] = \lim_{t'\to -i\infty} \lim_{t\to i\infty} N \int \mathcal{D}(q_i) e^{i\int_t^{t'} d\tau [L(q_i,\dot{q}_i)+\sum_{i=1}^N J_i q_i]} \tag{5.70}$$

Comments

- In the previous equation the time limits for times t and t' are imaginary. This means that these Green functions are in Euclidean space. For field theory this corresponds to the prescription $m^2 \to m^2 - i\epsilon$.
- In equation 5.70 we do not explicitly wrote the normalization. It should be chosen in such a way that $Z[0,\ldots,0]=1$. However, as we will see, for the connected Green functions in Quantum Field Theory the normalization it is not relevant, and therefore we will not worry about it.

5.5.2 Field theory

To get the generating functional in Quantum Field Theory we proceed in the usual heuristic way, by making the following equivalences,

$$t \;\to\; x^\mu$$

$$
\begin{aligned}
q(t) &\rightarrow \phi(x) \\
\mathcal{D}(q) &\rightarrow \mathcal{D}(\phi) \\
L(q_i, \dot{q}_i) &\rightarrow \int d^3x \mathcal{L}(\phi, \partial_\mu \phi)
\end{aligned} \tag{5.71}
$$

We get therefore

$$
Z[J] = N \int \mathcal{D}(\phi) e^{i \int d^4x [\mathcal{L}(\phi, \partial_\mu \phi) + J(x)\phi(x)]} \tag{5.72}
$$

This is the starting point in the path integral quantization. We will see in the next lecture how to use it. A more rigorous derivation of the above result, Eq. 5.72, be found in appendix B.

5.5.3 Applications

Once we know the generating functional $Z[J]$ we also know all the Green functions of the theory and therefore how to address any problem in Quantum Field Theory. We can ask in which conditions can we evaluate $Z[J]$? The answer is that as we only know how to do Gaussian path integrals we can only do either free fields or perturbation theory. However there are two main advantages in the method that we will now discuss:

- **Perturbation Theory**
  The expression for $Z[J]$ allows to establish the perturbative expansion and find the Feynman rules for any theory.

- **Formal Manipulations**
  Relations among the Green functions that are a consequence of symmetry properties of the theory (for instance Ward identities) are much simpler in terms of the generating functionals. Here the expression of $Z[J]$ in terms of a path integral, Eq. 5.72 is particularly useful as we will discuss later.

We will use the scalar field as an example to illustrate the first point, that is how to establish the perturbative expansion. We consider a real scalar field, described by the Lagrangian,

$$
\mathcal{L}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_I(\phi) \tag{5.73}
$$

where $\mathcal{L}_0(\phi)$ is quadratic in the fields, that is

$$
\mathcal{L}_0(\phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \tag{5.74}
$$

We can then write,

$$
\begin{aligned}
Z[J] &= N \int \mathcal{D}(\phi) e^{i \int d^4x [\mathcal{L}_0(\phi) + \mathcal{L}_I(\phi) + J\phi]} \\
&= N \int \mathcal{D}(\phi) e^{i \int d^4x [\mathcal{L}_I(\phi)]} \; e^{i \int d^4x [\mathcal{L}_0(\phi) + J\phi]}
\end{aligned} \tag{5.75}
$$

This last integral can formally be written in the form,

$$
Z[J] = \exp\left[i \int d^4x \mathcal{L}_I \left(\frac{\delta}{i\delta J} \right) \right] \; Z_0[J] \tag{5.76}
$$

where

$$Z_0[J] = N \int \mathcal{D}(\phi) e^{i \int d^4x [\mathcal{L}_0 + J\phi]} \; . \tag{5.77}$$

The usefulness of this expression results from the two following points:

- $Z_0[J]$ can be exactly calculated because is quadratic in the fields (Gaussian integral).
- If $\mathcal{L}_I(\phi)$ has a small parameter, the exponential can be developed in a power series in this parameter and the generating functional $Z[J]$ can be obtained order by order in perturbation theory, as the integrals will be Gaussian integrals with polynomials.

5.5.4 Example: perturbation theory for $\lambda\phi^4$

To see the connection with the usual results let us consider as an example the derivation of the Feynman rules for a real scalar theory with the interaction,

$$\mathcal{L}_I = -\frac{\lambda}{4!} \, \phi^4 \; . \tag{5.78}$$

The generating functional $Z[J]$ is

$$Z[J] = \mathcal{N} \exp\left\{ (-i\lambda) \frac{1}{4!} \int d^4x \left(\frac{\delta}{i\delta J} \right)^4 \right\} Z_0[J] \tag{5.79}$$

where (see Problems)

$$Z_0[J] = \exp\left\{ -\frac{1}{2} \int d^4x d^4y J(x) \Delta(x,y) J(y) \right\} \tag{5.80}$$

The normalization $\mathcal{N}$ is to be chosen such that $Z[0]=1$, as we will see later. We expand in a series in the coupling constant,

$$Z[J] = \mathcal{N} Z_0[J] \left\{ 1 + (-i\lambda) Z_1'[J] + (-i\lambda)^2 Z_2'[J] + \cdots \right\} \tag{5.81}$$

where

$$Z_1'[J] \equiv Z_0^{-1}[J] \left\{ \frac{1}{4!} \int d^4x \left(\frac{\delta}{\delta J} \right)^4 \right\} Z^0[J] \tag{5.82}$$

and

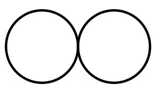
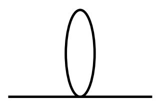
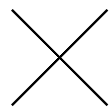
$$\begin{aligned} Z_2'[J] &\equiv \frac{1}{2} Z_0^{-1}[J] \left\{ \frac{1}{4!} \int d^4x \left(\frac{\delta}{\delta J} \right)^4 \right\}^2 Z^0[J] \\ &= \frac{1}{2} Z_0^{-1}[J] \left\{ \frac{1}{4!} \int d^4x \left(\frac{\delta}{\delta J} \right)^4 \right\} (Z_0 Z_1') \\ &= \frac{1}{2} \left(Z_1'[J] \right)^2 + \frac{1}{2} Z_0^{-1}[J] \, \frac{1}{4!} \int d^4x \left\{ 4 \frac{\delta^3 Z_0}{\delta J^3(x)} \frac{\delta Z_1'}{\delta J(x)} \right. \end{aligned}$$

$$+6\frac{\delta^2 Z_0}{\delta J^2(x)}\frac{\delta^2 Z_1'}{\delta J^2(x)} + 4\frac{\delta Z_0}{\delta J(x)}\frac{\delta^3 Z_1'}{\delta J^3(x)} + Z_0\frac{\delta^4 Z_1'}{\delta J^4(x)}\Bigg\} \tag{5.83}$$

We get for the first order

$$\begin{aligned}
&Z_1'[J] = \\
&= \frac{1}{4!}\int d^4x \Bigg[3\Delta(x,x)\Delta(x,x) - 3!\Delta(x,x)\int d^4y_1 d^4y_2 \Delta(x,y_1)\Delta(x,y_2)J(y_1)J(y_2) \\
&\quad + \int d^4y_1 \cdots d^4y_4 \Delta(x,y_1)\Delta(x,y_2)\Delta(x,y_3)\Delta(x,y_4)J(y_1)J(y_2)J(y_3)J(y_4)\Bigg]
\end{aligned} \tag{5.84}$$

This result can be represented diagrammatically in the form,

$Z_1' = \frac{1}{8}$  $-\frac{1}{4}$  $+\frac{1}{4!}$  (5.85)

For Z_2' we get

$$\begin{aligned}
&Z_2'[J] \\
= \;& \frac{1}{2}\left(Z_1'[J]\right)^2 + \frac{1}{2}\left(\frac{1}{4!}\right)^2 4!\int d^4x_1 d^4x_2 \Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_1,x_2) \\
&+ \left(\frac{1}{4!}\right)^2 \Bigg[-72\int d^4x_2 \int d^4x_1 \Delta(x_1,x_2)\int d^4y_1 \Delta(x_1,y_1)J(y_1) \\
&\qquad \Delta(x_1,x_2)\Delta(x_2,x_2)\int d^4y_2 \Delta(x_2,y_2)J(y_2) \\
&\qquad +24\int d^4x_2 d^4x_1 \Delta(x_1,x_1)\int d^4y_1 \Delta(x_1,y_1)J(y_1)\Delta(x_1,y_2) \\
&\qquad \int d^4y_2 \Delta(x_2,y_1)J(y_2)\int d^4y_3 \Delta(x_2,y_3)J(y_2)\int d^4y_4 \Delta(x_1,y_4)J(y_4) \\
&\qquad +24\int d^4x_2 \int d^4x_1 \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 \Delta(x_1,x_2)\Delta(x_1,y_1) \\
&\qquad \Delta(x_1,y_2)\Delta(x_1,y_2)\Delta(x_2,x_2)\Delta(x_2,y_4)J(y_1)\cdots J(y_4) \\
&\qquad -8\int d^4x^2 \int d^4x_1 \int d^4y_1 \cdots d^4y_6 \Delta(x_1,x_2)\Delta(x_1,y_1)\Delta(x_1 y_2) \\
&\qquad \Delta(x_1,y_3)\Delta(x_2,y_4)\Delta(x_2,y_5)\Delta(x_2,y_6)J(y_1)\cdots J(y_6) \\
&\qquad +36\int d^4x_2 d^4x_1 \Delta(x_1,x_1)\Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_2,x_2)
\end{aligned}$$

$$-36\int d^4x_2d^4x_1\Delta(x_1,x_1)\Delta(x_1,x_2)\Delta(x_1,x_2)\int d^4y_1d^4y_2\Delta(x_2,y_1)$$
$$\Delta(x_2,y_2)J(y_1)J(y_2)$$
$$-36\int d^4x_2d^4x_1d^4y_1d^4y_2\Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_2,x_1)$$
$$\Delta(x_1,y_1)\Delta(x_1,y_2)J(y_1)J(y_2)$$
$$+36\int d^4x_2d^4x_1d^4y_1\cdots d^4y_4\Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_1,y_1)$$
$$\Delta(x_1,y_2)\Delta(x_2,y_3)\Delta(x_2,y_4)J(y_1)\cdots J(y_4)$$
$$-48\int d^4x_2dy_1d^4y_1d^4y_2\Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_1,x_2)$$
$$\Delta(x_1,y_1)\Delta(x_2,y_2)J(y_1)J(y_2)\Big] \tag{5.86}$$

This can also be written as,

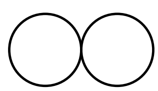
$$Z_2'[J]$$
$$=\frac{1}{2}\left(Z_1'[J]\right)^2+\frac{1}{2\cdot 4!}\int d^4x_1d^4x_2\Delta^4(x_1,x_2)$$
$$+\frac{3}{2\cdot 4!}\int d^4x_1d^4x_2\Delta(x_1,x_1)\Delta^2(x_1,x_2)\Delta(x_2,x_2)$$
$$-\frac{1}{2\cdot 3!\cdot 3!}\int d^4x_1d^4x_2d^4y_1\cdots d^4y_6\Delta(y_1,x_1)\Delta(y_2,x_1)\Delta(y_3,x_1)$$
$$\Delta(x_1,x_2)\Delta(x_2,y_4)\Delta(x_2,y_5)\Delta(x_2,y_6)J(y_1)\cdots J(y_6)$$
$$+\frac{2}{4!}\int d^4x_1d^4x_2d^4y_1\cdots d^4y_4\Delta(y_1,x_1)\Delta(x_1,x_1)\Delta(x_1,x_2)$$
$$\Delta(x_2,y_2)\Delta(x_2,y_3)\Delta(x_2,y_4)J(y_1)\cdots J(y_4)$$
$$+\frac{3}{2\cdot 4!}\int d^4x_1d^4x_2d^4y_1\cdots d^4y_4\Delta(y_1,x_1)\Delta(y_2,x_1)\Delta^2(x_1,x_2)$$
$$\Delta(x_2,y_3)\Delta(x_2,y_4)J(y_1)\cdots J(y_4)$$

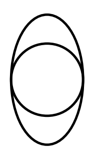
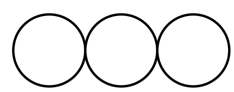
$$
\begin{aligned}
&-\frac{1}{8}\int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \Delta(y_1,x_1)\Delta(x_1,x_1)\Delta(x_1,x_2)\Delta(x_2,x_2)\\
&\qquad \Delta(x_2,y_2)J(y_1)J(y_2)\\
&-\frac{1}{8}\int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \Delta(y_1,x_1)\Delta^2(x_1,x_2)\Delta(x_2,x_2)\Delta(x_1,y_2)J(y_1)J(y_2)\\
&-\frac{1}{12}\int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \Delta(y_1,x_1)\Delta^3(x_1,x_2)\Delta(x_2,y_2)J(y_1)J(y_2)
\end{aligned}
\tag{5.87}
$$

Let us now evaluate the normalization up to second order in perturbation theory. The condition $Z[0]=1$ gives,

$$
1=\mathcal{N}\left[1+(-i\lambda)n_1+(-i\lambda)^2 n_2+\cdots\right] \tag{5.88}
$$

where

$n_1 \;=\; \dfrac{1}{8}$  (5.89)

$n_2 \;=\; \dfrac{1}{2}n_1^2 + \dfrac{1}{2\cdot 4!}$  $+\; \dfrac{3}{2\cdot 4!}$  (5.90)

We then get,

$$
\begin{aligned}
\mathcal{N} &= \frac{1}{1+(-i\lambda)n_1+(-i\lambda)^2 n_2+\cdots}\\
&= 1-(-i\lambda)n_1-(-i\lambda)^2(n_2-n_1^2)+\cdots
\end{aligned}
\tag{5.91}
$$

Putting everything together we have

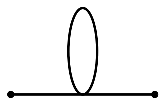
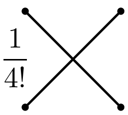
$$
\begin{aligned}
Z[J] &= Z_0[J]\left\{1-(-i\lambda)n_1-(-i\lambda)^2(n_2-n_1^2)+\cdots\right\}\\
&\qquad \left\{1+(-i\lambda)Z_1'+(-i\lambda)^2 Z_2'+\cdots\right\}\\
&= Z_0[J]\left\{1+(-i\lambda)(Z_1'-n_1)+(-i\lambda)^2(Z_2'-n_2+n_1^2-n_1 Z_1')+\cdots\right\}
\end{aligned}
\tag{5.92}
$$

Defining now

$$
Z_1 \equiv Z_1'-n_1
$$

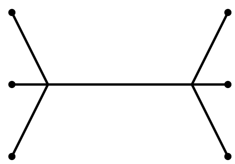
$$Z_2 \equiv Z_2' - n_2 + n_1^2 - n_1 Z_1' = Z_2' - n_2 - n_1 Z_1 \tag{5.93}$$

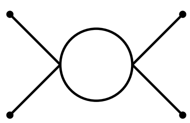
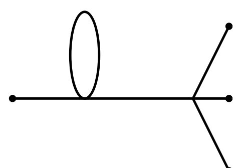
we get

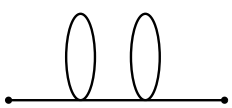
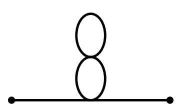
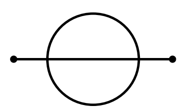
$Z_1[J] = -\frac{1}{4}$  $+\frac{1}{4!}$  (5.94)

and

$Z_2[J]$

$= \frac{1}{2}\left(Z_1[J]\right)^2 + \frac{-1}{2!\,3!\,3!}$ 

$+\frac{3}{2!\,4!}$  $+\frac{2}{4!}$ 

$-\frac{1}{8}$  $-\frac{1}{8}$  $-\frac{1}{12}$  (5.95)

with $Z_1[0] = Z_2[0] = 0$. Therefore the generating functional

$$Z[J] = Z_0[J]\left\{1 + (-i\lambda)Z_1[J] + (-i\lambda)^2 Z_2[J] + \cdots\right\} \tag{5.96}$$

is automatically correctly normalized if we neglect all the vacuum amplitudes, know as *bubbles*. To verify that this expression reproduces the perturbation theory results, let us evaluate the propagator up to second order in λ^2. We get

$$\Delta'(x_1, x_2) = \left.\frac{\delta^2 Z[J]}{i\delta J(x_1) i\delta J(x_2)}\right|_{J=0}$$

$$= -\left.\frac{\delta^2 Z_0[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J=0} - (-i\lambda)\left.\frac{\delta^2 Z_1[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J=0} - (-i\lambda)^2\left.\frac{\delta^2 Z_2[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J=0}$$

$$
\begin{aligned}
&= \Delta(x_1,x_2) \ + \ (-i\lambda)\frac{1}{2}\int d^4y \Delta(x_1,y)\Delta(x_2,y)\Delta(y,y) \\
&+(-i\lambda)^2 \int d^4y_1 d^4y_2 \left[\frac{1}{4}\Delta(x_1,y_1)\Delta(y_1,y_1)\Delta(y_1,y_2)\Delta(y_2,y_2)\Delta(y_2,x_2)\right. \\
&\left. +\frac{1}{4}\Delta(x_1,y_1)\Delta^2(y_1,y_2)\Delta(y_2,y_2)\Delta(y_1,x_2)+\frac{1}{6}\Delta(x_1,y_1)\Delta^3(y_1,y_2)\Delta(y_2,x_2)\right]
\end{aligned}
\tag{5.97}
$$

In diagrammatic form we have the situation of Figure 5.15.

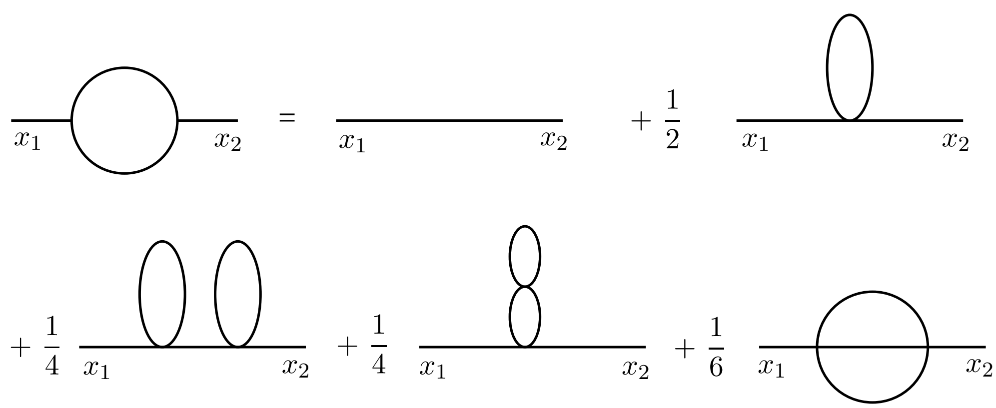


Figure 5.15:

Continuing with the scalar $\lambda\phi^4$ theory as an example, let analyze the generating functional for the connected Green functions, $W[J]$. It is easy to see that the terms like $Z_1^2[J]$ correspond to disconnected diagrams which are part of $Z[J]$. Let us see how they disappear in $W[J]$. We have

$$
\begin{aligned}
iW[J] &= \ln Z[J] = \\
&= \ln Z_0[J] + \ln\left\{1+(-i\lambda)Z_1[J]+(-i\lambda)^2 Z_2[J]+\cdots\right\} \\
&= iW_0[J]+(-i\lambda)Z_1[J]-\frac{1}{2}(-i\lambda)^2\left(Z_1[J]\right)^2+(-i\lambda)^2 Z_2[J]+\cdots \\
&= iW_0[J]+(-i\lambda)Z_1[J]+\left\{(-i\lambda)^2(Z_2[J]-\frac{1}{2}\left(Z_1[J]\right)^2\right\}+\cdots \\
&\equiv i\left\{W_0[J]+(-i\lambda)W_1[J]+(-i\lambda)^2 W_2[J]+\cdots\right\}
\end{aligned}
\tag{5.98}
$$

with

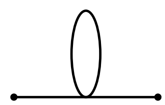
$$
iW_1[J]=Z_1[J], \quad iW_2[J]=Z_2[J]-\frac{1}{2}\left(Z_1[J]\right)^2 \tag{5.99}
$$

Therefore the disconnected diagrams contained in $Z_2[J]$ are subtracted and W_1 and W_2 have only the connected diagrams.

5.5.5 Symmetry factors

After doing the derivatives in order to J to obtain a given Green function, we have some diagrams multiplied by numbers known as symmetry factors. For instance for the one-loop correction to the propagator we get,

$$
\begin{aligned}
\Delta'(x_1, x_2) &= \left. \frac{\delta^2 Z}{i\delta J(x_1) i\delta J(x_2)} \right|_{J=0} = \\
&= \left. \frac{\delta^2 Z_0}{i\delta J(x_1) i\delta J(x_2)} \right|_{J=0} + (-i\lambda) \left. \frac{\delta^2 Z_1}{i\delta J(x_1) i\delta J(x_2)} \right|_{J=0} + \cdots \\
&= \Delta(x_1, x_2) + \frac{1}{2}
\end{aligned}
$$



$$+ \cdots \tag{5.100}$$

The factor $\frac{1}{2}$ is the symmetry factor that corresponds to that diagram. The method of the generating functional gives automatically the correct symmetry factors. However in practical applications it is normally easier to have a rule to obtain these symmetry factors.

Rule for the Symmetry Factors

The symmetry factor S of a diagram it is given by

$$S = \frac{N}{D} \tag{5.101}$$

where N is the # of different ways of forming the diagram, and D it is the product of the symmetry factors of each vertex by the number of permutations of equal vertices.

As an example take the diagram that contributes to the propagator at one-loop represented in Fig. 5.16. Then according to the rule we have,

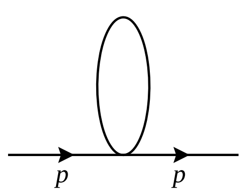


Figure 5.16:

$$S = \frac{4 \times 3}{4!} = \frac{1}{2} \tag{5.102}$$

5.5.6 A comment on the normal ordering

In the previous example we have diagrams like in Fig. 5.16, that we generically denote by *"tadpoles"*, that connect fields that belong to the same vertex, and that we saw that were excluded by the normal ordering. This difference comes from the fact that we were not very rigorous in the definition of the Lagrangian to include in the path formalism. If we would we would concluded that we should have to do the normal ordering in the Lagrangian to include in $e^{i\int d^4x\mathcal{L}(\phi)}$ This would make the Lagrangian $\mathcal{L}(\phi)$ to include in the path integral different from the classical Lagrangian. Let us look at ϕ^4 as an example. We will use the relations

$$\hat{\phi}(x)\hat{\phi}(x) =: \hat{\phi}(x)\hat{\phi}(x) : + \langle 0|\, \hat{\phi}(x)\hat{\phi}(x)\, |0\rangle \tag{5.103}$$

or symbolically

$$: \hat{\phi}^2 := \hat{\phi}^2 - \langle 0|\, \hat{\phi}^2\, |0\rangle \ . \tag{5.104}$$

In a similar way

$$\hat{\phi}^4 =: \hat{\phi}^4 : +6 : \hat{\phi}^2 : \ \langle 0|\, \hat{\phi}^2\, |0\rangle + 6 \langle 0|\, \hat{\phi}^2\, |0\rangle \ \langle 0|\, \hat{\phi}^2\, |0\rangle \tag{5.105}$$

Therefore we get

$$: \hat{\phi}^4 := \hat{\phi}^4 - 6 : \hat{\phi}^2(x) : \ \langle 0|\, \hat{\phi}^2(x)\, |0\rangle - 6 \left(\langle 0|\, \hat{\phi}^2\, |0\rangle\right)^2 \tag{5.106}$$

or

$$: \hat{\phi}^4 := \hat{\phi}^4 - 6\ \hat{\phi}^2 \langle 0|\, \hat{\phi}^2\, |0\rangle \tag{5.107}$$

This means that the quantum Lagrangian should be written as

$$\begin{aligned} \mathcal{L}^Q_{Int} &= -\frac{\lambda}{4!} : \hat{\phi}^4 : \\ &= -\frac{\lambda}{4!}\hat{\phi}^4 + \frac{\lambda}{4}\hat{\phi}^2 I \end{aligned} \tag{5.108}$$

where

$$\begin{aligned} I &\equiv \langle 0|\, \hat{\phi}^2(x)\, |0\rangle \\ &= \int d\tilde{k}_1 d\tilde{k}_2 \, \langle 0| \left(a^\dagger(k_1)e^{ik_1\cdot x} + a(k_1)e^{-ik_1\cdot x}\right) \left(a^\dagger(k_2)e^{ik_2\cdot x} + a(k_2)e^{-ik_2\cdot x}\right) |0\rangle \\ &= \int d\tilde{k}_1 d\tilde{k}_2 \, \langle 0|\, a(k_1)a^\dagger(k_2)\, |0\rangle \, e^{i(k_2-k_1)\cdot x} \\ &= \int d\tilde{k}_1 = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k} \end{aligned} \tag{5.109}$$

In the previous expression we have used the relations

$$\left[a(k), a^+(k')\right] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}') \tag{5.110}$$

$$\omega_k = \sqrt{k_0^2 + |\vec{k}|^2} \tag{5.111}$$

The integral I is divergent and in fact it is equal to the one-loop integral. In fact

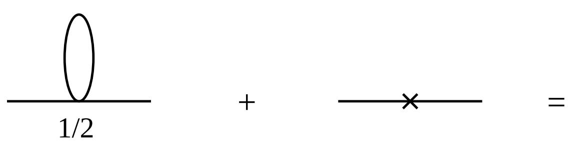
$$\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\varepsilon}$$

$$
\begin{aligned}
&= \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} dk_0 \, \frac{i}{(k_0-\omega_k)(k_0+\omega_k)} \\
&= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} = I
\end{aligned}
\tag{5.112}
$$

Therefore if we had been rigorous we should have to include the term, $\frac{\lambda}{4}\phi^2 I$ in the interaction. The Lagrangian to use in the path integral should then be,

$$
\mathcal{L}_{Int}^{IC} = -\frac{\lambda}{4!}\phi^4 + \frac{\lambda}{4}\phi^2 I \tag{5.113}
$$

It is easy to verify that the additional term cancels the *tadpole*. In fact we have



$$
\begin{aligned}
&= \frac{i}{p^2-m^2}\left[-i\frac{\lambda}{2}\, I + i\frac{\lambda}{2}\, I\right] \frac{i}{p^2-m^2} \\
&= 0
\end{aligned}
\tag{5.114}
$$

and therefore the *tadpoles* would not appear. However many times we do not worry about this considerations and just use the classical Lagrangian in the path integral. This can be done because the *tadpole* gives an infinite contribution to the mass, that can be absorbed in the renormalization process.

In QED the same occurs, we should have as interaction Lagrangian

$$
\mathcal{L}_{Int}^{IC} = -e\overline{\psi}\gamma^\mu\psi A_\mu + eA_\mu \langle 0|\, \overline{\psi}\gamma^\mu\psi\, |0\rangle \tag{5.115}
$$

and the second term would remove the *tadpole* shown in Fig. 5.17,

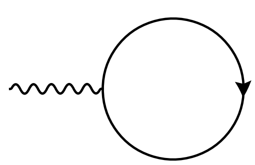


Figure 5.17: Tadpole for `QED`.

However, due the Lorentz invariance of the theory, one can show that this *tadpole* vanishes to all orders and therefore we do have to worry, just use the classical Lagrangian.

5.5.7 Generating functionals for fermions

For theories with fermion fields we introduce Grassmann variables. These anti-commuting variables are in some sense the the classical limit of fermionic quantum fields. The details of this construction are explained in the Appendices A and B. Here we just review our conventions. Due to the anti-commuting character it is necessary to specify the order of the derivatives.

We will take the convention that derivatives are left derivatives, that is they obey,

$$\begin{aligned}
\frac{\delta}{\delta\overline{\eta}(x)}\int d^4y\overline{\eta}(y)\psi(y) &= \psi(x)\\
\frac{\delta}{\delta\eta(x)}\int d^4y\overline{\psi}(y)\eta(y) &= -\overline{\psi}(x)
\end{aligned}\tag{5.116}$$

In the Green functions the order of the derivatives is such that

$$\begin{aligned}
G^{2n}(x_1,\ldots,y_n) &= \langle 0|\,T\psi(x_1)\cdots\psi(x_n)\overline{\psi}(y_1)\cdots\overline{\psi}(y_n)\,|0\rangle\\
&\equiv \frac{\delta^{2n}Z[\eta,\overline{\eta}]}{i\delta\eta(y_n)\cdots i\delta\eta(y_1)i\delta\overline{\eta}(x_n)\cdots i\delta\overline{\eta}(x_1)}\\
&\equiv \frac{\delta}{i\delta\eta(y_n)}\cdots\frac{\delta}{i\delta\overline{\eta}(x_1)}Z[\eta,\overline{\eta}]
\end{aligned}\tag{5.117}$$

where we have defined the generating functional for fermion fields as

$$\begin{aligned}
Z[\eta,\overline{\eta}] &= \langle 0|\,Te^{i\int d^4x\left[\overline{\eta}(x)\psi(x)+\overline{\psi}(x)\eta(x)\right]}\,|0\rangle\\
&= \int\mathcal{D}(\psi,\overline{\psi})e^{i\int d^4x\left[\mathcal{L}+\overline{\eta}(x)\psi(x)+\overline{\psi}(x)\eta(x)\right]}
\end{aligned}\tag{5.118}$$

Examples of these results will be given in the Problems at the end of the chapter.

5.6 Change of variables in path integrals. Applications

5.6.1 Introduction

One of the great advantages of having an expression for the generating functional $Z[J]$ in terms of a path integral is that a great number of manipulations that are familiar for the usual integrals, (change of variables, integration by parts, ...) can also be applied here. Let us see the implications of changing integration variables.

Let us consider an infinitesimal transformation of the form,

$$\phi_i \to \phi_i + \varepsilon F_i(\phi)\tag{5.119}$$

where

$$F_i(\phi) = f_i + f_{ij}\phi_j + \cdots\tag{5.120}$$

Then we should have, in first order,

$$\begin{aligned} \mathcal{D}(\phi) \to &\quad \mathcal{D}(\phi) \det \left| \delta_{ij} + \varepsilon \tfrac{\delta F_i}{\delta \phi_j} \right| \\ = &\quad \mathcal{D}(\phi) \left(1 + \varepsilon \tfrac{\delta F_i}{\delta \phi_i} \right) \end{aligned} \tag{5.121}$$

On the other hand

$$e^{i(S(\phi)+J_i\phi_i)} \to e^{i(S(\phi)+J_i\phi_i)} \left[1 + i\varepsilon \left(\frac{\delta S}{\delta \phi_i} + J_i \right) F_i(\phi) \right] \tag{5.122}$$

As the integral for $Z(J)$ should be independent of the change of variables we get

$$0 = \int \mathcal{D}(\phi) \left[i \left(\frac{\delta S}{\delta \phi_i} + J_i \right) F_i + \frac{\delta F_i}{\delta \phi_i} \right] e^{i(S[\phi]+J_i\phi_i)} \tag{5.123}$$

Using $\phi_i \to \frac{\delta}{i\delta J_i}$ we get a more compact expression,

$$\left\{ i \, \left[\frac{\delta S}{\delta \phi_i} \left(\frac{\delta}{i\delta J_i} \right) + J_i \right] F_i \left(\frac{\delta}{i\delta J_i} \right) + \frac{\delta F_i}{\delta \phi_i} \left(\frac{\delta}{i\delta J} \right) \right\} Z(J) = 0 \tag{5.124}$$

This the general expression that we are going to apply to two important particular cases, the Dyson-Schwinger equations and the Ward identities.

5.6.2 Dyson-Schwinger equations

Let $F_i = f_i$ independent of ϕ_i, that is a simple translation of the fields. Then the previous master equation simplifies to

$$\left(\frac{\delta S}{\delta \phi_i} \left[\frac{\delta}{i\delta J} \right] + J_i \right) Z(J) = 0 \tag{5.125}$$

We will see below that this the expression for the Dyson-Schwinger (DS) equations for the generating functional of the full Green functions. In this way, the DS equations are a consequence of the path integral for constant field translations. This equations can further be written as

$$J_k = -\frac{1}{Z} E \left[\frac{\delta}{i\delta J_k} \right] Z[J] \tag{5.126}$$

where the functional $E[\phi]$ is the Euler-Lagrange equation of motion,

$$E\left[\phi_k\right] \equiv \frac{\delta S}{\delta \phi_k} \ . \tag{5.127}$$

For many applications it is more convenient to write the Dyson-Schwinger equations for the connected and proper (one particle irreducible) Green functions. For this we have to write the corresponding equations for the functional W and Γ. Using the identity

$$\frac{1}{Z} \frac{\delta}{i\delta J_k} \left(Z[J] f[J] \right) = \left(\frac{\delta i W}{i\delta J_k} + \frac{\delta}{i\delta J_k} \right) f[J] \tag{5.128}$$

we can write

$$\frac{1}{Z}E\left[\frac{\delta}{i\delta J_k}\right]Z[J]=E\left[i\frac{\delta W}{i\delta J_k}+\frac{\delta}{i\delta J_k}\right]1 \tag{5.129}$$

Therefore the DS equation for the generating functional of the connected Green functions can be written as,

$$J_k=-E\left[i\frac{\delta W}{i\delta J_k}+\frac{\delta}{i\delta J_k}\right]1 \tag{5.130}$$

Let us now find the corresponding equation for the proper, or one particle irreducible, Green functions. To obtain this equation we use the following relations

$$\begin{cases} \phi_k=\dfrac{i\,\delta W}{i\delta J_k} & J_k=-\dfrac{\delta\Gamma}{\delta\phi_k} \\[2ex] \dfrac{\delta}{i\delta J_k}=G_{km}\dfrac{\delta}{\delta\phi_m} & \dfrac{\delta}{\delta\phi_k}=-i\Gamma_{kr}\dfrac{\delta}{i\delta J_r} \end{cases}$$

We get

$$\frac{\delta\Gamma}{\delta\phi_k}=E\left[\phi_k+G_{km}\frac{\delta}{\delta\phi_m}\right]1 \tag{5.131}$$

It is in the form of Eq. 5.131, that the DS are more useful.

Example : Self-energy in ϕ^3

Let us start with the example of the self-energy in ϕ^3. The action for this theory is, using our compact notation,

$$S[\phi]=\frac{1}{2}\phi_k(-\Box-m^2)\delta_{km}\phi_m-\frac{\lambda}{3!}(\phi_k)^3 \tag{5.132}$$

Therefore the equation of motion is

$$E[\phi_k]=(-\Box-m^2)\phi_k-\frac{\lambda}{2}(\phi_k)^2 \tag{5.133}$$

We get therefore, expanding the functional,

$$E\left[\phi_k+G_{km}\frac{\delta}{\delta\phi_m}\right]1=-(\Box+m^2)\phi_k-\frac{\lambda}{2}\left(\phi_k+G_{kr}\frac{\delta}{\delta\phi_r}\right)\phi_k \tag{5.134}$$

We get therefore for the DS equation

$$\frac{\delta\Gamma}{\delta\phi_k}=-(\Box+m^2)\phi_k-\frac{\lambda}{2}\left(\phi_k^2+G_{kr}\delta_{rk}\right) \tag{5.135}$$

By taking functional derivatives with respect to ϕ_m we get the DS for the various Green functions, all derived from just one master equation. For instance, for the self-energy we get,

$$\frac{\delta^2\Gamma}{\delta\phi_k\delta\phi_m}=-(\Box+m^2)\delta_{km}-\frac{\lambda}{2}\left(2\phi_k\delta_{km}-i\Gamma_{mn}G_{krn}\delta_{rk}\right) \tag{5.136}$$

Setting $\phi_k = 0$ we get

$$\begin{aligned}
\Gamma_{km} - (-\Box - m^2)\delta_{km} &= i\frac{\lambda}{2}\ \Gamma_{mn} G_{krn} \delta_{rk} \\
&= \frac{\lambda}{2}\ \Gamma_{mn} G_{krn} \Gamma_{rs} G_{sk} \\
&= i\frac{\lambda}{2}\ \Gamma_{mn} \Gamma_{rs} G_{sk} G_{kk'} G_{rr'} G_{nn'} \Gamma_{k'r'n'} \\
&= -i\frac{\lambda}{2}\ G_{kk'} G_{ks} \Gamma_{k'sm} \qquad (5.137)
\end{aligned}$$

where we have repeatedly used the relation

$$\Gamma_{ij} G_{jk} = i\delta_{ik} \qquad (5.138)$$

By definition of self-energy we have,

$$\Gamma_{km} - (-\Box - m^2)\delta_{km} \equiv -\Sigma_{km} \qquad (5.139)$$

Therefore

$$-i\Sigma_{km} = -i\frac{\lambda}{2}\ G_{kk'} G_{ks} i\Gamma_{k'sm} \qquad (5.140)$$

as shown in the Fig. 5.18, We see that the DS equation is no more than the statement

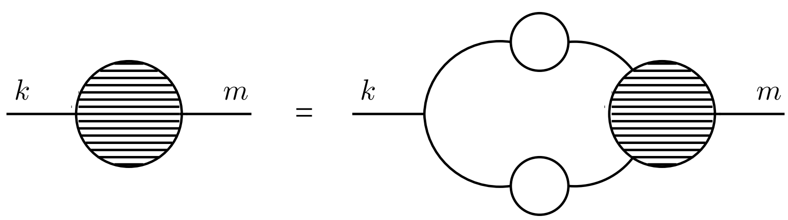


Figure 5.18: Dyson-Schwinger equation for ϕ^3.

that the vertex of the theory is $\frac{\lambda}{3!}\phi^3$.

Example: Self-energy in ϕ^4

In the case the action is

$$S[\phi] = \frac{1}{2}\phi_k(-\Box - m^2)\delta_{km}\phi_m - \frac{\lambda}{4!}\ (\phi_k)^4\ . \qquad (5.141)$$

Therefore the equation of motion is

$$E[\phi] = (-\Box - m^2)\phi_k - \frac{\lambda}{3!}\ (\phi_k)^3\ . \qquad (5.142)$$

We then get

$$E\left[\phi_k + G_{km}\frac{\delta}{\delta\phi_m}\right] 1 = -(\Box + m^2)\phi_k - \frac{\lambda}{3!}\left(\phi_k + G_{km}\frac{\delta}{\delta\phi_m}\right)\left(\phi_k + G_{kn}\frac{\delta}{\delta\phi_n}\right)\phi_k$$

$$
\begin{aligned}
&= -(\Box + m^2)\phi_k - \frac{\lambda}{3!}\left(\phi_k + G_{km}\frac{\delta}{\delta\phi_m}\right)\left(\phi_k^2 + G_{kn}\delta_{nk}\right) \\
&= -(\Box + m^2)\phi_k - \frac{\lambda}{3!}\Big(\phi_k^3 + \phi_k G_{kn}\delta_{nk} + 2G_{km}\phi_k\delta_{km} \\
&\quad - iG_{km}\Gamma_{m\ell}G_{kn\ell}\delta_{nk}\Big)
\end{aligned}
\tag{5.143}
$$

The master equation for the DS equations then reads,

$$
\frac{\delta\Gamma}{\delta\phi_k} = -(\Box + m^2)\phi_k - \frac{\lambda}{3!}\ \left(\phi_k^3 + \phi_k G_{kn}\delta_{nk} + 2G_{km}\phi_k\delta_{km} - iG_{km}\Gamma_{m\ell}G_{kn\ell}\delta_{nk}\right) \tag{5.144}
$$

To obtain the DS for the self-energy we take the derivative with respect to ϕ_j make all $\phi = 0$ after derivation. We obtain,

$$
\begin{aligned}
\Gamma_{kj} \quad &- \quad (-\Box - m^2)\delta_{kj} \\
&= \quad -\frac{\lambda}{3!}\ (G_{kn}\delta_{nk}\delta_{kj} + 2G_{km}\delta_{km}\delta_{kj} - iG_{km}\Gamma_{m\ell j}G_{kn\ell}\delta_{nk} \\
&\qquad -G_{kmp}\Gamma_{pj}\Gamma_{m\ell}G_{kn\ell}\delta_{nk} - G_{km}\Gamma_{m\ell}G_{kn\ell p}\Gamma_{pj}\delta_{nk})
\end{aligned}
\tag{5.145}
$$

This equation can in turn be written as,

$$
\begin{aligned}
-i\Sigma_{kj} \quad &= \quad -i\frac{\lambda}{2}G_{kk}\delta_{kj} + i\frac{\lambda}{3!}G_{km}i\Gamma_{m\ell j}G_{kk'}G_{nn'}G_{\ell\ell'}i\Gamma_{k'n'\ell'}\delta_{nk} \\
&\quad +i\frac{\lambda}{3!}G_{kk'}G_{mm'}G_{pp'}i\Gamma_{k'm'p'}\Gamma_{pj}\Gamma_{m\ell}G_{kk''}G_{nn'}G_{\ell\ell'}i\Gamma_{k''n'\ell'}\delta_{nk} \\
&\quad +i\frac{\lambda}{3!}G_{km}\Gamma_{m\ell}G_{kn\ell p}\Gamma_{pj}\delta_{nk} \\
&= \quad -i\frac{\lambda}{2}G_{kk}\delta_{kj} + i\frac{\lambda}{3!}\delta_{k\ell}\delta_{nk}G_{kn\ell p}i\Gamma_{pj}
\end{aligned}
\tag{5.146}
$$

For the ϕ^4 theory we have $\Gamma_{ijk} = 0$ and therefore

$$
G_{kn\ell p} = G_{kk'}G_{nn'}G_{\ell\ell'}G_{pp'}\ i\Gamma_{k'n'\ell'p'} \tag{5.147}
$$

We finally get,

$$
-\,i\Sigma_{kj} = -i\frac{\lambda}{2}G_{kk}\delta_{kj} - i\frac{\lambda}{3!}G_{kk'}G_{kn'}G_{k\ell'}\ i\Gamma_{k'n'\ell'j} \tag{5.148}
$$

that we represent diagrammatically in Fig. 5.19 Again the DS equation for the self-energy

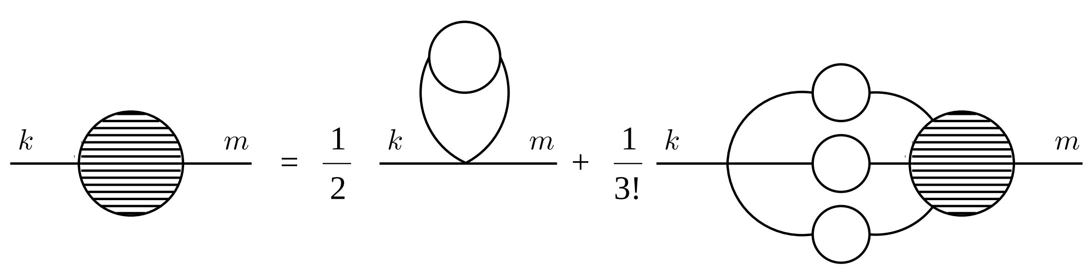


Figure 5.19: Dyson-Schwinger equation for ϕ^4.

is nothing else than the identification of the vertex of the theory.

5.6.3 Ward identities

Consider a theory with some symmetry. This symmetry is expressed as an invariance of the action, that is

$$\frac{\delta S[\phi]}{\delta \phi_i} F_i(\phi) = 0 \tag{5.149}$$

where we considered the previously defined infinitesimal transformations. If this symmetry transformation also leaves invariant the integration measure $\mathcal{D}(\phi)$ the we get simply the expression,

$$J_i F_i \left[\frac{\delta}{i\delta J}\right] Z(J) = 0 \tag{5.150}$$

This expression is known as *Ward Identity*. Derivation in order to the external sources will lead to relations among different Green functions as a consequence of the symmetry of the theory. For gauge theories the correct expression is a bit more complicated. The reason being, as we shall see, that in the quantization of gauge theories one normally has to introduce terms that break the symmetry, known as gauge fixing terms. In this case we can write,

$$S_{eff} = S_I + S_{NI} \tag{5.151}$$

where $\frac{\delta S_I}{\delta \phi_i} F_i = 0$ and $\frac{\delta S_{NI}}{\delta \phi_i} F_i \neq 0$. Then, if the measure is still invariant, we should have the more complicated expression for the Ward identities,

$$\left(\frac{\delta S_{NI}}{\delta \phi_i}\left[\frac{\delta}{i\delta J}\right] + J_i\right) F_i \left[\frac{\delta}{i\delta J}\right] Z(J) = 0 \tag{5.152}$$

In the next chapter we will apply this expression to obtain the Ward identities for QED and for the non-Abelian gauge theories. For these last ones the question of the invariance of the measure is more subtle and will be discussed there, after we have learned how to quantize these theories.

Problems for Chapter 5

5.1 Evaluate G_c^4 starting from 5.21 and show that it is indeed the connected Green function with four external legs.

5.2 Show that for a real scalar field we have

$$Z_0[J] = \exp\left\{-\frac{1}{2}\int d^4x d^4y J(x)\Delta^{(0)}(x,y)J(y)\right\} \tag{5.153}$$

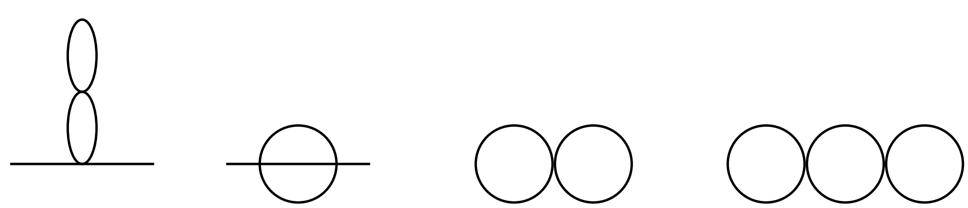
where

$$\Delta^{(0)}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \frac{i}{k^2 - m^2 + i\epsilon} \tag{5.154}$$

Hint: Use a convenient generalization of the result

$$\int_{-\infty}^{+\infty} dx_1 \cdots dx_N e^{-\frac{1}{2}x_i M_{ij} x_j + b_i x_i} = \pi^{N/2}(\det M)^{-1/2} e^{\frac{1}{2}b_i (M^{-1})_{ij} b_j} \tag{5.155}$$

5.3 Determine the symmetry factors for the following diagrams:



5.4 Consider the theory ϕ^3, that is, $V(\phi) = \frac{\lambda}{3!}\phi^3$. Using

$$Z[J] = \exp\left\{-i\int d^4x V\left[\frac{\delta}{i\delta J}\right]\right\} Z_0(J) \tag{5.156}$$

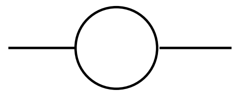
where

$$Z_0(J) = exp[-\frac{1}{2}\int d^4x d^4x' J(x) G_F^{(0)}(x-x') J(x')] \tag{5.157}$$

and

$$G_F^{(0)}(x-x') = i\int d^4k e^{ik(x-x')} \frac{1}{k^2 - m^2 + i\varepsilon} \tag{5.158}$$

show that the symmetry factor of the diagram



is $S = \frac{1}{2}$.

5.5 Given the Lagrangian for the free Dirac field

$$\mathcal{L}_0 = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \ , \tag{5.159}$$

show that the generating functional for the Green functions is

$$Z_0[\eta, \overline{\eta}] = e^{-\int d^4x d^4y \ \overline{\eta}(x) S_F^0(x,y) \eta(y)} \tag{5.160}$$

where

$$\begin{aligned} S^0_{F\alpha\beta}(x,y) &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \left(\frac{i}{\not{p} - m + i\varepsilon} \right)_{\alpha\beta} \\ &= \frac{\delta^2 Z_0}{i\delta\eta_\alpha(y) \ i\delta\overline{\eta}_\beta(x)} \\ &= \langle 0| \, T\psi_\beta(x)\overline{\psi}_\alpha(y) \, |0\rangle \ . \end{aligned}$$

5.6 As we will show in Chapter 6 the generating functional for the Green functions in QED is given by,

$$Z(J_\mu, \eta, \overline{\eta}) = \int \mathcal{D}(A_\mu, \psi, \overline{\psi}) \ e^{i\int d^4x (\mathcal{L}_{QED} + \mathcal{L}_{GF} + J^\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta)} \ . \tag{5.161}$$

where

$$\begin{aligned} &\mathcal{L}_{QED} = -\frac{1}{4} \ F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi \\ &\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial \cdot A)^2 \\ &D_\mu = \partial_\mu + ieA_\mu \ . \end{aligned}$$

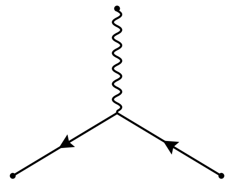
a) Determine $Z_0[J^\mu, \eta, \overline{\eta}]$
b) Show that

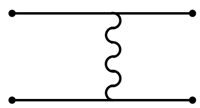
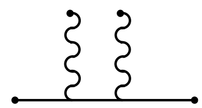
$$Z[J^\mu, \eta, \overline{\eta}] = \exp\left\{ (-ie) \int d^4x \frac{\delta}{\delta\eta_\alpha(x)} \, (\gamma^\mu)_{\alpha\beta} \, \frac{\delta}{\delta\overline{\eta}_\beta(x)} \frac{\delta}{\delta J_\mu(x)} \right\} Z_0[J^\mu, \eta, \overline{\eta}] \ . \tag{5.162}$$

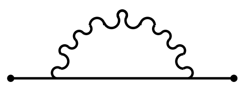
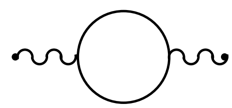
c) Expand

$$Z = Z_0 \left[1 + (-ie)Z_1 + (-ie)^2 Z_2 + \cdots \right] \tag{5.163}$$

where we have subtracted the vacuum-vacuum amplitudes in Z_i, that is, $Z_i[0] = 0 \to Z[0] = 1$. Show that

$Z_1 = -i$  (5.164)

$Z_2 = \frac{1}{2} Z_1^2 \quad + \quad \frac{1}{2}$  $+$ 

$-$  $+ \quad \frac{1}{2}$  (5.165)

d) Discuss the numerical factors and signs of the previous diagrams.
e) Evaluate in lowest order

$$\langle 0 | \, T A^\mu(x) \psi_\beta(y) \overline{\psi}_\alpha(z) \, |0\rangle = \frac{\delta^3 Z}{i\delta\eta_\alpha(z) i\delta\overline{\eta}_\beta(y) i\delta J_\mu(x)} \tag{5.166}$$

and verify that it coincides with the Feynman rules for the vertex
f) Determine the amplitude for the Compton scattering in lowest order, that is,

$$\langle 0 | \, T A^\mu(x) A^\nu(y) \psi_\beta(z) \overline{\psi}_\alpha(w) \, |0\rangle = \frac{\delta^4 Z}{i\delta\eta_\alpha(w) i\delta\overline{\eta}_\beta(z) i\delta J_\nu(y) i\delta J_\mu} \tag{5.167}$$

and verify that it reproduces the result obtained from the usual Feynman rules.

5.7 The Ward identities for QED derived in section 5.7 have not the form

$$J_i F_i \left[\frac{\partial}{i\partial J} \right] \; Z(J) = 0 \tag{5.168}$$

where $\delta\phi_i = F_i[\phi]$ because

$$S_{GF} = \int d^4x \left(-\frac{1}{2\xi} \; (\partial \cdot A)^2 \right) \tag{5.169}$$

it is not gauge invariant. Introduce the functional

$$Z'(J_\mu, \eta, \overline{\eta}) = \int \mathcal{D}(A_\mu, \psi, \overline{\psi}, \omega, \overline{\omega}) \; e^{i \int d^4x (\mathcal{L}_{eff} + J^\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta)} \tag{5.170}$$

where

$$\mathcal{L}_{eff} = \mathcal{L}_{QED} + \mathcal{L}_{GF} + \mathcal{L}_G \tag{5.171}$$

and

$$\mathcal{L}_G = -\overline{\omega} \Box \omega \; . \tag{5.172}$$

where ω and $\overline{\omega}$ are anti-commutative scalar fields.
a) Show that

$$Z'(J_\mu, \eta, \overline{\eta}) = \mathcal{N}\ Z(J_\mu, \eta, \overline{\eta}) \tag{5.173}$$

where $\mathcal{N}$ do not depend neither on the fields nor on the sources. Explain why this normalization does not affect the Green functions. Therefore either Z or Z' are good for its calculation. b) Show that the integration measure $\mathcal{D}(A_\mu, \psi, \overline{\psi}, \omega, \overline{\omega})$ and $\int d^4x \mathcal{L}_{eff}$ are invariants under the transformation

$$\begin{aligned}
&\delta\psi = -ie\omega\theta\psi \qquad \delta\overline{\psi} = ie\overline{\psi}\omega\theta \\
&\delta A_\mu = \partial_\mu \omega\theta \\
&\delta\overline{\omega} = \tfrac{1}{\xi}(\partial \cdot A)\theta \quad \delta\omega = 0
\end{aligned} \tag{5.174}$$

where θ is an anti-commutative parameter (Grassmann variable). c) Introduce the anti-commutative sources for the fields, ω and $\overline{\omega}$, that is

$$\overline{Z}(J_\mu, \eta, \overline{\eta}, \zeta, \overline{\zeta}) = \int \mathcal{D}(A_\mu, \psi, \overline{\psi}, \omega, \overline{\omega}) e^{i\int d^4x(\mathcal{L}_{eff} + J^\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta + \overline{\omega}\zeta + \overline{\zeta}\omega)} \tag{5.175}$$

Show that

$$\overline{Z}(J_\mu, \eta, \overline{\eta}, \zeta, \overline{\zeta}) = Z_G(\zeta, \overline{\zeta})\ Z(J_\mu, \eta, \overline{\eta}) \tag{5.176}$$

where

$$Z(J_\mu, \eta, \overline{\eta}) = \int \mathcal{D}(A_\mu, \psi, \overline{\psi})\ e^{i\int d^4x(\mathcal{L}_{QED} + \mathcal{L}_{GF} + J^\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta)}\ . \tag{5.177}$$

Consider the functionals $\overline{W}$, W_G and W as well $\overline{\Gamma}$,Γ_G and Γ defined in a similar way. What is the relation between $\overline{W}$, W_G and W and the set $\overline{\Gamma}$, Γ_G and Γ . d) Show that the Dyson-Schwinger equation for the fields ω and $\overline{\omega}$ is

$$\frac{\delta\overline{\Gamma}}{\delta\overline{\omega}} = -\Box\omega\ . \tag{5.178}$$

e) Show that the Ward identities can now be written as

$$J_i F_i[\frac{\delta}{i\delta J}]\overline{Z} = 0\ . \tag{5.179}$$

Write the Ward identities for $\overline{\Gamma}(A_\mu, \psi, \overline{\psi}, \omega, \overline{\omega})$. Show that you recover the known results.
f) Show that a mass term for the photon, although it breaks the gauge symmetry does not spoil the Ward identities, if the ghosts also have mass. If the photon mass term were $\frac{1}{2}\ \mu^2 A_\mu A^\mu$ what would be the mass of the ghosts?

Chapter 6

Non-Abelian Gauge Theories

6.1 Classical theory

6.1.1 Introduction

We will start by reviewing briefly how to construct the classical action for a non-abelian (Yang-Mills) theory. Let us consider a compact group G corresponding to some internal symmetry. Let ϕ_i, $(i = 1, \cdots, N)$ be a set of fields that transform under G in a representation of dimension N.

$$\phi(x) \to \phi'(x) = U(g)\phi(x) \tag{6.1}$$

where $U(g)$ is a $N \times N$ matrix. In an infinitesimal transformation

$$g = 1 - i\alpha^a t^a \qquad\qquad a = 1, \cdots, r \tag{6.2}$$

where α^a are infinitesimal parameters and t^a are the generators of the group. For the fundamental representation they satisfy

$$\begin{aligned} \left[t^a, t^b\right] &= i f^{abc} t^c \\ Tr\left(t^a t^b\right) &= \frac{1}{2}\delta^{ab} \end{aligned} \tag{6.3}$$

Examples of these generators are

$$\begin{aligned} &SU(2) \qquad t^a = \frac{\sigma^a}{2} \quad ; \quad a = 1, 2, 3 \\ &SU(3) \qquad t^a = \frac{\lambda^a}{2} \quad ; \quad a = 1, \cdots, 8 \end{aligned} \tag{6.4}$$

where σ^a and λ^a are the Pauli and Gell-Mann matrices, respectively. In the representation associated with the fields ϕ_i, the matrices T^a are of dimension $(N \times N)$ and they form a representation of the Lie algebra, that is

$$[T^a, T^b] = i f^{abc} T^c \tag{6.5}$$

Its normalization is given by

$$Tr(T^a T^b) = \delta^{ab} T(R) \tag{6.6}$$

where $T(R)$ is a number that characterizes the representation R. For a given representation one can show the identity (see Problem 6.1),

$$T(R)\ r = d(R) C_2(R) \tag{6.7}$$

where r is the dimension of the group G and $d(R)$ is the dimension of the representation R. In an infinitesimal transformation

$$\delta\phi = -i\alpha^a T^a \phi \equiv -i\utilde{\alpha}\phi \tag{6.8}$$

where we have introduced the useful notation $\utilde{\alpha} \equiv \alpha^a T^a$.

6.1.2 Covariant derivative

In a local gauge theory we have the usual problem that the derivative does not transform as the fields, that is,

$$\partial_\mu \phi' \neq U \partial_\mu \phi \tag{6.9}$$

because the parameters depend on the coordinates, x^α. To solve this we introduce the covariant derivative,

$$D_\mu \phi = (\partial_\mu - ig\utilde{A}_\mu)\phi \qquad ; \qquad \utilde{A}_\mu = A^a_\mu T^a \tag{6.10}$$

where A^a_μ are the gauge fields, in equal number to the generators of the group. The transformation properties of A^a_μ are obtained requiring that $D_\mu\phi$ transforms in the same way as ϕ, that is,

$$\begin{aligned} (D_\mu\phi)' &= (\partial_\mu - ig\utilde{A}'_\mu)\phi' = (\partial_\mu - ig\utilde{A}'_\mu)U\phi \\ &= \partial_\mu U\phi + U\partial_\mu\phi - ig\utilde{A}'_\mu U\phi \\ &= UD_\mu\phi + (igU\utilde{A}_\mu - ig\utilde{A}'_\mu U + \partial_\mu U)\phi \end{aligned} \tag{6.11}$$

Therefore $(D_\mu\phi)' = U(D_\mu\phi)$ requires

$$\utilde{A}'_\mu = U\utilde{A}_\mu U^{-1} - \frac{i}{g}\partial_\mu U U^{-1} \tag{6.12}$$

For infinitesimal transformations $U \simeq 1 - i\utilde{\alpha}$ and we get

$$\delta\utilde{A}_\mu \equiv \utilde{A}'_\mu - \utilde{A}_\mu = -i\left[\utilde{\alpha}, \utilde{A}_\mu\right] - \frac{1}{g}\partial_\mu\utilde{\alpha} \tag{6.13}$$

This can be written in components

$$\delta A^a_\mu = -\frac{1}{g}\partial_\mu\alpha^a + f^{bca}\alpha^b A^c_\mu = -\frac{1}{g}(\partial_\mu\alpha^a - gf^{bca}\alpha^b A^c_\mu) \tag{6.14}$$

As in the adjoint representation, $(T^c)_{ab} = -if^{bca}$, we get

$$\delta A^a_\mu = -\frac{1}{g}\left(\partial_\mu\delta_{ab} - ig(T^c)_{ab}A^c_\mu\right)\alpha^b = -\frac{1}{g}(D_\mu\alpha)^a \tag{6.15}$$

that is, the gauge fields transform proportionally to the covariant derivative of the parameters of the gauge transformation.

6.1.3 Tensor $F_{\mu\nu}$

Let us calculate the commutator of two covariant derivatives,

$$\begin{aligned}
[D_\mu, D_\nu]\,\phi &= \left[\partial_\mu - ig\underset{\sim}{A}_\mu, \partial_\nu - ig\underset{\sim}{A}_\nu\right]\phi \\
&= -ig\left(\partial_\mu \underset{\sim}{A}_\nu - \partial_\nu \underset{\sim}{A}_\mu - ig\left[\underset{\sim}{A}_\mu, \underset{\sim}{A}_\nu\right]\right)\phi \\
&\equiv -ig\ \underset{\sim}{F}_{\mu\nu}\ \phi
\end{aligned} \tag{6.16}$$

We have defined the tensor $\underset{\sim}{F}_{\mu\nu} \equiv F^a_{\mu\nu} T^a$, known as curvature,

$$\underset{\sim}{F}_{\mu\nu} = \partial_\mu \underset{\sim}{A}_\nu - \partial_\nu \underset{\sim}{A}_\mu - ig\left[\underset{\sim}{A}_\mu, \underset{\sim}{A}_\nu\right] \tag{6.17}$$

In components

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{6.18}$$

which shows that it is a generalization of the Maxwell tensor. Let us see how $F_{\mu\nu}$ transforms under gauge transformations,

$$\begin{aligned}
\underset{\sim}{F}'_{\mu\nu} &= \partial_\mu \underset{\sim}{A}'_\nu - \partial_\nu \underset{\sim}{A}'_\mu - ig\left[\underset{\sim}{A}'_\mu, \underset{\sim}{A}'_\nu\right] \\
&= \left[\partial_\mu (U\underset{\sim}{A}_\nu U^{-1}) - \frac{i}{g}\partial_\mu(\partial_\nu U U^{-1}) - (\mu \leftrightarrow \nu)\right] \\
&\quad -igU[\underset{\sim}{A}_\mu, \underset{\sim}{A}_\nu]U^{-1} - \left[\partial_\mu U U^{-1}, U\underset{\sim}{A}_\nu U^{-1}\right] \\
&\quad - \left[U\underset{\sim}{A}_\mu U^{-1}, \partial_\nu U U^{-1}\right] + \frac{i}{g}\left[\partial_\mu U U^{-1}, \partial_\nu U U^{-1}\right]
\end{aligned} \tag{6.19}$$

Using

$$\partial_\mu U^{-1} = -U^{-1}\partial_\mu U U^{-1} \tag{6.20}$$

we get

$$\underset{\sim}{F}'_{\mu\nu} = U\underset{\sim}{F}_{\mu\nu} U^{-1} \tag{6.21}$$

For infinitesimal transformations this gives,

$$\delta \underset{\sim}{F}_{\mu\nu} = -i\left[\underset{\sim}{\alpha}, \underset{\sim}{F}_{\mu\nu}\right] \tag{6.22}$$

It easy to see that one can construct an invariant with the tensor $F_{\mu\nu}$. In fact the quantity,

$$\mathrm{Tr}(\underset{\sim}{F}'_{\mu\nu} \underset{\sim}{F}'^{\mu\nu}) = \mathrm{Tr}(\underset{\sim}{F}_{\mu\nu} \underset{\sim}{F}^{\mu\nu}) = \frac{1}{2} F^a_{\mu\nu} F^{a\mu\nu} \tag{6.23}$$

is an invariant and can be used to construct the action. Generalizing the Maxwell action for the Yang-Mills theories we get,

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\mathrm{Tr}(\underset{\sim}{F}_{\mu\nu} \underset{\sim}{F}^{\mu\nu}) = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \tag{6.24}$$

6.1.4 Choice of gauge

As we will see later in this chapter, gauge invariance can be used to choose some particular configuration, or gauge for the gauge fields. We call *pure gauge* to the field $\underset{\sim}{A}^\mu$ such that $\underset{\sim}{F}_{\mu\nu} = 0$. One can easily show that

$$F_{\mu\nu} = 0 \Longleftrightarrow \exists U : \underset{\sim}{A}_\mu = \partial_\mu U U^{-1} \tag{6.25}$$

Two important examples of gauge choices are, the *Axial gauge* defined by,

$$n^\mu A^a_\mu(x) = 0 \tag{6.26}$$

where n^μ is a constant four vector, and the *Lorenz gauge* defined by,

$$\partial^\mu A^a_\mu(x) = 0 \tag{6.27}$$

6.1.5 The action and the equations of motion

The action for the pure gauge theory (without matter fields), is

$$S = -\frac{1}{2}\int d^4x \mathrm{Tr}(\underset{\sim}{F}_{\mu\nu}\underset{\sim}{F}^{\mu\nu}) = -\frac{1}{4}\int d^4x F^a_{\mu\nu}F^{\mu\nu a} \tag{6.28}$$

and it is invariant under gauge transformations, because $\mathrm{Tr}(\underset{\sim}{F}_{\mu\nu}\underset{\sim}{F}^{\mu\nu})$ is. The Euler-Lagrange equations

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu A^a_\nu)} - \frac{\delta \mathcal{L}}{\delta A^a_\nu} = 0 \tag{6.29}$$

can be easily obtained noticing that we have

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu A^a_\nu)} = \frac{\delta \mathcal{L}}{\delta F^b_{\rho\sigma}}\frac{\delta F^b_{\rho\sigma}}{\delta(\partial_\mu A^a_\nu)} = -F^{a\mu\nu} \tag{6.30}$$

and

$$\frac{\delta \mathcal{L}}{\delta A^a_\nu} = \frac{\delta \mathcal{L}}{\delta F^b_{\rho\sigma}}\frac{\delta F^b_{\rho\sigma}}{\delta A^a_\nu} = g f^{bca} A^b_\mu F^{c\mu\nu} \tag{6.31}$$

We get therefore

$$\partial_\mu F^{\mu\nu a} + g f^{bca} A^b_\mu F^{\mu\nu c} = 0 \tag{6.32}$$

As we have in the adjoint representation, $(T^c)_{ab} = -i f^{bca}$, we get

$$(\partial_\mu \delta_{ab} - ig(T^c)_{ab} A^c_\mu) F^{\mu\nu b} = 0 \tag{6.33}$$

that is

$$D^{ab}_\mu F^{\mu\nu b} = 0 \tag{6.34}$$

which is the equivalent of the Maxwell equations in the absence of external sources. As in the Maxwell theory, from the antisymmetry of $F^{\mu\nu a}$ we can derive the Bianchi identities,

$$D^{ab}_\mu F_{\nu\rho b} + D^{ab}_\nu F_{\rho\mu b} + D^{ab}_\rho F_{\mu\nu b} = 0 \tag{6.35}$$

which are equivalent to the homogeneous Maxwell equations.

6.1.6 Energy–momentum tensor

As in the case of Electromagnetism, the canonical energy momentum tensor is not gauge invariant. In fact [1]

$$\begin{aligned}\tilde{\theta}^{\mu\nu} &= -\frac{\delta\mathcal{L}}{\delta(\partial_\mu A^a_\rho)}\partial^\nu A^a_\rho + g^{\mu\nu}\mathcal{L} \\ &= F^{\mu\rho a}\partial^\nu A^a_\rho - \frac{1}{4}g^{\mu\nu}F^{\rho\sigma a}F^a_{\rho\sigma} \end{aligned} \tag{6.36}$$

To make it gauge invariant we proceed as in the Electromagnetism. We subtract from $\tilde{\theta}^{\mu\nu}$ a quantity that is a four divergence, in such a way that the conservation laws are not changed. The relevant quantity is,

$$\begin{aligned}\Delta\theta^{\mu\nu} &= \partial_\rho(F^{\mu\rho a}A^{\nu a}) \\ &= \partial_\rho F^{\mu\rho a}A^{\nu a} + F^{\mu\rho a}\partial_\rho A^{\nu a} \\ &= g f^{bca}A^b_\rho F^{\rho\mu c}A^{\nu a} + F^{\mu\rho a}\partial_\rho A^{\nu a} \\ &= F^{\mu\rho a}(-F^{\nu a}_\rho + \partial^\nu A^a_\rho)\end{aligned} \tag{6.37}$$

We get therefore

$$\begin{aligned}\theta^{\mu\nu} &\equiv \tilde{\theta}^{\mu\nu} - \Delta\theta^{\mu\nu} \\ &= F^{\mu\rho a}F^{\nu a}_\rho - \frac{1}{4}g^{\mu\nu}F^{\rho\sigma a}F^a_{\rho\sigma}\end{aligned} \tag{6.38}$$

which is analogous to the electromagnetism. Introducing the analog of the electric and magnetic fields,

$$E^i_a = F^{i0}_a \ ; \ B^k_a = -\frac{1}{2}\varepsilon_{ijk}F^{ij}_a \qquad i,j,k = 1,2,3 \tag{6.39}$$

we get

$$\begin{cases}\theta^{00} = \frac{1}{2}(\vec{E}^a\cdot\vec{E}^a + \vec{B}^a\cdot\vec{B}^a) \\ \theta^{0i} = (\vec{E}^a\times\vec{B}^a)^i\end{cases} \tag{6.40}$$

with an interpretation similar to the case of the electromagnetism.

6.1.7 Hamiltonian formalism

From the component θ^{00} we get for the Hamiltonian

$$H = \int d^3x \frac{1}{2}(\vec{E}^a\cdot\vec{E}^a + \vec{B}^a\cdot\vec{B}^a) = \int d^3x\mathcal{H} \tag{6.41}$$

[1] One should note an overall sign difference with respect to the general definition of Eq. (1.75). This is to maintain the component θ^{00} with the meaning of a positive energy density. Obviously, the overall sign in Eq. (1.75), has no meaning prior to make contact with the model.

where $\mathcal{H}$ is the Hamiltonian density. We are now going to show that the relation between the Hamiltonian and the Lagrangian is not the usual one. For this it is convenient to write the action using the so-called first order formalism,

$$S = \int d^4x \left\{ -\frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu) F^{\mu\nu a} + \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \right\} \tag{6.42}$$

where A^a_μ e $F^a_{\mu\nu}$ are now taken as independent variables. It is easy to show that the variation of S in order to $F^a_{\mu\nu}$ gives back its definition,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\nu + g f^{abc} A^b_\mu A^c_\nu \tag{6.43}$$

and therefore if we substitute in S, Eq. 6.42, we get back the usual action. Using the definitions of $\vec{E}^a$ and $\vec{B}^a$ we get

$$\begin{aligned} S &= \int d^4x - (\partial^0 \vec{A}^a + \vec{\nabla} A^{0a} - g f^{abc} A^{0b} \vec{A}^c) \cdot \vec{E}^a - \frac{1}{2}(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a) \\ &= \int d^4x \left\{ -\partial^0 \vec{A}^a \cdot \vec{E}^a - \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + A^{0a}(\vec{\nabla} \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c) \right\} \end{aligned} \tag{6.44}$$

The Lagrangian density is then

$$\mathcal{L} = -E^{ka} \partial^0 A^{ka} - \mathcal{H}(E^{ka}, A^{ka}) + A^{0a} C^a \tag{6.45}$$

where

$$\begin{cases} \mathcal{H} \equiv \frac{1}{2}(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a) \\ B^{ka} \equiv -\frac{1}{2} \epsilon^{kmn} F^{mna} \\ C^a = \vec{\nabla} \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c \end{cases} \tag{6.46}$$

The variables A^a_k and $-E^a_k$ are canonical conjugate variables, $\mathcal{H}(E^a_k, A^a_k)$ is the Hamiltonian density. The variables A^{0a} play the role of Lagrange multipliers for the conditions,

$$\vec{\nabla} \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c = 0 \tag{6.47}$$

which are just the equations of motion for $\nu = 0$ (Gauss's Law). If we introduce an equal time Poisson bracket

$$\{A^{ia}(x), E^{jb}(y)\}_{x_0=y_0} = \delta^{ij} \delta^{ab} \delta^3(\vec{x} - \vec{y}) \tag{6.48}$$

one can show that

$$\{C^a(x), C^b(y)\}_{x_0=y_0} = -g f^{abc} C^c(x) \delta^3(\vec{x} - \vec{y}) \tag{6.49}$$

$$\{\mathcal{H}, C^a(x)\} = 0 \tag{6.50}$$

This shows that gauge theories, both abelian and non-abelian, are what it is known as *Hamiltonian Generalized Systems*, first introduced by Dirac.

To define these systems let us consider a system with canonical variables (p_i, q_i) that generate the phase space Γ^{2n} $(i = 1, \ldots, n)$. Then the action of the *Hamiltonian Generalized Systems* is given by,

$$S = \int L(t)dt \quad \text{where} \quad L(t) = \sum_{i=1}^{n} p_i \dot{q}_i - h(p,q) - \sum_{\alpha=1}^{m} \lambda^\alpha \varphi_\alpha(p,q) \tag{6.51}$$

The variables $\lambda^\alpha (\alpha = 1, ...m)$ are Lagrange multipliers and φ^α are the constraints. To be an Hamiltonian generalized system the following conditions should be verified,

$$\begin{aligned} \{\varphi^\alpha, \varphi^\beta\} &= \sum_\gamma f^{\alpha\beta\gamma}(p,q)\varphi^\gamma && (6.52) \\ \{h, \varphi^\alpha\} &= f^{\alpha\beta}(p,q)\varphi^\beta && (6.53) \end{aligned}$$

The case of gauge theories is a particular case with $f^{\alpha\beta} = 0$. Therefore to be able to quantize gauge theories we have to learn first how to quantize Hamiltonian generalized systems.

6.2 Quantization

6.2.1 Systems with n degrees of freedom

Let us consider an *Hamiltonian Generalized Systems*, described before. The Lagrangian is

$$L(t) = p_i \dot{q}_i - h(p,q) - \lambda^\alpha \varphi^\alpha(p,q) \tag{6.54}$$

which leads to the following equations of motion,

$$\begin{cases} \dot{q}_i = \dfrac{\partial h}{\partial p_i} + \lambda^\alpha \dfrac{\partial \varphi^\alpha}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial h}{\partial q_i} - \lambda^\alpha \dfrac{\partial \varphi^\alpha}{\partial q_i} \\ \varphi^\alpha(p,q) = 0 \qquad \alpha = 1, \ldots, m \end{cases} \tag{6.55}$$

One can show that an *Hamiltonian Generalized System*, (**HGS**) is equivalent to an usual Hamiltonian system (**HS**) defined in a phase space $\Gamma^{*2(n-m)}$. That is, a **HGS** it is equivalent to an **HS** with $n - m$ degrees of freedom. To prove this we construct explicitly the **HS** Γ^*. For this consider m conditions,

$$\chi^\alpha(p,q) = 0 \quad ; \quad \alpha = 1, \ldots, m \tag{6.56}$$

such that they satisfy,

$$\left\{\chi^\alpha, \chi^\beta\right\} = 0 \tag{6.57}$$

and

$$\det \left| \{\varphi^\alpha, \chi^\beta\} \right| \neq 0 \tag{6.58}$$

Then, the subspace of Γ^{2n} defined by the conditions

$$\left\{ \begin{array}{rcl} \chi^\alpha(p,q) & = & 0 \\ \varphi^\alpha(p,q) & = & 0 \end{array} \right. \qquad \alpha = 1,\ldots,m \tag{6.59}$$

is the space $\Gamma^{*2(n-m)}$. The canonical variables p^* and q^* in $\Gamma^{*2(n-m)}$ can be found in the following way. Due to the requirement $\left\{\chi^\alpha,\chi^\beta\right\}=0$ we can choose the variables q_i in Γ^{2n} in such a way that the χ^α coincide with the first m variables of the coordinate type, that is

$$\underbrace{q}_{n} \equiv (\underbrace{\chi^\alpha}_{m}, \underbrace{q^*}_{n-m}) \tag{6.60}$$

Let now $p=(p^\alpha,p^*)$ be the corresponding conjugate momenta. In these variables, the condition in the determinant takes the form,

$$\det\left|\frac{\partial\varphi^\alpha}{\partial p^\beta}\right| \neq 0 \tag{6.61}$$

This means that, in principle the conditions $\varphi^\alpha(p,q)=0$ can be solved for p^α, that is,

$$p^\alpha = p^\alpha(p^*,q^*) \ . \tag{6.62}$$

The subspace Γ^* it is therefore defined by the conditions,

$$\left\{ \begin{array}{rcl} \chi^\alpha & \equiv & q^\alpha = 0 \\ p^\alpha & = & p^\alpha(p^*,q^*) \end{array} \right. \ . \tag{6.63}$$

The variables p^* and q^* are canonical and the Hamiltonian is given by

$$h^*(p^*,q^*) = h(p,q)\left|_{(\chi=0\ ;\ \varphi=0)}\right. \ . \tag{6.64}$$

The equations of motion are now

$$\dot{q}^* = \frac{\partial h^*}{\partial p^*} \qquad \dot{p}^* = -\frac{\partial h^*}{\partial q^*} \ , \tag{6.65}$$

in a total of $2(n-m)$ equations. The fundamental result can be formulated in a form of a theorem.

Theorem 6.1
The two representations lead to the same equations of motion and are therefore equivalent.

Proof:

The relations $q^\alpha=0 \Longrightarrow \dot{q}^\alpha=0$, which means, in the (p,q) description,

$$\frac{\partial h}{\partial p_\alpha} + \lambda^\beta\frac{\partial\varphi_\beta}{\partial p_\alpha} = 0 \quad ; \quad \alpha=1,\ldots,m \ . \tag{6.66}$$

Let us now consider the equations of motion in for the coordinates q^* in the two representations

$$
\begin{aligned}
\dot{q}^* &= \frac{\partial h}{\partial p^*} + \lambda^\alpha \frac{\partial \varphi_\alpha}{\partial p^*} \\
\dot{q}^* &= \frac{\partial h^*}{\partial p^*} = \frac{\partial h}{\partial p^*} + \frac{\partial h}{\partial p^\alpha}\frac{\partial p_\alpha}{\partial p^*}
\end{aligned}
\tag{6.67}
$$

The two equations will be equivalent if

$$
\lambda^\alpha \frac{\partial \varphi_\alpha}{\partial p^*} = \frac{\partial h}{\partial p^\alpha}\frac{\partial p_\alpha}{\partial p^*}
\tag{6.68}
$$

which means, using the previous relations,

$$
\lambda^\alpha \left(\frac{\partial \varphi_\alpha}{\partial p^*} + \frac{\partial \varphi_\alpha}{\partial p_\beta}\frac{\partial p_\beta}{\partial p^*} \right) = 0
\tag{6.69}
$$

But this relation is true due to the constraints $\varphi_\alpha = 0$. Therefore the two representations are equivalent which proves the theorem (the equations for p^* could be treated in a similar way).

If we want to quantize these systems we can use the expressions for the evolution operator in terms of a path integral in the variables (p^*, q^*) as these correspond to an usual Hamiltonian system. We have then,

$$
U(q_f^*, q_i^*) = \int \prod_t \frac{dp^* dq^*}{(2\pi)} e^{i\int [p^*\dot{q}^* - h(p^*,q^*)]dt} \ .
\tag{6.70}
$$

Although this is a possible way of quantizing the theory, it is normally not very convenient in most situations. This is because in real situations it is difficult to invert the relations $\varphi^\alpha = 0$ to get $p^\alpha = p^\alpha(p^*, q^*)$. It is normally more convenient to use the variables (p, q) with appropriate restrictions. This can be easily done substituting,

$$
\prod_t \frac{dp^* dq^*}{(2\pi)} \to \prod_t \frac{dpdq}{2\pi} \prod_t \delta(q^\alpha)\delta(p^\alpha - p^\alpha(p^*, q^*)) \ .
\tag{6.71}
$$

Then

$$
U(q_f, q_i) = \int \prod_t \frac{dpdq}{2\pi} \prod_t \delta(q^\alpha)\delta(p^\alpha - p^\alpha(p^*, q^*)) e^{i\int dt(p\dot{q} - h(p,q))} \ .
\tag{6.72}
$$

This expression can be written in terms of the constraints if we recall that

$$
\begin{aligned}
&\delta(q^\alpha) = \delta(\chi^\alpha) \\
&\delta(p^\alpha - p^\alpha(p^*, q^*)) = \delta(\varphi^\alpha) \det \left| \frac{\partial \varphi_\alpha}{\partial p_\beta} \right| \ .
\end{aligned}
\tag{6.73}
$$

Then we get

$$
\prod_t \delta(q^\alpha)\delta(p^\alpha - p^\alpha(p^*, q^*)) = \prod_t \delta(\varphi^\alpha)\delta(\chi^\alpha) \det |\{\varphi_\alpha, \chi_\beta\}| \ .
\tag{6.74}
$$

Finally we use the identity

$$\delta(\varphi^\alpha) = \int \frac{d\lambda}{2\pi} e^{-i\int dt \lambda^\alpha \varphi_\alpha} \ , \tag{6.75}$$

to get

$$U(q_f, q_i) = \int \prod_t \frac{dpdq}{2\pi} \frac{d\lambda}{2\pi} \prod_{t,x} \delta(\chi^\alpha) \det |\{\varphi^\alpha, \chi_\beta\}| \, e^{iS(p,q,\lambda)} \ , \tag{6.76}$$

where

$$S(p, q, \lambda) = \int [p\dot{q} - h(p, q) - \lambda\varphi] dt \ . \tag{6.77}$$

It will be this expression in Eq. (6.76) that we will apply to the gauge theories. Note that the expression in the parenthesis is precisely the Lagrangian for generalized Hamiltonian systems, Eq. (6.54). It can be shown that the physical results do not depend on the auxiliary conditions $\chi^\alpha = 0$. In gauge theories these are known as the *gauge choice.*

6.2.2 QED as a simple example

Let us consider the electromagnetic field coupled to an external conserved current, $J^\mu = (\rho, \vec{J})$, with $\partial_\mu J^\mu = 0$. The Lagrangian is,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu \ . \tag{6.78}$$

The action can be written using the first order formalism,

$$S = \int d^4x \left[-\vec{E} \cdot (\vec{\nabla} A^0 + \dot{\vec{A}}) - \vec{B} \cdot \vec{\nabla} \times \vec{A} + \frac{\vec{B}^2 - \vec{E}^2}{2} - \rho A^0 + \vec{J} \cdot \vec{A} \right] \ . \tag{6.79}$$

The equations of motion are obtained by varying with respect to $\vec{E}$ and $\vec{B}$

$$\left\{ \begin{array}{rcl} \vec{E} &=& -(\vec{\nabla} A^0 + \dot{\vec{A}}) \\ \vec{B} &=& \vec{\nabla} \times \vec{A} \end{array} \right. \rightarrow \left\{ \begin{array}{rcl} \vec{\nabla} \cdot \vec{B} &=& 0 \ , \\ \vec{\nabla} \times \vec{E} &=& -\dfrac{\partial \vec{B}}{\partial t} \ . \end{array} \right. \tag{6.80}$$

Varying with respect to A^0 and $\vec{A}$,

$$\left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{E} = \rho \ , \\ \vec{\nabla} \times \vec{B} - \dfrac{\partial \vec{E}}{\partial t} = \vec{J} \ . \end{array} \right. \tag{6.81}$$

If we substitute $\vec{B} = \vec{\nabla} \times \vec{A}$ we get after an integration by parts,

$$S = \int d^4x \left\{ -\vec{E} \cdot \dot{\vec{A}} - \left(\frac{\vec{E}^2 + (\vec{\nabla} \times A)^2}{2} - \vec{J} \cdot \vec{A} \right) + A^0 (\vec{\nabla} \cdot \vec{E} - \rho) \right\} \ . \tag{6.82}$$

It is clear that A^0 plays the role of a Lagrange multiplier. The canonical variables are $\vec{A}$ and $\vec{E}$ but they are not free, as there exists one constraint to be obeyed, $\vec{\nabla} \cdot \vec{E} = \rho$. This constraint is linear in the fields. Here resides the big simplification of the electromagnetism. If we choose a linear gauge condition, then $\det\{\varphi^\alpha, \chi_\beta\}$ will not depend either in $\vec{E}$ or $\vec{A}$ and will be a constant that only will modify the normalization. Such a gauge condition is obtained, for instance, with the *Lorenz gauge*

$$\chi = \partial_\mu A^\mu - c(\vec{x}, t) \ , \tag{6.83}$$

where $c(\vec{x}, t)$ is an arbitrary function. Then the generating functional for Green functions is (the term that comes from $\det\{\varphi^\alpha, \chi_\beta\}$ is absorbed in the normalization)

$$Z[J^\mu] = \mathcal{N} \int \mathcal{D}(\vec{E}, \vec{A}, A^0) \prod_x \delta(\partial_\mu A^\mu - c(x)) e^{iS} \ , \tag{6.84}$$

where

$$\begin{aligned} S &= \int d^4x \left\{ -\vec{E} \cdot \dot{\vec{A}} - \left[\frac{E^2 + (\vec{\nabla} \times A)^2}{2} + (\vec{J} \cdot \vec{A}) \right] + A^0 (\vec{\nabla} \cdot \vec{E} - \rho) \right\} \\ &= \int d^4x \left\{ -\frac{E^2}{2} - \vec{E} \cdot (\vec{\nabla} A^0 + \dot{\vec{A}}) - \frac{(\vec{\nabla} \times A)^2}{2} - J_\mu A^\mu \right\} \ . \end{aligned} \tag{6.85}$$

The integration in $\vec{E}$ is Gaussian and can be done immediately (we keep the notation $\mathcal{N}$ although this normalization will be different after the integration)

$$Z[J^\mu] = \mathcal{N} \int \mathcal{D}(A_\mu) \prod_x \delta(\partial_\mu A^\mu - c(x)) e^{iS} \ , \tag{6.86}$$

where now

$$\begin{aligned} S &= \int d^4x \left[-\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - J_\mu A^\mu \right] \\ &= \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \right] \ . \end{aligned} \tag{6.87}$$

As the functions $c(x)$ are arbitrary we can integrate over them with a weight

$$\exp\left(-\frac{1}{2\xi} \int d^4x c^2(x) \right) \ . \tag{6.88}$$

We get then the familiar result,

$$Z[J^\mu] = \mathcal{N} \int \mathcal{D}(A_\mu) e^{i \int d^4x \left[-\frac{1}{4} F^2 - \frac{1}{2\xi} (\partial \cdot A)^2 - J \cdot A \right]} \ . \tag{6.89}$$

As we will see later, if we had chosen a non-linear gauge condition the $\det |\{q, \chi\}|$ would depend on $\vec{E}$ or $\vec{A}$ and it would not be possible to absorb it in the normalization (which is irrelevant as we can always choose it such that $Z[0] = 1$). In that case it would be necessary to use the methods of non-abelian gauge theories that we will discuss in the next section.

6.2.3 Non abelian gauge theories. Non covariant gauges

We have seen before that the action for the non-abelian gauge theories, could be written in the form,

$$S = -2\int d^4x \ \text{Tr}\left[\vec{\underset{\sim}{E}}\cdot\partial^0\vec{\underset{\sim}{A}} + \frac{1}{2}(\vec{\underset{\sim}{E}}^2 + \vec{\underset{\sim}{B}}^2) - \underset{\sim}{A}^0(\vec{\nabla}\cdot\vec{\underset{\sim}{E}} + g[\vec{\underset{\sim}{A}},\vec{\underset{\sim}{E}}])\right] \tag{6.90}$$

$$= \int d^4x \left[-E_k^a\partial^0 A_k^a - \mathcal{H}(E_k, A_k) + A^{0a}C^a\right] \ , \tag{6.91}$$

where

$$C^a = \vec{\nabla}\cdot\vec{E}^a - gf^{abc}\vec{A}^b\cdot\vec{E}^c \ . \tag{6.92}$$

If we introduce the equal time Poisson brackets

$$\left\{-E_a^i(x), A_b^j(y)\right\}_{x^0=y^0} = \delta^{ij}\delta_{ab}\delta^3(\vec{x}-\vec{y}) \ , \tag{6.93}$$

one can then show that

$$\begin{aligned}
&\left\{C^a(x), C^b(y)\right\}_{x_0=y_0} = -gf^{abc}C^c(x)\delta^3(\vec{x}-\vec{y}) \\
&\{H, C^a(x)\} = 0 \ ,
\end{aligned} \tag{6.94}$$

where

$$H = \int d^3x\mathcal{H}(E_k, A_k) = \frac{1}{2}\int d^3x\left[(E^{ka})^2 + (B^{ka})^2\right] \ . \tag{6.95}$$

We see then that the non-abelian gauge theories are an example of generalized Hamiltonian systems, like we saw with the electromagnetism. The variables of the type coordinate are A_k^a, and the conjugate momenta are $-E_k^a$. The variables A^{0a} are Lagrange multipliers for the constraints,

$$\vec{\nabla}\cdot\vec{E}^a - gf^{abc}\vec{A}^b\cdot\vec{E}^c = 0 \ , \tag{6.96}$$

which are part of the equations of motion.

To proceed with the quantization we have to use the formalism of the **HGS**. For that we have to impose r auxiliary conditions (where r is the dimension of the Lie group and therefore of its adjoint representation where the gauge fields are), that is, as many as the constraints $C^a(x) = 0$, $a = 1, \ldots, r$. Choose these conditions is what is known as choosing or fixing the gauge. This choice is arbitrary, and the physical results should not depend on it. However intermediate expressions as, for instance, the Feynman rules can depend on the choice.

As we saw in the case of the electromagnetism if it is possible to make a gauge fixing that is linear in the dynamical variables, $\vec{A}^a$ and $\vec{E}^a$, then the path integral will simplify because the determinant will not depend on these variables and can be absorbed into the normalization. For the non-abelian case, a gauge where this is possible is the axial gauge that we now study.

Axial Gauge

It is always possible to make a gauge transformation such that the component of $\vec{A}^a$ along some direction vanishes in all points, that is,

$$A^{3a} = 0 \qquad\qquad a = 1, \ldots, r \ , \tag{6.97}$$

where we have chosen the direction along the z axis. These r conditions are our auxiliary conditions necessary to proceed with the quantization of the theory. The advantage of this gauge choice is the following. If we calculate $\{C^a, A^{3b}\}$ we get

$$\begin{aligned}\{C_a(x), A_b^3(y)\} &= \{\partial_k E_a^k(x), A_b^3(y)\} - g f_{adc} A_d^k \{E_c^k(x), A_b^3(y)\} \\ &= -g\ \delta_{ab} \frac{\partial}{\partial x^3} \delta^3(\vec{x} - \vec{y}) = \frac{\delta}{\delta \alpha^a(x)} (\delta A_b^3(y)) \ ,\end{aligned} \tag{6.98}$$

where we have used the fact that $A_b^3 = 0$. We see then that $\{C^a, A_b^3\}$ does not depend on $\vec{A}_a$ and $\vec{E}_a$ and the determinant that appears in the expression for the path integral can be absorbed in the normalization. We can then write the generating functional for the Green functions in this gauge as

$$Z[J^{\mu a}] = \int \mathcal{D}(\vec{E}, \vec{A}, A^0) \prod_x \delta(A^3) e^{iS(\vec{E}, \vec{A}, A^0, J^\mu)} \ , \tag{6.99}$$

where

$$S(\vec{E}, \vec{A}, A^0, J^\mu) = \int d^4x \left[-\vec{E}^a \cdot \partial^0 \vec{A}^a - \frac{1}{2} \left[(\vec{E}^a)^2 + (\vec{B}^a)^2 \right] + A^{0a} C^a + A^a \cdot J^a \right] \ , \tag{6.100}$$

and

$$C^a = \vec{\nabla} \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c \ . \tag{6.101}$$

As the integration in $\vec{E}$ is Gaussian we can easily get

$$Z_A[J^{\mu a}] = \int \mathcal{D}(A^\mu) \prod_x \delta(A^3) e^{i \int d^4x \left[\mathcal{L}(x) + A^a \cdot J^a \right]} \ . \tag{6.102}$$

After the integration the Lagrangian is then

$$\mathcal{L} = -\frac{1}{4} (F_{\mu\nu}^a F^{a\mu\nu}) \ . \tag{6.103}$$

The index A in $Z_A[J^{\mu a}]$ reminds us that this generating functional corresponds to the axial gauge. Although the expression for the generating functional can be easily written in this gauge it has the disadvantage that the Feynman rules are not covariant. Before we introduce the covariant gauges, let us look at another non-covariant gauge, the so-called *Coulomb gauge*.

Coulomb Gauge

This gauge is defined by the auxiliary conditions,

$$\vec{\nabla} \cdot \vec{A}_a = 0 \qquad\qquad a = 1, \ldots, r \; . \tag{6.104}$$

These auxiliary conditions have a non-trivial Poisson bracket with the constraints $C^a(x)$. In fact one can show (see problems) that

$$\delta \vec{A}_a = -\frac{1}{g} \int d^3y \left\{ \vec{A}_a(x), \alpha^b(y) C_b(y) \right\}_{x_0=y_0} \; . \tag{6.105}$$

Therefore

$$\left\{ \vec{A}_a(x), C_b(y) \right\}_{x_0=y_0} = -g \frac{\delta}{\delta \alpha_b(y)} (\delta \vec{A}_a(x)) \; , \tag{6.106}$$

and

$$\left\{ \vec{\nabla} \cdot \vec{A}_a(x), C_b(y) \right\}_{x_0=y_0} = -g \frac{\delta}{\delta \alpha_b(y)} \vec{\nabla} \cdot (\delta \vec{A}_a(x)) \; . \tag{6.107}$$

As we have

$$\delta \vec{A}_a(x) = \frac{1}{g} \vec{\nabla} \alpha_a(x) + f_{abc} \alpha^b(x) \vec{A}^c(x) \; , \tag{6.108}$$

we get (with the condition $\vec{\nabla} \cdot \vec{A}_a = 0$)

$$-\, g \vec{\nabla} \cdot (\delta \vec{A}_a(x)) = -\nabla_x^2 \alpha_a(x) - g f_{abc} \vec{A}_c(x) \cdot \vec{\nabla} \alpha_b(x) \; . \tag{6.109}$$

This gives finally

$$\left\{ \vec{\nabla} \cdot \vec{A}_a(x), C_b(y) \right\} = \left[-\nabla_x^2 \delta_{ab} - g f_{abc} \vec{A}_c(x) \cdot \vec{\nabla}_x \right] \delta^3(\vec{x} - \vec{y}) \equiv \mathcal{M}^c_{ab}(x,y) \; . \tag{6.110}$$

As $\det \mathcal{M}$, although depending on $\vec{A}$ it does not depend on $\vec{E}$, the Gaussian integration in $\vec{E}$ can still be done and we get

$$Z_C[J^{\mu a}] = \int \mathcal{D}(A_\mu) \prod_x [\det \mathcal{M}_C \prod_x \delta(\vec{\nabla} \cdot \vec{A})] e^{i \int d^4x [\mathcal{L} + A^a \cdot J^a]} \; . \tag{6.111}$$

Now it is not possible to absorb $\det \mathcal{M}_\mathcal{C}$ in the normalization. The Feynman rules that can be obtained from $Z_C[J^\mu]$ are again non-covariant.

6.2.4 Non abelian gauge theories in covariant gauges

The gauge conditions chosen up to now (axial and Coulomb gauges) lead to Feynman rules where the Lorentz covariance is lost. Of course the final physical results should not depend on this, but the non-covariance in the intermediate stages of the calculations is a complication. We are now going to generalize the previous results to covariant gauges. The method to follow will be a sub-product of the answer to the following question: How can we show the equivalence between the axial and Coulomb gauges? For the argument that follows it is convenient to work with gauge invariant quantities. Then, instead of the functional $Z_A[J^\mu]$ we are going to consider the integral $Z_A[J=0]$ that, as we have seen, has the meaning of a vacuum $\to$ vacuum transition in the absence of external sources,

$$Z_A[0] = \int \mathcal{D}(A_\mu) \prod_{x,a} \delta(A^{3a}(x)) \exp\{iS[A_\mu]\} \ , \tag{6.112}$$

where $S[A_\mu]$ is the action. In a gauge transformation,

$$\underset{\sim}{A}_\mu \to \underset{\sim}{A}'_\mu = {}^g\underset{\sim}{A}_\mu = U(g)\underset{\sim}{A}_\mu U^{-1}(g) - \frac{i}{g}\partial_\mu U U^{-1} \ , \tag{6.113}$$

the action $S[A_\mu]$ and the measure $\mathcal{D}(A_\mu)$ are invariant, therefore we get

$$Z_A[J=0] = \int \mathcal{D}(A_\mu) \prod_{x,a} \delta({}^gA^{3a}(x)) \exp\{iS[A_\mu]\} \ . \tag{6.114}$$

We define now the functional $\Delta_C[A_\mu]$ through the relation

$$\Delta_C^{-1}[A_\mu] = \int \mathcal{D}(g) \prod_{x,a} \delta(\vec{\nabla} \cdot {}^g\vec{A}^a) \ , \tag{6.115}$$

where $\mathcal{D}(g)$ represents the infinite product of the invariant measures for the group G at each space-time point, that is

$$\mathcal{D}(g) = \prod_x dg(x) \ . \tag{6.116}$$

The invariance of the integration measure of the group G, $\mathcal{D}g' = \mathcal{D}(gg')$ has the consequence that Δ_C is gauge invariant. In fact

$$\begin{aligned} \Delta_C^{-1}[{}^gA_\mu] &= \int \mathcal{D}(g') \prod_{x,a} \delta(\vec{\nabla} \cdot {}^{g'g}\vec{A}^a) \\ &= \int \mathcal{D}(g'g) \prod_{x,a} \delta(\vec{\nabla} \cdot {}^{g'g}\vec{A}^a) \\ &= \Delta_C^{-1}[A_\mu] \ . \end{aligned} \tag{6.117}$$

We introduce now in the expression for $Z_A[J=0]$ the identity

$$1 = \Delta_C[A_\mu] \int \mathcal{D}(g) \prod_{x,a} \delta(\vec{\nabla} \cdot {}^g\vec{A}^a) \ . \tag{6.118}$$

We therefore get

$$\begin{aligned} Z_A(J=0) &= \int \mathcal{D}A_\mu e^{iS[A_\mu]} \prod_{x,a} \delta\left(A^{3a}(x)\right) \Delta_C[A_\mu] \int \mathcal{D}(g) \prod_{y,b} \delta(\vec{\nabla} \cdot {}^g\vec{A}^b) \\ &= \int \mathcal{D}A_\mu e^{iS[A_\mu]} \Delta_C[A_\mu] \prod_{y,b} \delta\left(\vec{\nabla} \cdot \vec{A}^b\right) \int \mathcal{D}(g) \prod_{x,a} \delta({}^{g^{-1}}A^{3a}) \ , \end{aligned} \tag{6.119}$$

where we have used the gauge invariance of $\mathcal{D}$, $S[A_\mu]$ and $\Delta_C[A_\mu]$. As the measure is invariant, we can write in the last integral $g^{-1} \to gg_0$. Then

$$\int \mathcal{D}(g) \prod_{x,a} \delta\left({}^{g^{-1}}\!A^{3a}(x)\right) = \int \mathcal{D}(g) \prod_{x,a} \delta\left({}^{gg_0}\!A^{3a}(x)\right) \ . \tag{6.120}$$

where g_0 is the gauge transformation that takes from the gauge $\vec{\nabla} \cdot \vec{A} = 0$ to the gauge $A'^3 = 0$, that is

$$A'^3 = {}^{g_0}\!A^3 = 0 \ , \tag{6.121}$$

with $\vec{\nabla} \cdot \vec{A} = 0$. We still have to calculate the integral over the group, that now takes the form,

$$\int \mathcal{D}(g) \prod_{x,a} \delta\left({}^{g}\!A'^{3a}(x)\right) \ , \tag{6.122}$$

with $A'^{3a} = 0$. As $A'^{3a} = 0$ it is enough to consider infinitesimal transformations in the vicinity of the unit,

$$g(x) = e - i\underset{\sim}{\alpha}(x) = e - i\alpha^a(x)t^a \ , \tag{6.123}$$

where $\alpha^a(x)$ are infinitesimal. In these conditions, the integration measure $dg(x)$ is given by,

$$dg(x) = \prod_a d\alpha^a(x) \ . \tag{6.124}$$

On the other hand to first order in α^a we have

$${}^{g}\!A'^{3a}(x) = \frac{1}{g}\frac{\partial \alpha^a}{\partial x^3} \ , \tag{6.125}$$

and therefore the integral is now

$$\int \mathcal{D}(g) \prod_{x,a} \delta\left({}^{g}\!A'^{3a}(x)\right) = \int \mathcal{D}(\alpha) \prod_{x,a} \delta\left(\frac{1}{g}\frac{\partial \alpha^a}{\partial x^3}\right) = \mathcal{N} \ . \tag{6.126}$$

The integral is independent of A_μ and therefore it can be absorbed in the normalization. We get then

$$Z_A[J=0] = \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_C[A_\mu] \prod_{x,b} \delta(\vec{\nabla} \cdot \vec{A}^b) e^{iS[A_\mu]} \ . \tag{6.127}$$

We have obtained before an expression for $Z_C[J=0]$, which was,

$$Z_C[J=0] = \int \mathcal{D}(A_\mu) \prod_x \det \mathcal{M}_C \prod_{x,b} \delta(\vec{\nabla} \cdot \vec{A}^b) e^{iS[A_\mu]} \ . \tag{6.128}$$

Therefore to show that the two path integrals, that represent the the *vacuum* $\to$ *vacuum* amplitudes in the absence of external sources, we still have to show that $\Delta_C[A_\mu] = \det \mathcal{M}_C$. This can be easily shown true. In fact,

$$\Delta_C^{-1}[A_\mu] \ = \ \int \mathcal{D}(g) \prod_{x,a} \delta\left(\vec{\nabla} \cdot {}^{g}\!\vec{A}^a\right)$$

$$
\begin{aligned}
&= \int \mathcal{D}(\alpha) \prod_{x,a} \delta \left[\vec{\nabla} \cdot \left(\frac{1}{g} \vec{\nabla} \alpha^a(x) + f^{abc} \alpha^b \vec{A}^c \right) \right] \\
&= \int \mathcal{D}(\alpha) \prod_{x,a} \delta \left(\frac{1}{g} \nabla_x^2 \alpha^a(x) + f^{abc} \vec{\nabla} \alpha^b \cdot \vec{A}^c \right) \\
&\propto \det{}^{-1} \mathcal{M}_C \ , \qquad (6.129)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{M}_C^{ab}(x,y) &= -g \frac{\delta}{\delta \alpha^b(y)} \left(\vec{\nabla} \cdot {}^g\vec{A}^a \right)_{\alpha=0} \\
&= \left(-\nabla_x^2 \delta_{ab} - g f^{abc} \vec{A}_c \cdot \vec{\nabla}_x \right) \delta^3(\vec{x} - \vec{y}) \ . \qquad (6.130)
\end{aligned}
$$

Therefore $\Delta_C[A_\mu] \propto \det \mathcal{M}_C$, and except for an irrelevant normalization we have, $Z_A[0] = Z_C[0]$.

The way we have shown this equivalence between the axial and Coulomb gauges suggests a way to define the *vacuum* $\rightarrow$ *vacuum* amplitude for an arbitrary gauge defined by the gauge conditions,

$$
F^a[A_\mu] = 0 \qquad\qquad a = 1, \ldots, r \qquad (6.131)
$$

For that we define $\Delta_F[A_\mu]$ by the expression

$$
\Delta_F^{-1}[A_\mu] = \int \mathcal{D}(g) \prod_{x,a} \delta \left(F^a[{}^gA_\mu] \right) \qquad (6.132)
$$

and, like before, we introduce

$$
1 = \Delta_F[A_\mu] \int \mathcal{D}(g) \prod_{x,a} \delta \left(F^a[{}^gA_\mu] \right) \ , \qquad (6.133)
$$

in the expression for $Z_A[J=0]$. We therefore get

$$
\begin{aligned}
Z_A[J=0] =& \int \mathcal{D}(A_\mu) \prod_{x,a} \delta \left(A^{3a}(x) \right) e^{iS[A_\mu]} \Delta_F[A_\mu] \int \mathcal{D}(g) \prod_{y,b} \delta \left(F^b[{}^g A_\mu] \right) \\
=& \int \mathcal{D}(A_\mu) \prod_{y,b} \delta \left(F^b[A_\mu] \right) \Delta_F[A_\mu] e^{iS[A_\mu]} \int \mathcal{D}(g) \prod_{x,a} \delta \left({}^{g^{-1}} A^{3a}(x) \right) \\
=& \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_F[A_\mu] \prod_{x,a} \delta \left(F^b[A_\mu] \right) e^{iS[A_\mu]} \\
=& \mathcal{N} Z_F[J=0] \ , \qquad (6.134)
\end{aligned}
$$

showing that the axial gauge and general gauges of the type F are equivalent. The *vacuum* $\rightarrow$ *vacuum* amplitude in the gauge $F^a = 0$, is therefore given by

$$
Z_F[J=0] = \int \mathcal{D}(A_\mu) \Delta_F[A_\mu] \prod_{x,a} \delta \left(F^a[A_\mu] \right) e^{iS[A_\mu]} \ . \qquad (6.135)
$$

To finish we still have to be able to evaluate $\Delta_F[A_\mu]$. As in the definition $\Delta_F[A_\mu]$ appears multiplied by $\prod \delta\left(F^a[A_\mu]\right)$, we only need to know $\Delta_F[A_\mu]$ for A_μ that satisfy $F^a[A_\mu]=0$. Then for g in the vicinity of the identity we have

$$\begin{aligned} F^a[{}^g\!A^b_\mu] &= F^a[A^b_\mu] + \frac{\delta F^a}{\delta A^b_\mu}\delta A^b_\mu \\ &= -\frac{1}{g}\,\frac{\delta F^a}{\delta A^b_\mu}\left(D_\mu\alpha\right)^b \ , \end{aligned} \tag{6.136}$$

where we have used $F^a[A^b_\mu]=0$ and $\delta A^b_\mu = -\frac{1}{g}(D_\mu\alpha)^b$. Let us calculate Δ_F. We get

$$\begin{aligned} \Delta_F^{-1}[A_\mu] &= \int \mathcal{D}(g)\prod_{x,a}\delta\left(F^a[{}^g\!A^b_\mu]\right) \\ &= \int \mathcal{D}(\alpha)\prod_{x,a}\delta\left(-\frac{1}{g}\frac{\delta F^a}{\delta A^b_\mu}(D_\mu\alpha)^b\right) \\ &\propto \det{}^{-1}\mathcal{M}_F \ , \end{aligned} \tag{6.137}$$

where

$$\mathcal{M}_F^{ab}(x,y) = \frac{\delta F^a}{\delta A^c_\mu(x)} D^{cb}_\mu \delta^4(x-y) = -g\frac{\delta F^a[{}^g A(x)]}{\delta\alpha^b(y)} \ , \tag{6.138}$$

and therefore

$$\Delta_F[A_\mu] = \det\mathcal{M}_F = \det\left(-g\frac{\delta F^a(x)}{\delta(\alpha^b(y))}\right) \ . \tag{6.139}$$

We have discovered how to write the *vacuum* $\to$ *vacuum* amplitude in the absence of external sources. However this is not the more interesting quantity, but rather the *vacuum* $\to$ *vacuum* amplitude in the presence of sources, $Z_F[J]$ because this is the one that generates the Green functions of the theory. In all this discussion the source terms, $\int d^4x J^a_\mu A^{\mu a}$, were put to zero because they are not gauge invariant, and our derivation relied upon gauge invariance. If we define $Z_F[J^a_\mu]$ by the relation

$$Z_F[J^a_\mu] \equiv \int \mathcal{D}(A_\mu)\Delta_F[A_\mu]\prod_{x,a}\delta(F^a[A^b_\mu(x)])e^{i(S[A_\mu]+\int d^4x J^a_\mu A^{\mu a})} \ , \tag{6.140}$$

then it is clear that the functional Z_F will not be equivalent for different choices of $F^a=0$. This means that the Green functions obtained from $Z_F[J^\mu]$ will depend on the gauge $F^a=0$. In the section 6.2.5 we will show that although the Green functions depend on the gauge, this is not really a problem, because the physical results for the elements of the renormalized S matrix are gauge independent and these are the ones that we compare with the experiments.

Before we finish let us make a transformation in the functional $Z_F[J^a_\mu]$ to get rid of the δ function. For the calculations it is important to exponentiate $\prod\delta(F^a[A_\mu])$. This can be done in the following way. We start by defining a more general gauge condition,

$$F^a[A^b_\mu] - c^a(x) = 0 \ , \tag{6.141}$$

where $c^a(x)$ are arbitrary functions of space-time but that do not depend on the fields. Then $\Delta_F[A]$ will not be changed and we write,

$$Z_F[J_\mu^a] = \mathcal{N}\int \mathcal{D}(A_\mu)\Delta_F[A_\mu]\prod \delta(F^a[A_\mu]-c^a)e^{i(S[A_\mu]+\int d^4x J_\mu^a A^\mu)} \ . \tag{6.142}$$

The left side of this equation does not depend on $c^a(x)$ and therefore we can integrate over $c^a(x)$ with a convenient weight, more specifically,

$$\exp\left\{-\frac{i}{2}\int d^4x\ c_a^2(x)\right\} \ , \tag{6.143}$$

where x is a real parameter. We finally get

$$\begin{aligned} Z_F[J_\mu^a] &= \mathcal{N}\int \mathcal{D}(A_\mu)\Delta_F[A_\mu]e^{i\left(S[A_\mu]+\int d^4x\left(-\frac{1}{2}F_a^2+J^{\mu a}A_\mu^a\right)\right)} \\ &= \mathcal{N}\int \mathcal{D}(A_\mu)\Delta_F[A_\mu]e^{i\int d^4x[\mathcal{L}(x)-\frac{1}{2}F_a^2+J^{\mu a}A_\alpha^a]} \ . \end{aligned} \tag{6.144}$$

This expression is the starting point for the calculation of the Green functions in an arbitrary gauge defined by the gauge fixing F^a. To be able to establish the Feynman rules for this theory we still have to exponentiate $\Delta_F[A_\mu]$. This will be done in the section 6.2.6 with the introduction of the Faddeev-Popov *ghosts*.

6.2.5 Gauge invariance of the S matrix

In the previous section we have defined the generating functional for the Green functions, $Z_F[J_\mu^a]$, for a gauge condition given by the function, $F^a[A_\mu^b]$, through the relation,

$$Z_F[J_\mu^a] \equiv \mathcal{N}\int \mathcal{D}(A_\mu)\Delta_F[A]\prod_{x,a}\delta(F^a[A_\mu^b(x)])e^{i(S[A_\mu]+\int d^4x J_\mu^a A^{\mu a})} \ . \tag{6.145}$$

We have shown the equivalence between different gauges in the case of vanishing sources. We are now going to show what happens when $J_\mu^a \neq 0$. For this we will go back and redo the proof of the equivalence in the presence of the source terms. We choose for this, the case of the Coulomb and Lorenz gauges defined by

$$\left\{\begin{array}{lcll} F^a &=& \vec{\nabla}\cdot\vec{A}^a & \textit{Coulomb gauge} \\ \\ F^a &=& \partial_\mu A^{\mu a} & \textit{Lorenz gauge}\ . \end{array}\right. \tag{6.146}$$

We define the generating functionals $Z_C[j_\mu^a]$ and $Z_L[J_\mu^a]$ by the expressions

$$Z_C[j_\mu^a] \equiv \mathcal{N}\int \mathcal{D}(A_\mu)\Delta_c[A]\prod_{x,a}\delta(\vec{\nabla}\cdot\vec{A}^a)e^{i(S[A]+\int d^4x j_\mu^a A^{\mu a})} \ , \tag{6.147}$$

and

$$Z_L[J_\mu^a] = \mathcal{N}\int \mathcal{D}(A_\mu)\Delta_L[A]\prod_{x,a}\delta(\partial_\mu A^{\mu a})e^{i(S[A]+\int d^4x J_\mu^a A^{\mu a})} \ . \tag{6.148}$$

Let us derive the relation between them. Following the methods of last section we introduce in $Z_C[j^a_\mu]$ the identity given by,

$$1 = \Delta_L[A] \int \mathcal{D}(g) \prod_{x,a} (\partial_\mu {}^g A^{\mu a}) \ . \tag{6.149}$$

We get then,

$$\begin{aligned}
&Z_C[j^a_\mu] \\
&= \mathcal{N} \textstyle\int \mathcal{D}(A_\mu)\Delta_C[A] \prod_{x,a} \delta(\vec{\nabla}\cdot\vec{A}^a) e^{i(S[A]+\int d^4x j^a_\mu A^{\mu a})} \Delta_L[A] \int \mathcal{D}(g) \prod_{y,b} \delta(\partial_\mu \ {}^g\!A^{\mu a}) \\
&= \mathcal{N} \textstyle\int \mathcal{D}(A_\mu)\Delta_L[A] \prod_{y,b} \delta(\partial_\mu A^{\mu b}) e^{iS[A]} \Delta_C[A] \int \mathcal{D}(g) \prod_{x,a} \delta(\vec{\nabla}\cdot \ {}^{g^{-1}}\!\vec{A}^a) e^{i\int d^4x j^a_\mu \ {}^{g^{-1}}\!A^{\mu a}} \\
&= \mathcal{N} \textstyle\int \mathcal{D}(A_\mu)\Delta_L[A] \prod_{y,b} \delta(\partial_\mu A^{\mu b}) e^{iS[A]} \Delta_C[A] \int \mathcal{D}(g) \prod_{x,a} \delta(\vec{\nabla}\cdot \ {}^{gg^0}\!\vec{A}^a) e^{i\int d^4x j^a_\mu \ {}^{gg^0}\!A^{\mu a}}
\end{aligned} \tag{6.150}$$

where g^0 is the gauge transformation that goes from the gauge $\partial_\mu A^{\mu a} = 0$ to the gauge $\vec{\nabla}\cdot\vec{A}'^a = 0$, $\vec{A}'^a = \ {}^{g^0}\!\vec{A}$. It is obtained solving the equation

$$\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\left[U(g^0)\vec{A}U^{-1}(g^0) - \frac{i}{g}\vec{\nabla}U(g^0)U^{-1}(g^0)\right] = 0 \tag{6.151}$$

where $\partial_\mu A^{\mu a} = 0$. Due to the factor $\prod_x \delta(\vec{\nabla}\cdot \ {}^g\!\vec{A}')$ we are only interested in infinitesimal transformations, and therefore

$$Z_C[j^a_\mu] = \mathcal{N} \int \mathcal{D}(A_\mu)\Delta_L[A] \prod_{y,b} \delta(\partial_\mu A^{\mu b}) e^{iS[A]} e^{i\int d^4x j^a_\mu \ {}^{g^0}\!A^\mu} \ , \tag{6.152}$$

where we have used, like before, the result

$$\int \mathcal{D}(g) \prod_{x,a} \delta(\vec{\nabla}\cdot \ {}^g\!\vec{A}') = \Delta_C^{-1}[A] \ . \tag{6.153}$$

To compare with $Z_L[J^a_\mu]$ it is necessary to write ${}^{g^0}\!A^\mu$ as a function of A^μ, solving the equation for g^0. This can be done formally in a series in the potentials A^μ. We should then have

$$A'_i = \left(\delta_{ij} - \nabla_i \frac{1}{\nabla^2}\nabla_j\right) A_j + O(A^2_\lambda) \ . \tag{6.154}$$

If we restrict the Coulomb source to be transverse, $j^0 = 0$ and $\vec{\nabla}\cdot\vec{j} = 0$, we can then write,

$$Z_C[j^a_\mu] = \mathcal{N} \int \mathcal{D}(A_\mu)\Delta_L[A] \prod_{y,b} \delta(\partial_\mu A^{\mu b}) e^{iS[A]+\int d^4x F^a_\mu j^{\mu a}} \ , \tag{6.155}$$

where $F^C_\mu[A] = A^a_\mu + O(A^2_\lambda)$. Comparing with the expression for the functional $Z_L[J^a_\mu]$ we finally get,

$$Z_C[j^a_\mu] = \exp\left\{ i \int d^4x j^a_\mu F^{\mu a}\left[\frac{\delta}{i\delta J^b}\right]\right\} Z_L[J^a_\mu] \ . \tag{6.156}$$

This is the expression that relates Z_C with Z_L. As $F_\mu[A]$ is a complicated functional, we see that the expressions will be different in the two gauges. But what has a physical meaning and can be compared with the experiment are the matrix elements of the renormalized S matrix. The equivalence theorem that we will prove next shows that these matrix elements are gauge invariant. For simplicity we will make the proof for the $\lambda\phi^4$ case, but the reasoning also applies to the gauge theories.

Theorem 6.2
If two generating functionals, Z and $\tilde{Z}$, differ only by the terms of the external sources, then they will lead to the same renormalized S matrix.

Proof:
Let us consider the generating functional of the Green functions,

$$Z[J] = \mathcal{N} \int \mathcal{D}(\phi) e^{i(S[\phi]+\int d^4x J\phi)} \;, \tag{6.157}$$

where

$$S[\phi] + \int d^4x\, J\phi = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi \right] \;. \tag{6.158}$$

What happens if we couple the external source to $\phi + \phi^3$ instead of coupling to just ϕ?
The generating functional $\tilde{Z}[j]$ will then be,

$$\tilde{Z}[j] = \mathcal{N} \int \mathcal{D}(\phi) e^{i[S[\phi]+\int d^4x\, j(\phi+\phi^3)]} \;. \tag{6.159}$$

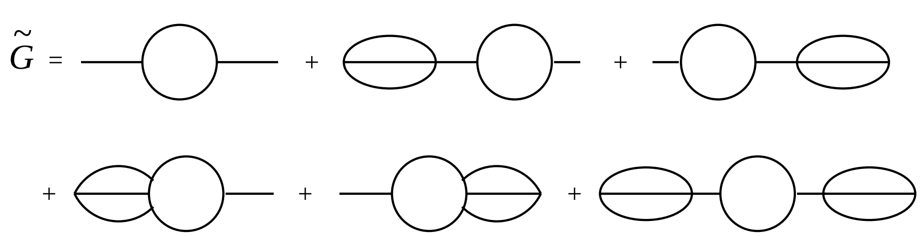
We can write$\tilde{Z}[j]$ in terms of $Z[J]$ using the usual trick,

$$\tilde{Z}[j] = \exp\left\{ i \int d^4x\, j(x) F\left[\frac{\delta}{i\delta J}\right] \right\} Z[J] \;, \tag{6.160}$$

where $F[\phi] = \phi + \phi^3$. Let us now consider the four-point Green function, $\tilde{G}(1,2,3,4)$ generated by $\tilde{Z}[j]$

$$\tilde{G}(1,2,3,4) = (-i)^4 \frac{\delta^4 \tilde{Z}[j]}{\delta j(1)\delta j(2)\delta j(3)\delta j(4)} \;. \tag{6.161}$$

A typical diagram that contributes to $\tilde{G}(1,2,3,4)$ is shown in Fig.6.1, where the part inside the square corresponds to a Green function generated by $Z[J]$.
Let us consider now the propagators $\tilde{G}(1,2)$ and $G(1,2)$ generated by $\tilde{Z}[j]$ and $Z[J]$ respectively. We get the following expansion of $\tilde{G}(1,2)$ in terms of $G(1,2)$



(6.162)

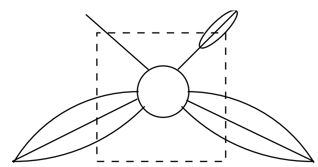
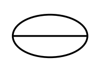
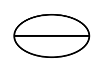


Figure 6.1: Green functions generated by $Z[J]$ and $\tilde{Z}[j]$.

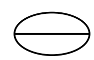
If we examine the propagators near the physical mass pole, we get (Z_2 and $\tilde{Z}_2$ are the renormalization constants in the schemes)

$$\lim_{p^2\to m_R^2} \tilde{G} = \frac{i\tilde{Z}_2}{p^2 - m_R^2} \qquad ; \qquad \lim_{p^2\to m_R^2} G = \frac{iZ_2}{p^2 - m_R^2} \ . \tag{6.163}$$

Now, if we multiply the previous expansion near the mass pole by $p^2 - m_R^2$ and take the limit $p^2 \to m_R^2$ we get,

$$\tilde{Z}_2 = Z_2 \left[1 + 2 \right.$$  $$+ \left(\right.$$  $$\left. \right)^2 + \cdots \left. \right] \ . \tag{6.164}$$

From here we get

$$\sigma \equiv \left(\frac{\tilde{Z}_2}{Z_2} \right)^{1/2} = 1 +$$  $$+ \cdots \ , \tag{6.165}$$

The unrenormalized S matrix is given by,

$$S^{\mathrm{NR}}(k_1, \ldots, k_n) = \prod_{i=1}^{n} \lim_{k_i^2\to m_R^2} (k_i^2 - m_R^2) G(k_1, \ldots, k_n) \ , \tag{6.166}$$

for the Green functions obtained through $Z[J]$. We define in the same way,

$$\tilde{S}^{\mathrm{NR}}(k_1, \ldots, k_n) = \prod_{i=1}^{n} \lim_{k_i^2\to m_R^2} (k_i^2 - m_R^2) \tilde{G}(k_1, \ldots, k_n) \ , \tag{6.167}$$

for the Green functions calculated from $\tilde{Z}[j]$. In these expressions n is the number of external particles. From the argument used to relate $\lim(k^2 - m_R^2)\tilde{G}(k_1, \ldots, k_n)$ with $\lim(k^2 - m_R^2)G(k_1, \ldots, k_n)$ it is easy to see that in relating $\prod \lim(k_i^2 - m_R^2)\tilde{G}$ with $\prod \lim(k_i^2 - m_R^2)G$ only contribute the diagrams with poles in the variables k_i^2. Therefore we obtain

$$\begin{aligned} \prod_{i=1}^{n} \lim_{k_i^2\to m_R^2} (k_i^2 - m_R^2)\tilde{G} &= \left(\frac{Z}{\tilde{Z}} \right)^{-\frac{n}{2}} \prod_{i=1}^{n} \lim_{k_i^2\to m_R^2} (k_i^2 - m_R^2) G \\ &= \sigma^n \prod_{i=1}^{n} \lim_{k_i^2\to m_R^2} (k_i^2 - m_R^2) G \ . \end{aligned} \tag{6.168}$$

From this we get a relation between the unrenormalized S matrices in the two schemes,

$$\tilde{S}^{\ \mathrm{NR}} = \sigma^n S^{\ \mathrm{NR}} \ , \tag{6.169}$$

which van be written as,

$$\frac{1}{\tilde{Z}^{\frac{n}{2}}}\tilde{S}^{\rm NR}(k_1,\ldots,k_n) = \frac{1}{Z^{\frac{n}{2}}}S^{\rm NR}(k_1,\ldots,k_n) \ . \tag{6.170}$$

But $\frac{1}{Z^{\frac{n}{2}}}S^{\rm NR}(k_1,\ldots,k_n)$ *it is precisely the definition of the renormalized* S *matrix, so we get*

$$\tilde{S}^{\rm R} = S^{\rm R} \ . \tag{6.171}$$

We conclude that two generating functionals that only differ by the coupling to the external source lead to the same renormalized S *matrix and then to the same physical quantities. This completes the proof of the equivalence theorem.*

The application of our result is now clear, because

$$Z_C[j^a_\mu] = \exp\left\{ i\int d^4x j^a_\mu F^{\mu a}\left[\frac{\delta}{i\delta J_x}\right]\right\} Z_L[J^c_\mu] \ , \tag{6.172}$$

where $F^a_\mu[A] = A^a_\mu + O(A^2_\lambda)$. The difference between $Z_C[j_\mu]$ and $Z_L[J_\mu]$ lies in the coupling to the external source, and although the Green functions are in general gauge dependent, the renormalized S matrix is gauge independent and hence physical.

6.2.6 Faddeev-Popov ghosts

Having shown the gauge invariance of the renormalized S matrix, let us go back to the generating functional in an arbitrary gauge defined by the gauge condition $F^a[A^b_\mu]$. We have seen in section 6.2.4 that this functional in given in the form,

$$Z_F[J^a_\mu] = \mathcal{N}\int \mathcal{D}(A_\mu)\Delta_F[A]e^{i\int d^4x[\mathcal{L}(x)-\frac{1}{2\xi}(F^a)^2+J^a_\mu A^{\mu a}]} \ , \tag{6.173}$$

where

$$\Delta_F[A] = \det \mathcal{M}_F = \det\left(-g\frac{\delta F^a(x)}{\delta\alpha^b(y)}\right) \ . \tag{6.174}$$

In this form the Feynman rules are complicated because $\det \mathcal{M}_F$ will lead to a non-local interaction among the gauge fields. If, in some way, we could exponentiate the determinant and add it to the action, that would solve our problem.

The idea to exponentiate a determinant comes from using Gaussian integrals with Grassmann variables we have (see Appendix),

$$\int \mathcal{D}(\overline{\omega},\omega)e^{-\int d^4x\overline{\omega}\mathcal{M}_F\omega} = \det \mathcal{M}_F \ . \tag{6.175}$$

Using this result, and making the change $\mathcal{M}_F \to i\mathcal{M}_F$ (an irrelevant change in the normalization), we get

$$Z_F[J^a_\mu] = \mathcal{N}\int \mathcal{D}(A_\mu,\overline{\omega},\omega)e^{i\int d^4x[\mathcal{L}_{eff}+J^a_\mu A^{\mu a}]} \ , \tag{6.176}$$

where $\overline{\omega}$ and ω are anti-commutative scalar fields and $\mathcal{L}_{eff}$ is given by,

$$\mathcal{L}_{eff} = \mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_G \ , \tag{6.177}$$

with

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \\
\mathcal{L}_{GF} &= -\frac{1}{2\xi} (F^a)^2 \\
\mathcal{L}_G &= -\overline{\omega}^a \mathcal{M}_F^{ab} \omega^b \ .
\end{aligned} \tag{6.178}$$

The fields ω and $\overline{\omega}$ are auxiliary fields and are called *Faddeev-Popov* ghosts. The name comes from the wrong spin-statistics connection, but there is no problem with this, as they are not physical fields.

Let us now evaluate more explicitly the ghost part of the Lagrangian,

$$\mathcal{M}_F^{ab}(x,y) = -g \frac{\delta F^a(x)}{\delta \alpha^b(y)} = \frac{\delta F^a[A(x)]}{\delta A^c_\mu(y)} D^{cb}_\mu \ , \tag{6.179}$$

where we have used,

$$\delta A^a_\mu(y) = -\frac{1}{g} D^{ab}_\mu \alpha^b \ .$$

We get

$$\int d^4x d^4y \overline{\omega}^a(x) \mathcal{M}_F^{ab}(x,y) \omega^b(y) = \int d^4x \int d^4y \ \overline{\omega}^a(x) \frac{\delta F^a(x)}{\delta A^c_\mu(y)} D^{cb}_\mu \omega_b(y) \ , \tag{6.180}$$

or

$$\mathcal{L}_G(x) = -\int d^4y \ \overline{\omega}^a(x) \frac{\delta F^a(x)}{\delta A^b_\mu(y)} D^{bc}_\mu \omega_c(y) \ . \tag{6.181}$$

To have a more explicit form we have to specify the gauge. In the Lorenz gauge $F^a = \partial_\mu A^{\mu a}$ and therefore

$$\begin{aligned}
\mathcal{L}_G(x) &= -\int d^4y \overline{\omega}^a(x) \partial^\mu_x \left[\delta^4(x-y)\right] D^{ab}_\mu \omega^b(y) \\
&= \partial^\mu \overline{\omega}^a(x) D^{ab}_\mu \omega^b(x) \ .
\end{aligned} \tag{6.182}$$

In the last step we have used integration by parts and that the covariant derivative in the adjoint representation where the ghosts, like the gauge fields, are is given by,

$$D^{ab}_\mu = \partial_\mu \delta^{ab} - g f^{abc} A^c_\mu \ . \tag{6.183}$$

6.2.7 Feynman rules in the Lorenz gauge

We are now ready to write the Feynman rules that will enable us to evaluate, in perturbation theory any process in a theory that can be described by an non-abelian gauge theory. All the work done so far just lead us to an effective Lagrangian with which we can obtain the Feynman rules as if it was a normal theory without the problem of the mismatch of the degrees of freedom. Our effective Lagrangian is then

$$\mathcal{L}_{eff} = \mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_G \ , \tag{6.184}$$

where

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \qquad ; \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\nu + g f^{bca} A^b_\mu A^c_\nu \\ \mathcal{L}_{GF} &= -\frac{1}{2\xi} (F_a)^2 \\ \mathcal{L}_G &= -\overline{\omega}^a \int d^4 y \frac{\delta F^a}{\delta A^b_\mu} D^{bc}_\mu \omega_c \ . \end{aligned} \tag{6.185}$$

The structure constants f^{abc} are defined by the commutation of the generators of the group G. Our conventions are,

$$\begin{aligned} [t^a, t^b] &= i f^{abc} t^c \\ Tr(t^a t^b) &= \frac{1}{2} \delta^{ab} \ . \end{aligned} \tag{6.186}$$

To fix things, let us consider the Lorenz gauge, defined by

$$F^a[A] = \partial_\mu A^{\mu a}(x) \ . \tag{6.187}$$

We get then

$$\mathcal{L}_{eff} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - \frac{1}{2\xi} (\partial_\mu A^{\mu a})^2 + \partial^\mu \overline{\omega}^a D^{ab}_\mu \omega^b \ . \tag{6.188}$$

Using the fact that the ghosts are in the adjoint representation of the group, we get

$$(D_\mu \omega)^a = \left(\partial_\mu \delta^{ab} - i g A^c_\mu (T^c)^{ab} \right) \omega^b \ , \tag{6.189}$$

with

$$(T^c)^{ab} \equiv -i f^{bca} = -i f^{abc} \ . \tag{6.190}$$

Therefore we have

$$D^{ab}_\mu \omega^b = (\partial_\mu \delta^{ab} - g f^{abc} A^c_\mu) \omega^b \ . \tag{6.191}$$

We can therefore separate the Lagrangian in kinetic and interaction parts

$$\mathcal{L}_{eff} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}} \ , \tag{6.192}$$

where

$$\begin{aligned} \mathcal{L}_{\text{kin}} &= -\frac{1}{4} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi} (\partial_\mu A^{\mu a})^2 + \partial_\mu \overline{\omega}^a \partial^\mu \omega^a \\ &= \frac{1}{2} A^{\mu a} \left[\Box g_{\mu\nu} - \left(1 - \frac{1}{\xi} \right) \partial_\mu \partial_\nu \right] \delta^{ab} A^{\nu b} - \overline{\omega}^a \Box \ \delta^{ab} \omega^b \ , \end{aligned} \tag{6.193}$$

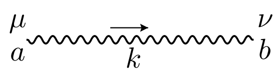
where we have done integrations by parts and neglected total divergences. For the interaction Lagrangian we have

$$\mathcal{L}_{\text{int}} = -g f^{abc} \partial_\mu A^a_\nu A^{\mu b} A^{\nu c} - \frac{1}{4} g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e} + g f^{abc} \partial^\mu \overline{\omega}^a A^b_\mu \omega^c \ . \tag{6.194}$$

We are now in position to get the Feynman rules for a non-abelian gauge theory. We get with the usual conventions,

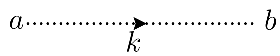
Propagators

i) Gauge fields



$$-\,i\delta_{ab}\left[\frac{g^{\mu\nu}}{k^2+i\epsilon}-(1-\xi)\,\frac{k^\mu k^\nu}{(k^2+i\epsilon)^2}\right] \tag{6.195}$$

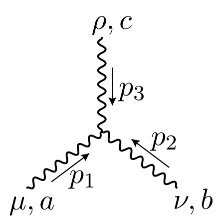
ii) Ghosts



$$\frac{i}{k^2+i\epsilon}\;\delta_{ab} \tag{6.196}$$

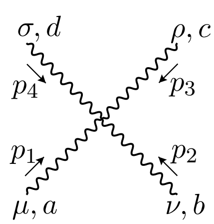
Vertices

i) Gauge bosons triple vertex



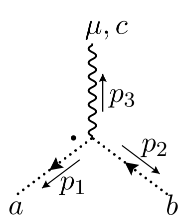
$$\begin{aligned}-gf^{abc}[\quad & g^{\mu\nu}(p_1-p_2)^\rho+g^{\nu\rho}(p_2-p_3)^\mu \\ & +g^{\rho\mu}(p_3-p_1)^\nu]\end{aligned} \tag{6.197}$$

ii) Gauge bosons quartic vertex



$$\begin{aligned}-ig^2\Big[\quad & f^{eab}f^{ecd}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}) \\ & +f^{eac}f^{edb}(g_{\mu\sigma}g_{\rho\nu}-g_{\mu\nu}g_{\rho\sigma}) \\ & +f^{ead}f^{ebc}(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\rho}g_{\nu\sigma})\Big]\end{aligned} \tag{6.198}$$

iii) Gauge boson-Ghost interaction



$$g\;f^{abc}p_1^\mu \tag{6.199}$$

Notes:

1. The dot in the vertex of the ghosts with the gauge fields corresponds to the leg where the derivative applies. This corresponds to the line exiting the diagram (the ghost lines are oriented as they have ghost number, as we will see)

2. The other rules are as usual not forgetting the minus sign for the ghost loop because of their anti-commuting character.

6.2.8 Feynman rules for the interaction with matter

We have just seen the Feynman rules for the pure gauge theory, without interaction with matter. This already non trivial due to the non-abelian character. Interaction with matter is done in the usual way, changing normal derivatives into covariant derivatives. In general matter is described by scalar fields, as

$$\phi_i \quad ; \quad i = 1, \ldots M \tag{6.200}$$

We consider also fermion fields

$$\psi_j \quad ; \quad j = 1, \ldots N \tag{6.201}$$

They belong to the representations of dimension M and N, respectively. The Lagrangian will then be

$$\begin{aligned} \mathcal{L}_{\text{matter}} &= (D_\mu \phi)^\dagger D^\mu \phi - m_\phi^2 \phi^\dagger \phi - V(\phi) \\ &\quad + i\overline{\psi} D^\mu \gamma_\mu \psi - m_\psi \overline{\psi}\psi \\ &\equiv \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}} \ . \end{aligned} \tag{6.202}$$

The interaction Lagrangian between matter and the gauge fields is easily obtained from the covariant derivative

$$D^\mu_{ij} = \partial_\mu \delta_{ij} - igA^a_\mu T^a_{ij} \tag{6.203}$$

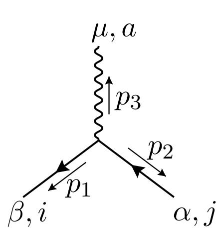
where T^a_{ij} are the generators in the representations appropriate for the matter fields ϕ and ψ.

We get then,

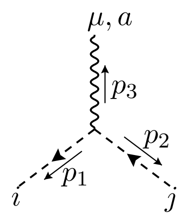
$$\begin{aligned} \mathcal{L}_{\text{int}} &= ig\phi_i^* (\overrightarrow{\partial} - \overleftarrow{\partial})^\mu \phi_j T^a_{ij} A_{\mu a} + g^2 \phi_i^* T^a_{ij} T^b_{jk} \phi_k A^a_\mu A^{\mu b} \\ &\quad + g\overline{\psi}_i \gamma^\mu \psi_j T^a_{ij} A^a_\mu \end{aligned} \tag{6.204}$$

This leads to the following vertices:

Triple Vertices

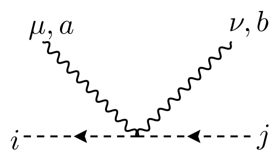


$$ig(\gamma^\mu)_{\beta\alpha} T^a_{ij} \tag{6.205}$$



$$ig(p_1 - p_2)^\mu T^a_{ij} \tag{6.206}$$

Quartic Vertex



$$ig^2 g_{\mu\nu}\{T^a, T^b\}_{ij} \tag{6.207}$$

Group Factors

The factors f^{abc} and T^a_{ij} that appear in the Feynman rules do not need in fact to be known. In the calculations in the end will appear combinations of those that can expressed in terms of invariant quantities that characterize the group and the representation. Our generators are hermitian ($T^{a\dagger} = T^a$) and satisfy the normalization conditions,

$$\begin{aligned} &[T^a, T^b] = i f^{abc} T^c \\ &Tr(T^a T^b) = \delta^{ab} T(R) \end{aligned} \tag{6.208}$$

In these definitions $T(R)$ is a number that characterizes the representation R. Other frequently used quantity is the Casimir of the representation, defined by

$$\sum_{a,k} T^a_{ik} T^a_{kj} = \mathrm{Tr}[T^a T^a] = \delta_{ij} C_2(R) \tag{6.209}$$

For the adjoint representation we get

$$f^{acd} f^{bcd} = \delta^{ab} C_2(G)\ . \tag{6.210}$$

$T(R)$ and $C_2(R)$ are not independent, they obey the relation

$$T(R) r = d(R) C_2(R) \tag{6.211}$$

where r is the dimension of the group G (number of generators) and $d(R)$ is the dimension of the representation R. For the adjoint representation we have

$$d(\mathrm{Adjoint}) = r$$

In many applications we are interested in the $SU(N)$ family of Lie groups. For these we have the results

$$r = N^2 - 1 \; ; \; d(N) = N \; ; \; d(adj) \equiv d(G) = r \tag{6.212}$$

$$T(N) = \frac{1}{2} \; ; \; C_2(N) = \frac{N^2 - 1}{2N} \tag{6.213}$$

$$T(G) = C_2(G) = N \tag{6.214}$$

Symmetry Factors

For the calculation of some diagrams there appear symmetry factors different from one. They were discussed before, in section5.5.5, but for completeness we recall their definition here. The *symmetry factor* is given by the # of different ways in which the lines can be connected with the same final diagram, divided by the permutations factors for the vertices involved and by the permutation factors for the number of equal vertices.

6.3 Ward Identities

6.3.1 BRS transformation

We are now going to study the Ward identities[2] for the non-abelian gauge theories. The more convenient method is that of the Becchi, Rouet e Stora (BRS) transformations. The BRS transformations are a generalization of the gauge transformations that make invariant the effective action.

As we saw for non-abelian gauge theories the effective action is given by ($A =$ gauge field, $\phi =$ matter field)

$$S_{eff}[A, \phi] = S[A, \phi] - \frac{1}{2\xi} \int d^4x F_a^2[A, \phi] - \int d^4x\, \overline{\omega}^a \mathcal{M}_{ab} \;, \omega^b \tag{6.215}$$

where $S[A, \phi]$ is the classical action, invariant under the (infinitesimal) gauge transformations,

$$\begin{aligned} \delta A_\mu^a &= -\frac{1}{g} D_\mu^{ab} \alpha^b \\ \delta \phi_i &= -i (T^a)_{ij} \phi_j \alpha^a \;, \end{aligned} \tag{6.216}$$

and where $F_a[A, \phi]$ are the gauge conditions and the operator $\mathcal{M}_{ab}$ is such that

$$\mathcal{M}_{ab} \omega^b = \frac{\delta F_a}{\delta A_\mu^c} D_\mu^{cb} \omega^b + \frac{\delta F_a}{\delta \phi_i} ig (T^b)_{ij} \phi_j \omega^b \;. \tag{6.217}$$

The effective action, S_{eff}, is not invariant under gauge transformations due to the non-invariance of the gauge fixing term and of the ghost Lagrangian. This non-invariance can

[2] We use here the generic name of Ward identities for the more general identities for non-abelian gauge theories, discovered by Ward, Takahashi, Slavnov and Taylor.

disappear if we choose appropriate transformations for the ghosts in order to compensate the non-invariance of $\int d^4xF_a^2$. These transformations, known as BRS transformations, are defined by,

$$\left\{\begin{array}{lcl} \delta_{\rm BRS}A^a_\mu &=& D^{ab}_\mu\omega^b\theta \\ \delta_{\rm BRS}\phi_i &=& ig(T^b)_{ij}\phi_j\omega^b\theta \\ \delta_{\rm BRS}\overline{\omega}^a &=& \frac{1}{\xi}F_a[A,\phi]\theta \\ \delta_{\rm BRS}\omega^a &=& \frac{1}{2}gf^{abc}\omega^b\omega^c\theta \end{array}\right. \tag{6.218}$$

where θ is an anti-commuting parameter independent of the space-time point *(Grassmann variable)*.

We see that for the fields A^a_μ and ϕ_i the BRS transformations are gauge transformations with parameter $\alpha^a(x) = -g\omega^a(x)\theta$. Notice the anti-commuting character of θ is necessary for the product $\omega^a\theta$ to have a bosonic (commutative) character. To show the invariance of $S_{eff}[A,\phi]$ we are going to prove a series of theorems needed for the general proof. Before we do that it is convenient to introduce the Slavnov operator, s, defined by the relations,

$$\begin{array}{lclclcl} \delta_{\rm BRS}A^a_\mu &=& sA^a_\mu\theta & \qquad & \delta_{\rm BRS}\omega^a &=& s\omega^a\theta \\ \delta_{\rm BRS}\phi_i &=& s\phi_i\theta & & \delta_{\rm BRS}\overline{\omega}^a &=& s\overline{\omega}^a\theta \end{array} \tag{6.219}$$

This operator is distributive with respect to multiplication (like a derivative) and obeys the following relations,

$$\begin{aligned} &s(B_1B_2) = sB_1B_2 + B_1sB_2 \\ &s(F_1B_2) = sF_1B_2 + F_1sB_2 \\ &s(B_1F_2) = -sB_1F_2 + B_1sF_2 \\ &s(F_1F_2) = -sF_1F_2 + F_1sF_2 \end{aligned} \tag{6.220}$$

Theorem 6.3
The Slavnov operator s is nilpotent in the fields A^a_μ, ϕ_i e ω^a, that is $s^2A^a_\mu = s^2\phi_i = s^2\omega^a = 0$.

Proof: *We show for each case. We have*

a) $s^2A^a_\mu = 0$

$$\begin{aligned} s^2A^a_\mu =& s(D^{ab}_\mu\omega^b) = -\frac{\delta D^{ab}_\mu}{\delta A^c_\nu}sA^c_\nu\omega^b + D^{ab}_\mu s\omega^b \\ =& -\delta^\nu_\mu(-gf^{abc})D^{cd}_\nu\omega^d\omega^b + \frac{1}{2}gf^{bcd}D^{ab}_\mu(\omega^c\omega^d) \\ =& \left[gf^{abc}\partial_\mu\omega^c\omega^b + \frac{1}{2}gf^{acd}\partial_\mu\omega^c\omega^d + \frac{1}{2}gf^{acd}\omega^c\partial_\mu\omega^d\right] \\ & + \left[gf^{abc}(-g)f^{cde}A^e_\mu\omega^d\omega^b + \frac{1}{2}g(-g)f^{bcd}f^{abe}A^e_\mu\omega^c\omega^d\right] \end{aligned}$$

$$
\begin{aligned}
&=(gf^{abc}\partial_\mu\omega^c\omega^b - gf^{abc}\partial_\mu\omega^c\omega^b) \\
&\quad -\frac{1}{2}g^2(f^{abc}f^{cde} - f^{adc}f^{cbe} + f^{cdb}f^{ace})A^e_\mu\omega^d\omega^b = 0 \qquad (6.221)
\end{aligned}
$$

b) $s^2\phi_i = 0$

$$
\begin{aligned}
s^2\phi_i =& s\left[ig(T^a)_{ij}\phi_j\omega^a\right] \\
=& -ig(T^a)_{ij}s\phi_j\omega^a + ig(T^a)_{ij}\phi_j s\omega^a \\
=& g^2(T^a)_{ij}(T^b)_{jk}\phi_k\omega^b\omega^a + ig(T^a)_{ij}\phi_j\frac{1}{2}gf^{abc}\omega^b\omega^c \\
=& \frac{1}{2}g^2[T^c,T^b]_{ik}\omega^b\omega^c\phi_k + \frac{i}{2}g^2(T^a)_{ij}\phi_j f^{abc}\omega^b\omega^c \\
=& \frac{i}{2}g^2(T^a)_{ij}\phi_j(f^{acb}+f^{abc})\omega^b\omega^c \\
=& 0 \qquad (6.222)
\end{aligned}
$$

c) $s^2\omega^a = 0$

$$
\begin{aligned}
s^2\omega^a =& s\left(\frac{1}{2}gf^{abc}\omega^b\omega^c\right) \\
=& -\frac{1}{2}gf^{abc}s\omega^b\omega^c + \frac{1}{2}gf^{abc}\omega^b s\omega^c \\
=& -gf^{abc}s\omega^b\omega^c \\
=& -\frac{1}{2}g^2f^{abc}f^{bef}\omega^e\omega^t\omega^c \\
=& -\frac{1}{6}g^2(f^{abc}f^{bef} + f^{abe}f^{bfc} + f^{abf}f^{bce})\omega^e\omega^t\omega^c \\
=& 0 \qquad (6.223)
\end{aligned}
$$

where we have used the anti-commutation of the ghost fields and the Jacobi identity. The theorem is then proved.

For linear gauge fixing, we can show an important result that we also give in the form of a theorem.

Theorem 6.4
For linear gauge fixings, the Slavnov operator verifies the relation

$$
s(\mathcal{M}_{ab}\omega^b) = 0 \qquad (6.224)
$$

Proof:

We saw before that

$$\mathcal{M}_{ab}\omega^b(x) = \int d^4y \left[\frac{\delta F_a(x)}{\delta A_\mu^c(y)} D_\mu^{cb}\omega^b(y) + \frac{\delta F_a(x)}{\delta \phi_i(y)} ig(T^b)_{ij}\phi_j\omega^b(y)\right] \tag{6.225}$$

If we use the definitions of $\delta_{\rm BRS}$ *and of the Slavnov operator, we can write*

$$\mathcal{M}_{ab}\omega^b(x) = \int d^4y \left[\frac{\delta F_a(x)}{\delta A_\mu^c(y)} sA_\mu^c(y) + \frac{\delta F_a(x)}{\delta \phi_i(y)} s\phi_i(y)\right] \ . \tag{6.226}$$

If the gauge fixing is linear $\frac{\delta F_a}{\delta A^c_\mu}$ *and* $\frac{\delta F_a}{\delta \phi_i}$ *do not depend on the on the fields and then*

$$s\left[\mathcal{M}_{ab}\omega^b(x)\right] = \int d^4y \left[\frac{\delta F_a(x)}{\delta A_\mu(y)} s^2 A_\mu^c(y) + \frac{\delta F_a(x)}{\delta \phi_i(y)} s^2\phi_i(y)\right] = 0 \ . \tag{6.227}$$

where we have used the previous results. This proves the theorem.

Using theses results we can then show that the effective action is invariant under BRS transformations. We are going to show this result also in the form of a theorem.

Theorem 6.5
The action S_{eff} *is invariant under BRS transformations.*

Proof:

The effective action is

$$S_{eff}[A,\phi] = S[A,\phi] + \int d^4x \left[-\frac{1}{2\xi}F_a^2[A,\phi] - \overline{\omega}^a \mathcal{M}_{ab}\omega^b\right] \ . \tag{6.228}$$

As the classical action is invariant under gauge transformations we should have

$$s\left(S[A,\phi]\right) = 0 \ . \tag{6.229}$$

For the other terms we have

$$s\left(-\frac{1}{2\xi}F_a^2 - \overline{\omega}^a\mathcal{M}_{ab}\omega^b\right) = -\frac{1}{\xi}F_a sF_a + s\overline{\omega}^a\mathcal{M}_{ab}\omega^b - \overline{\omega}^a s(\mathcal{M}_{ab}\omega^b) \ . \tag{6.230}$$

But

$$sF_a(x) = \int d^4y \left[\frac{\delta F_a}{\delta A_\mu^b(y)} sA_\mu^b(y) + \frac{\delta F_a}{\delta \phi_i(y)} s\phi_i(y)\right] = \mathcal{M}_{ab}\omega^b(x) \ , \tag{6.231}$$

and using one of the previous theorems we get

$$s(\mathcal{M}_{ab}\omega^b) = 0 \ . \tag{6.232}$$

Therefore

$$s\left(-\frac{1}{2\xi}F_a^2-\overline{\omega}^a\mathcal{M}_{ab}\omega^b\right)=\left(-\frac{1}{\xi}F_a+s\overline{\omega}^a\right)\mathcal{M}_{ab}\omega^b=0\ , \tag{6.233}$$

where we have used the fact that $s\overline{\omega}^a=\frac{1}{\xi}F_a$*. Putting everything together we get,*

$$sS_{eff}[A,\phi]=0\ . \tag{6.234}$$

For the applications we still need another result on the invariance of the integration measure that we also present as a theorem.

Theorem 6.6
The measure $\mathcal{D}(A_\mu,\phi_i,\overline{\omega}^a,\omega^b)$ *is invariant under BRS transformations.*

Proof:

Simple calculations lead to the following relations

$$\begin{aligned}
&\frac{\delta(sA^a_\mu)}{\delta A^a_\mu}=-gf^{aba}\delta^\mu_\mu\omega^b=0\\
&\frac{\delta(s\phi_i)}{\delta\phi_i}=ig(T^a)_{ii}\omega^a=0\qquad ;\qquad (Tr(T^a)=0)\\
&\frac{\delta(s\omega^a)}{\delta\omega^a}=gf^{aac}\omega^c=0\\
&\frac{\delta(s\overline{\omega}^a)}{\delta\overline{\omega}^a}=0
\end{aligned} \tag{6.235}$$

As we saw in the last chapter these relations imply that the integration measure is invariant proving the theorem (see Eq. (5.121)).

6.3.2 Ward-Takahashi-Slavnov-Taylor identities

We are now in position to derive the generalization of the Ward-Takahashi identities to non-abelian gauge theories. This extension was done, among others, by Slavnov and Taylor, but frequently we use the short designation of Ward identities even for the non-abelian case. In a generic form Ward identities are relations among the Green functions that result from the gauge symmetry of the theory. As we discuss the more convenient way to describe them is through the generating functionals of the Green functions. Let us then consider a non-abelian gauge theory. For simplicity we just consider that the matter fields are scalars ϕ_i. Fermions can be introduced later easily. The generating functional is then

$$Z[J^a_\mu,J_i,\eta^a,\overline{\eta}^a]=\int\mathcal{D}(A_\mu,\phi_i,\overline{\omega},\omega)e^{i\int d^4x[\mathcal{L}_{eff}+J^a_\mu A^{\mu a}+J_i\phi_i+\overline{\eta}^a\omega^a+\overline{\omega}^a\eta^a]} \tag{6.236}$$

where we have also introduced sources for the ghosts.

A BRS transformation is a change of variables in the integral. The value of the integral should not be changed by this. As S_{eff} and the measure are invariant we should have the following theorem:

Theorem 6.7
Given any Green function

$$G(x_1, ..., y_1, ..., z_1, ..., w_1, ...) = \langle 0|\, TA^a_{\mu_1}(x_1)\cdots\phi_{i_1}(y_1)\cdots\overline{\omega}^a(z_1)\cdots\omega^b(w_1)\cdots|0\rangle \tag{6.237}$$

we have the following relations:

$$\begin{aligned} &i) \qquad s\,\langle 0|\, TA^a_\mu(x_1)\cdots\phi_{i_1}(y_1)\cdots\overline{\omega}^a(z_1)\cdots\omega^{b1}(w_1)\,|0\rangle = 0 \\ &ii) \qquad 0 = \langle 0|\, TsA^a_\mu(x_1)\cdots|0\rangle + \cdots + \langle 0|\, T\cdots s\phi_i\cdots|0\rangle + \cdots \\ &\qquad\qquad + \langle 0|\, T\cdots s\overline{\omega}^a\cdots|0\rangle\cdots + \langle 0|\, T\cdots s\omega^a\cdots|0\rangle \end{aligned} \tag{6.238}$$

Proof: *The proof is clear if we write*

$$\begin{aligned} &\langle 0|\, TA^a_\mu(x_1)\cdots\phi_{i_1}(y_1)\cdots\overline{\omega}^a(z_1)\cdots\omega^b(w_1)\,|0\rangle = \\ &= \int \mathcal{D}(A_\mu,\phi_i,\omega,\overline{\omega})A^a_\mu(x_1)\cdots\phi_{i1}(y_1)\cdots\overline{\omega}^a(z_1)\cdots\omega^b(w_1)e^{iS_{eff}}\ . \end{aligned} \tag{6.239}$$

Then the BRS transformation should leave the integral invariant proving the first relation. Then the second relation results from the first and from the invariance of the measure and of the effective action.

This theorem constitutes a quick way to establish relations among Green functions for particular cases, as we shall see below. However to establish general results for the renormalization and gauge invariance of the S matrix, we are interested in the Ward identities expressed in terms of the generating functionals. Using the invariance of the integral for a change of variables, the invariance of the measure $\mathcal{D}$ and of S_{eff}, we get for the Ward identity for the generating functional Z,

$$0 = \int \mathcal{D}(A_\mu,\phi_i,\omega,\overline{\omega})\int d^4x(J^{\mu a}sA^a_\mu + J_i s\phi_i + \overline{\eta}^a s\omega^a - s\overline{\omega}^a\eta^a)e^{i(S_{eff}+\text{sources})}\ . \tag{6.240}$$

The more useful Ward identities are for the functional Γ. The previous expression can not directly lead to Γ functional, because sA^a_μ, $s\phi_i$ and $s\omega^a$ are not linear in the fields. To solve this problem we introduce sources for these non-linear operators. We generalize the effective action defining a new quantity Σ such that

$$\begin{aligned} &\Sigma[A^a_\mu,\phi_i,\overline{\omega}^a,\omega^a,K^a_\mu,K_i,L^a] \\ &\qquad \equiv S_{eff}[A^a_\mu,\phi_i,\overline{\omega}^a,\omega^a] + \int d^4x(K^{a\mu}sA^a_\mu + K^i s\phi_i + L^a s\omega^a)\ , \end{aligned} \tag{6.241}$$

where $K^{a\mu}, K_i$ and L^a are sources for the non-linear operators $sA^a_\mu, s\phi_i$ and $s\omega^a$ respectively. Using the previous theorems it is easy to show that Σ is invariant under BRS transformations, that is

$$s\Sigma = 0\ . \tag{6.242}$$

Let us consider now the generating functional for the Green functions in the presence of the sources $J_\mu^a, J_i, \eta^a, \overline{\eta}^a, K^{\mu a}, K^i$ and L^a, that is

$$Z[J_\mu^a, J_i, \eta, \overline{\eta}, K_\mu, K_i, L] = \int \mathcal{D}(A_\mu, \phi_i, \overline{\omega}, \omega) e^{i\left[\Sigma + \int d^4x (J_\mu^a A^{\mu a} + J_i \phi_i + \overline{\eta}\omega + \overline{\omega}\eta)\right]} \tag{6.243}$$

We can now repeat the previous reasoning for invariance under BRS transformations. Like before we get (recall that $s\Sigma = 0$)

$$0 = \int \mathcal{D}(\cdots) \int d^4x [J_a^\mu s A_\mu^a + J^i s\phi_i + \overline{\eta}^a s\omega^a - s\overline{\omega}^a \eta^a] e^{i(\Sigma + \text{sources})} , \tag{6.244}$$

only now we have composite operators $sA, s\phi$ e $s\omega$, that is

$$sA_\mu^a = \frac{\delta\Sigma}{\delta K^{\mu a}}, \qquad s\phi_i = \frac{\delta\Sigma}{\delta K_i}, \qquad s\omega^a = \frac{\delta\Sigma}{\delta L^a}, \qquad s\overline{\omega}^a = \frac{1}{\xi} F^a \tag{6.245}$$

We get then

$$\int \mathcal{D}(\cdots) \int d^4x \left[J_a^\mu \frac{\delta\Sigma}{\delta K^{\mu a}} + J^i \frac{\delta\Sigma}{\delta K^i} + \overline{\eta}^a \frac{\delta\Sigma}{\delta L^a} - \frac{1}{\xi} F^a \eta^a \right] e^{i(\Sigma + \text{sources})} = 0 \tag{6.246}$$

or in another form

$$\int d^4x \left[J_a^\mu \frac{\delta}{i\delta K^{\mu a}} + J^i \frac{\delta}{i\delta K^i} + \overline{\eta}^a \frac{\delta}{i\delta L^a} - \frac{1}{\xi} F^a \left[\frac{\delta}{i\delta J_\mu}, \frac{\delta}{i\delta J_i} \right] \eta^a \right] e^{iW[J_\mu^a, J_i, \eta, \overline{\eta}, K_\mu, K_i, L]} = 0 \tag{6.247}$$

For a linear gauge condition all the differential operators in the square bracket are of first order and therefore we can write

$$\int d^4x \left[J_a^\mu \frac{\delta}{\delta K^{\mu a}} + J^i \frac{\delta}{\delta K_i} + \overline{\eta}^a \frac{\delta}{\delta L^a} - \frac{1}{\xi} F_a \eta^a \right] W = 0 \ . \tag{6.248}$$

This is the expression of the Ward identities for the generating functional of the connected Green functions, W. Normally the Ward identities are more useful for generating functional of the irreducible Green functions. This defined by,

$$\Gamma[A_\mu, \phi_i, \overline{\omega}, \omega, K_\mu, K_i, L] \equiv W[J_\mu, J_i, \eta, \overline{\eta}, K_\mu, K_i, L] - \int d^4x [J_\mu^a A^{a\mu} + J_i \phi_i + \overline{\eta}\omega + \overline{\omega}\eta] \tag{6.249}$$

with the usual relations

$$\begin{aligned} \phi_i &= \frac{\delta W}{\delta J_i} & \omega^a &= \frac{\delta W}{\delta \overline{\eta}^a} \\ A_\mu^a &= \frac{\delta W}{\delta J^{\mu a}} & \overline{\omega}^a &= -\frac{\delta W}{\delta \eta^a} \end{aligned} \tag{6.250}$$

as usual as the inverse relations,

$$\begin{aligned} J_i &= -\frac{\delta\Gamma}{\delta\phi_i} & \overline{\eta}^a &= \frac{\delta\Gamma}{\delta\omega^a} \\ J_\mu^a &= -\frac{\delta\Gamma}{\delta A^{\mu a}} & \eta^a &= -\frac{\delta\Gamma}{\delta\overline{\omega}^a} \end{aligned} \tag{6.251}$$

As the Legendre transform leaves the sources K^a_μ, K_i and L^a unchanged, we should have.

$$\frac{\delta W}{\delta K^a_\mu} = \frac{\delta \Gamma}{\delta K^a_\mu} \quad ; \quad \frac{\delta W}{\delta K_i} = \frac{\delta \Gamma}{\delta K_i} \quad ; \quad \frac{\delta W}{\delta L^a} = \frac{\delta \Gamma}{\delta L^a} \tag{6.252}$$

We then get easily

$$\int d^4x \left[\frac{\delta \Gamma}{\delta K^a_\mu(x)} \frac{\delta \Gamma}{\delta A^{\mu a}(x)} + \frac{\delta \Gamma}{\delta K_i(x)} \frac{\delta \Gamma}{\delta \phi_i(x)} - \frac{\delta \Gamma}{\delta L^a(x)} \frac{\delta \Gamma}{\delta \omega^a(x)} - \frac{1}{\xi} F^a \frac{\delta \Gamma}{\delta \overline{\omega}^a(x)} \right] = 0 \tag{6.253}$$

This is the generating functional for the Ward identities for the irreducible Green functions. The Ward identities for specific Green functions are obtained by taking appropriate functional derivatives of the fields.

In the applications the previous equation is used in connection with another functional identity, the equation of motion (or Dyson-Schwinger) for the ghosts. This can be easily obtained doing the following change of variables in the functional integral,

$$\left\{ \begin{array}{lcl} \delta A^a_\mu & = & \delta \phi_i = \delta \omega^a = 0 \\ \delta \overline{\omega}^a & = & f^a = \text{infinitesimal constant} \end{array} \right. \tag{6.254}$$

Then

$$\delta Z = 0 = \int \mathcal{D}(\cdots) \left(i \frac{\delta \Sigma}{\delta \overline{\omega}^a} + i \eta^a \right) f^a e^{i(\Sigma + \text{sources})} \tag{6.255}$$

but

$$\begin{aligned} \frac{\delta \Sigma}{\delta \overline{\omega}^a(x)} &= -\mathcal{M}_{ab} \omega^b(x) = -s F_a(x) \\ &= -\int d^4y \left[\frac{\delta F_a(x)}{\delta A^b_\mu(y)} s A^b_\mu(y) + \frac{\delta F_a(x)}{\delta \phi_i(y)} s \phi_i(y) \right] \\ &= -\int d^4y \left[\frac{\delta F_a(x)}{\delta A^b_\mu(y)} \frac{\delta \Sigma}{\delta K^{b\mu}(y)} + \frac{\delta F_a(x)}{\delta \phi_i(y)} \frac{\delta \Sigma}{\delta K_i(y)} \right] \end{aligned} \tag{6.256}$$

We therefore obtain

$$\begin{aligned} 0 &= \int \mathcal{D}(\cdots) \left\{ -i \int d^4y \left[\frac{\delta F_a(x)}{\delta A^b_\mu(y)} \frac{\delta \Sigma}{\delta K^{b\mu}(y)} + \frac{\delta F_a(x)}{\delta \phi_i(y)} \frac{\delta \Sigma}{\delta K_i(y)} \right] + i \eta^a(x) \right\} e^{i(\Sigma + \text{sources})} \\ &= \left\{ -\int d^4y \left[\frac{\delta F_a(x)}{\delta A^b_\mu(y)} \frac{\delta}{\delta K^{b\mu}(y)} + \frac{\delta F_a(x)}{\delta \phi_i(y)} \frac{\delta}{\delta K_i(y)} \right] + i \eta^a(x) \right\} e^{iW} \end{aligned} \tag{6.257}$$

Using now

$$\eta^a = -\frac{\delta \Gamma}{\delta \overline{\omega}^a} \tag{6.258}$$

we finally get (for linear gauges)

$$\int d^4y \left[\frac{\delta F_a(x)}{\delta A^b_\mu(y)} \frac{\delta \Gamma}{\delta K^{\mu b}(y)} + \frac{\delta F_a(x)}{\delta \phi_i(y)} \frac{\delta \Gamma}{\delta K_i(y)} \right] = -\frac{\delta \Gamma}{\delta \overline{\omega}^a(x)} \tag{6.259}$$

which the generating functional for the Dyson-Schwinger equations for the ghosts.

6.3.3 Example: Transversality of vacuum polarization

We are going to give an example of the application of the Ward identities. For this we will show that the vacuum polarization is transversal. As the pure gauge theory is already non-trivial, we only consider this case, the generalizations being straightforward. To show the details of the calculations that will shed some light on the more formal expression we just proved, we are going to do this example using two methods. The first one, that we will call *formal method*, will use the general expression for the Ward identities satisfied by the generating functional of the irreducible Green functions, Γ. The second method, which we call *practical method*, will use the results of one of the theorems on the BRS transformations that we proved before. The comparison between the two methods will be important to clarify the meaning of the expressions.

i) Formal Method

For the pure gauge theory case, the expression for the Ward identities for the generating functional Γ is,

$$\int d^4x \left[\frac{\delta\Gamma}{\delta K^a_\mu(x)} \frac{\delta\Gamma}{\delta A_{\mu a}(x)} - \frac{\delta\Gamma}{\delta L^a(x)} \frac{\delta\Gamma}{\delta\omega^a(x)} - \frac{1}{\xi} F^a(x) \frac{\delta\Gamma}{\delta\overline{\omega}^a(x)} \right] = 0 \tag{6.260}$$

where we will choose a covariant linear gauge,

$$F^a(x) = \partial_\mu A^{a\mu}(x) \tag{6.261}$$

To proceed it is necessary to know what is the meaning of the functional derivatives, $\frac{\delta\Gamma}{\delta K^a_\mu}$ and $\frac{\delta\Gamma}{\delta L^a}$. From their definition we have

$$\begin{aligned} \frac{\delta\Gamma}{\delta K^a_\mu(x)} &= \frac{\delta W}{\delta K^a_\mu} = \frac{\delta}{i\delta K^a_\mu} \ln Z = \frac{1}{Z} \frac{\delta Z}{i\delta K^a_\mu(x)} \\ &= \frac{1}{Z} \int \mathcal{D}(\cdots) s A^a_\mu(x) e^{i(\Sigma + \text{sources})} \end{aligned} \tag{6.262}$$

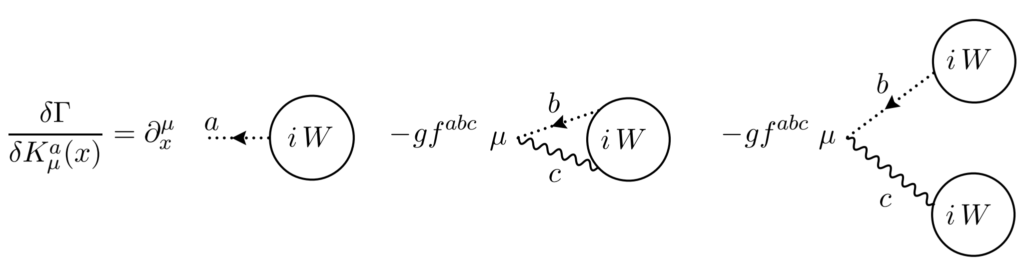
As $sA^a_\mu(x) = D^{ab}_\mu \omega^b = \partial_\mu \omega^a(x) - g f^{abc} \omega^b(x) A^c_\mu(x)$, we then get

$$\frac{\delta\Gamma}{\delta K^a_\mu(x)} = \partial^x_\mu \frac{1}{Z} \frac{\delta Z}{i\delta\overline{\eta}^a(x)} - g f^{abc} \frac{1}{Z} \frac{\delta^2 Z}{i\delta J^c_\mu(x) i\delta\overline{\eta}^b(x)} \tag{6.263}$$

Introducing now $Z \equiv \exp(iW)$, the previous expression becomes,

$$\frac{\delta\Gamma}{\delta K^a_\mu(x)} = \partial^\mu_x \frac{\delta(iW)}{i\delta\overline{\eta}^a(x)} - g f^{abc} \left[\frac{\delta^2 iW}{i\delta J^c_\mu(x) i\delta\overline{\eta}^b(x)} + \frac{\delta iW}{i\delta J^c_\mu(x)} \frac{\delta iW}{i\delta\overline{\eta}^b(x)} \right] , \tag{6.264}$$

which has the following diagrammatic representation



(6.265)

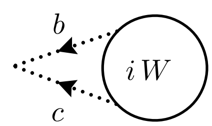
where W is the generating functional for the connected Green functions.

In a similar way we can show that ($s\omega^a = \frac{1}{2} g f^{abc}\omega^b\omega^c$)

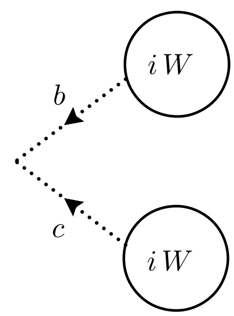
$$\begin{aligned}\frac{\delta\Gamma}{\delta L^a(x)} &= \frac{1}{2} g\ f^{abc}\ \frac{1}{Z}\ \frac{\delta^2 Z}{i\delta\overline{\eta}^c(x) i\delta\overline{\eta}^b(x)} \\ &= \frac{1}{2} g\ f^{abc}\left[\frac{\delta^2(iW)}{i\delta\overline{\eta}^c(x) i\delta\overline{\eta}^b(x)} + \frac{\delta(iW)}{i\delta\overline{\eta}^c(x)}\frac{\delta(iW)}{i\delta\overline{\eta}^b(x)}\right]\end{aligned} \tag{6.266}$$

In diagrammatic form this gives

$$\frac{\delta\Gamma}{\delta L^a(x)} = \frac{1}{2} g\ f^{abc}$$



$$+\frac{1}{2} g\ f^{abc}$$



(6.267)

We want to apply $\dfrac{\delta^2}{\delta\omega^b(y)\delta A^c_\nu(z)}$ to the original equation. We get

$$\frac{\delta^2}{\delta\omega^b(y)\delta A^c_\nu(z)}\left(\frac{\delta\Gamma}{\delta K^a_\mu(x)}\frac{\delta\Gamma}{\delta A^{\mu a}(x)}\right)\Bigg|_{=0} = \frac{\delta^2\Gamma}{\delta\omega^b(y)\delta K^a_\mu(x)}\Bigg|_{=0}\ \frac{\delta^2\Gamma}{\delta A^c_\nu(z)\delta A^{\mu a}(x)}\Bigg|_{=0} \tag{6.268}$$

But we have

$$\begin{aligned}\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta K^a_\mu(x)}\Bigg|_{=0} &= \int d^4w\left(-i\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^f(w)}\right)\left(\frac{\delta^2\Gamma}{i\delta\eta^f(w)\delta K^a_\mu(x)}\right)\Bigg|_{=0} \\ &= \partial^\mu_x\int d^4w\left(-i\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^f(w)}\right)\left(\frac{\delta^2(iW)}{i\delta\eta^f(w)i\delta\overline{\eta}^a(x)}\right)\Bigg|_{=0} \\ &- g\ f^{ab'c}\int d^4w\left(-i\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^f(w)}\right)\left(\frac{\delta^3 iW}{i\delta\eta^f(w)i\delta\overline{\eta}^{b'}(x)i\delta J^c_\mu(x)}\right)\Bigg|_{=0} \\ &= \partial^\mu_x\delta^4(x-y)\delta^{ab} - gf^{ab'c}\int d^4w\left(-i\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^f(w)}\right) \\ &\qquad\left(\frac{\delta^3 iW}{i\delta\eta^f(w)i\delta\overline{\eta}^{b'}(x)i\delta J^c_\mu(x)}\right)\Bigg|_{=0}\end{aligned} \tag{6.269}$$

In a similar way we have for the second term,

$$\frac{\delta^2}{\delta\omega^b(y)\delta A^c_\nu(z)}\left(\frac{\delta\Gamma}{\delta L^a}\frac{\delta\Gamma}{\delta\omega^a}\right)\Bigg|_{=0} = 0 \tag{6.270}$$

$$\frac{\delta^2}{\delta\omega^b(y)\delta A^c_\nu(z)}\left(\frac{1}{\xi}\partial_\rho A^{\rho a}(x)\frac{\delta\Gamma}{\delta\overline{\omega}^a(x)}\right)\Bigg|_{=0} = \frac{1}{\xi}\partial^\nu_x\delta^4(x-z)\ \frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^a(x)}\Bigg|_{=0} \tag{6.271}$$

Using these results we get

$$-\partial^y_\mu \frac{\delta^2\Gamma}{\delta A^b_\mu(y)\delta A^c_\nu(z)} - g f^{ade}\int d^4x d^4w \left(-i\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^f(w)}\right)$$
$$\left(\frac{\delta^3 iW}{i\delta\eta^f(w) i\delta\overline{\eta}^d(x) i\delta J^e_\mu(x)}\right)\left(\frac{\delta^2\Gamma}{\delta A^a_\mu(x)\delta A^c_\nu(z)}\right) + \frac{1}{\xi}\partial^\nu_z \frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^c(z)} = 0 \tag{6.272}$$

We now apply the Fourier transform, with the conventions shown in the Fig. 6.2, we

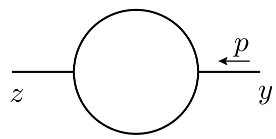


Figure 6.2: Momentum definition for the Fourier Transform.

get

$$-ip^\mu(i)G^{-1cb}_{\ \ \nu\mu}(p) - gf^{ade} iG^{-1ca}_{\ \ \nu\mu}(p)\Delta^{-1fb}X^{\mu def} + (-ip^\nu)\frac{i}{\xi}\Delta^{-1cb}(p) = 0 \tag{6.273}$$

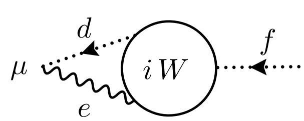
This can be written as

$$p^\mu G^{-1cb}_{\ \ \nu\mu} = -\frac{1}{\xi}\Delta^{-1cb}p_\nu + ig\ f^{ade}G^{-1ca}_{\ \ \nu\mu}(p)\ \Delta^{-1fb}X^{\mu def} \tag{6.274}$$

where

$$X^{\mu def} = \mathrm{FT}\left[< 0|T\omega^d(x)\overline{\omega}^f(w)A^{\mu e}(x)|0>_c\right]$$

$$\equiv \tag{6.275}$$



To prove the Transversality we also need the equation of motion for the ghosts. For our case this is

$$\frac{\delta\Gamma}{\delta\overline{\omega}^a(z)} = -\partial^\mu_z \frac{\delta\Gamma}{\delta K^{\mu a}(z)} \tag{6.276}$$

Applying the operator $\frac{\delta}{\delta\omega^b(y)}$, we get

$$\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^a(z)} = -\Box\ \delta^{ab}\delta^4(y-z)$$
$$+ gf^{adc}\int d^4w\left(-i\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^f(w)}\right)\partial^\mu_z\left(\frac{\delta^3 iW}{i\delta J^c_\mu(z) i\delta\eta^f(w) i\delta\overline{\eta}^d(z)}\right) \tag{6.277}$$

Applying now the Fourier transform, we get

$$i\Delta^{-1ab} = p^2\delta^{ab} + gf^{adc}(-ip^\mu)X_\mu^{dcf}\Delta^{-1fb} \tag{6.278}$$

The previous equations allow now to complete the proof of the transversality of the vacuum polarization. For this we write,

$$G^{-1ab}_{\ \ \mu\nu} = G^{-1ab}_{T\ \mu\nu} + i\frac{a}{\xi}\delta^{ab}p_\mu p_\nu \tag{6.279}$$

where $p^\mu G^{-1ab}_{T\ \mu\nu} = 0$. For the free propagator we have $a = 1$. To show the transversality we just have to show that the longitudinal part is not renormalized and that therefore the value of a remains always $a = 1$. Using

$$p^\mu G^{-1ab}_{\ \ \mu\nu} = i\frac{a}{\xi}\delta^{ab}p^2 p_\nu \tag{6.280}$$

and multiplying equation by p^ν we obtain

$$i\frac{a}{\xi}p^4\delta^{cb} = -\frac{1}{\xi}p^2\Delta^{-1cb} - \frac{a}{\xi}p^2 g\ f^{cde}p_\mu X^{\mu def}\Delta^{-1fb} \tag{6.281}$$

Using now equation Eq. (6.278) we get after some trivial algebra

$$0 = -\frac{1}{\xi}p^2\Delta^{-1cb} + \frac{a}{\xi}p^2\Delta^{-1cb} \tag{6.282}$$

This implies

$$a = 1 \tag{6.283}$$

as we wanted to shown.

2) Practical Method

Now we are going to use the so-called practical method based in Theorem 6.7 Using

$$s\overline{\omega}^b(x) = \frac{1}{\xi}\partial_\mu A^{\mu b}(x) \tag{6.284}$$

and

$$sA^a_\nu = \partial_\nu\omega^a - gf^{adc}\omega^d A^c_\nu \tag{6.285}$$

it is easy to see that the starting Green function should be $\left\langle 0|TA^a_\nu(x)\overline{\omega}^b(y)|0\right\rangle$. Then Theorem 6.7 tells us that

$$s\left\langle 0|TA^a_\mu(x)\overline{\omega}^b(y)|0\right\rangle = 0 \tag{6.286}$$

that is

$$\begin{aligned}\frac{1}{\xi}\left\langle 0|TA^a_\nu(x)\partial_\mu A^{\mu b}(y)|0\right\rangle &= \left\langle 0|T\partial_\nu\omega^a(x)\overline{\omega}^b(y)|0\right\rangle \\ &\quad -gf^{adc}\left\langle 0|T\omega^d(x)A^c_\nu(x)\overline{\omega}^b(y)|0\right\rangle\end{aligned} \tag{6.287}$$

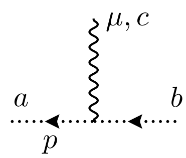
We now take the Fourier transform obtaining

$$\frac{i}{\xi}p^\rho G^{ab}_{\nu\rho}(p) = -ip_\nu \Delta^{ab}(p) - gf^{adc}X^{dcb}_\nu \tag{6.288}$$

where X^{dcb}_ν has been defined before. Multiplying by $G^{-1\nu\mu}\Delta^{-1}$ we then get

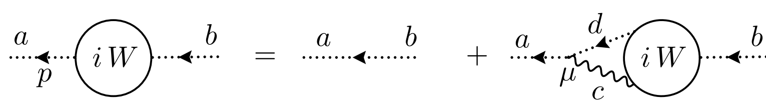
$$p^\mu G^{-1ac}_{\nu\mu} = -\frac{1}{\xi}p_\nu \Delta^{-1ac} + igf^{fde}\chi^{deb}_\nu \Delta^{-1bc}G^{-i\nu\mu af} \tag{6.289}$$

which is precisely Eq. (6.274) The result in Eq. (6.278) can be easily obtained knowing that the only vertex of the ghosts is



$$gf^{abc}p^\mu \tag{6.290}$$

Then



(6.291)

which means

$$\Delta^{ab}(p) = \frac{i}{p^2}\delta^{ab} + \frac{i}{p^2}gf^{adc}p^\mu X^{dcb}_\mu \tag{6.292}$$

or in another form

$$i\Delta^{-1ab} = p^2\delta^{ab} - igf^{adc}p^\mu X^{dcb'}_\mu \Delta^{-1b'b} \tag{6.293}$$

which is precisely Eq. (6.278). The proof of transversality follows now the same steps as in the formal case.

6.3.4 Gauge invariance of the S matrix

We have shown before the gauge invariance of the S matrix using the equivalence theorem and the fact that the generating functionals corresponding to different gauge conditions only differ in the source terms. The proof used some properties of the Coulomb gauge and this can raise some doubts about the genera validity of the argument.

We are going to show here, using the Ward identities, that the functionals Z_F and $Z_{F+\Delta F}$ corresponding to the gauge conditions F and $F + \Delta F$, respectively, only differ in the source terms. As F and ΔF are arbitrary the proof is general. We have

$$Z_F[J^a_\mu, J_i] = \int D(\cdots)e^{i[S_{eff}+\int d^4x(J^a_\mu A^{\mu a}+J_i\phi_i)]} \tag{6.294}$$

Then

$$Z_{F+\Delta F} - Z_F = \int D(\cdots)\int d^4x\ i\left[-\frac{1}{\xi}F^a\Delta F^a - \overline{\omega}^a\int d^4y\frac{\delta\Delta F^a_{(x)}}{\delta A^b_\mu(y)}sA^{b\mu}(y)\right.$$
$$\left. -\overline{\omega}^a\int d^4y\frac{\delta\Delta F^a(x)}{\delta\phi_i(y)}s\phi_i(y)\right] e^{i(S_{eff}+\text{sources})} \tag{6.295}$$

We use now the Ward identities in the form that corresponds to the generating functional Z, that is

$$0 = \int D(\cdots)\int d^4x[J^{\mu a}sA^a_\mu + J^i s\phi_i + \overline{\eta}s\omega - s\overline{\omega}\eta]\ e^{\{i(S_{eff}+J^a_\mu A^{\mu a}+J_i\phi_i+\overline{\omega}\eta+\overline{\eta}\omega)\}} \tag{6.296}$$

Taking the derivative in order to $\eta^a(x)$ and after setting the ghost sources to zero, we get

$$0 = \int D(\cdots)\left[\frac{1}{\xi}F^a(x) + i\overline{\omega}^a(x)\int d^4y[J^{\mu b}sA^b_\mu + J^i s\phi_i]\right] e^{i[S_{eff}+\int d^4x(J^a_\mu A^{\mu i}+J_i\phi_i)]} \tag{6.297}$$

or

$$\begin{aligned} -\tfrac{1}{\xi}F^a\left[\tfrac{\delta}{i\delta J}\right] &\int D(\cdots)e^{i(S_{eff}+\text{sources})} = \\ &= \int D(\cdots)i\overline{\omega}^a(x)\int d^4y[J^{\mu b}sA^b_\mu + J_i s\phi_i]e^{i(S_{eff}+\text{sources})} \end{aligned} \tag{6.298}$$

Then

$$\begin{aligned} \int &\quad \mathcal{D}(\cdots)\left(-\tfrac{1}{\xi}F^a\Delta F^a\right)e^{i(S_{eff}+\text{sources})} = \\ &= \quad \Delta F^a\left[\tfrac{\delta}{i\delta J}\right]\left(-\tfrac{1}{\xi}F^a\left[\tfrac{\delta}{i\delta J}\right]\right)\int D(\cdots)e^{i(S_{eff}+\text{sources})} \\ &= \quad \Delta F^a\left[\tfrac{\delta}{i\delta J}\right]\int D(\cdots)i\overline{\omega}^a(x)\int d^4y[J^{\mu b}sA^b_\mu + J^i s\phi_i]e^{i(S_{eff}+\text{sources})} \\ &= \quad \int\mathcal{D}(\cdots)\left\{\overline{\omega}^a(x)\int d^4y\left[\frac{\delta\Delta F^a(x)}{\delta A^b_\mu(y)}sA^b_\mu(y) + \frac{\delta\Delta F^a(x)}{\delta\phi_i(y)}s\phi_i(y)\right]\right. \\ &\qquad \left.+i\overline{\omega}^a(x)\Delta F^a(x)\int d^4y[J^{\mu b}sA^b_\mu + J^i s\phi_i]\right\}e^{i(S_{eff}+\text{sources})} \end{aligned} \tag{6.299}$$

We get therefore

$$\int\mathcal{D}(\cdots)\left(-\frac{1}{\xi}F^a\Delta F^a - \overline{\omega}^a(x)\int d^4y\left[\frac{\delta\Delta F^a(x)}{\delta A^b_\mu(y)}sA^b_\mu(\eta) + \frac{\delta\Delta F^a}{\delta\phi_i(y)}s\phi_i(y)\right]\right)e^{i(S_{eff}+\text{sources})}$$
$$= \int\mathcal{D}(\cdots)i\overline{\omega}^a(x)\Delta F^a(x)\int d^4y\left[J^{\mu b}sA^b_\mu + J^i s\phi_i\right]e^{i(S_{eff}+\text{sources})} \tag{6.300}$$

We can then write

$$
\begin{aligned}
Z_{F+\Delta F} & -Z_F \\
& = \int \mathcal{D}(\cdots) i \int d^4x \Big[i\overline{\omega}^a(x) \Delta F^a(x) \int d^4y (J^{\mu b} s A^b_\mu + J_i s\phi_i) \Big] e^{i(S_{eff}+\text{sources})} \\
& = \int \mathcal{D}(\cdots) e^{i\left\{S_{eff} + \int d^4y [J^a_\mu(y) \mathcal{A}^{\mu a}(y) + J_i \Phi_i(y)]\right\}}
\end{aligned}
\tag{6.301}
$$

where

$$
\Phi_i(y) \equiv \phi_i(y) + i \int d^4x [\overline{\omega}^a(x) \Delta F^a(x) s\phi_i(y)] \tag{6.302}
$$

and

$$
\mathcal{A}^a_\mu(y) \equiv A^a_\mu(y) + i \int d^4x [\overline{\omega}^b(x) \Delta F^b(x) s A^a_\mu(y)] \tag{6.303}
$$

The difference between the generating functionals $Z_{F+\Delta F}$ and Z_F is only in the functional form of the source terms. We can then use the equivalence theorem to show that the renormalized S matrix are equal in both cases.

$$
S^R_{F+\Delta F} = S^R_F \ . \tag{6.304}
$$

6.4 Ward Takahashi Identities in QED

6.4.1 Ward-Takahashi identities for the functional $Z[J]$

We will now derive again the Ward identities for QED, that we found in our study of renormalization, using now the functional methods. The generating functional for the Green functions for QED is given by, in a linear gauge,

$$
Z(J_\mu, \overline{\eta}, \eta) = \int \mathcal{D}(A_\mu, \psi, \overline{\psi}) e^{i(S_{eff} + \int d^4x (J_\mu A^\mu + \overline{\eta}\psi + \overline{\psi}\eta)} \tag{6.305}
$$

where J_μ, $\overline{\eta}$ e η are the sources for A_μ, ψ and $\overline{\psi}$ respectively. The effective action is given by,

$$
S_{eff} = \int d^4x \left[\mathcal{L}_{QED} - \frac{1}{2\xi} (\partial \cdot A)^2 \right] = S_{QED} + S_{GF} \tag{6.306}
$$

where

$$
\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi \ . \tag{6.307}
$$

where

$$
D_\mu = \partial_\mu - i\, e A_\mu \tag{6.308}
$$

with our convention that $e = |e|$ and the electron charge is given by $q_e = -e < 0$. S_{QED} is invariant under local gauge transformation of the group $U(1)$ that we write as,

$$
\begin{cases}
\delta A_\mu = \partial_\mu \Lambda \\
\delta \psi = ie\Lambda\psi \\
\delta \overline{\psi} = -ie\Lambda\overline{\psi}
\end{cases}
$$

The S_{eff} contains the part of the gauge fixing that it is not invariant under these transformations. Therefore the Ward identities take the form,

$$\left(\frac{\delta S_{GF}}{\delta\phi_i}\left[\frac{\delta}{i\delta J}\right]+J_i\right)F_i\left[\frac{\delta}{i\delta J}\right]Z(J)=0 \tag{6.309}$$

This can be written in our case as, putting back the explicit integrations,

$$0=\int d^4x\left[\frac{1}{\xi}\partial^\mu\partial_\nu\left(\frac{\delta}{i\delta J_\nu}\right)\partial_\mu\Lambda+J^\mu\partial_\mu\Lambda+ie\Lambda\overline{\eta}\frac{\delta}{i\delta\overline{\eta}}-ie\Lambda\eta\frac{\delta}{i\delta\eta}\right]Z(J^\mu,\overline{\eta},\eta) \tag{6.310}$$

After an integration by parts we get,

$$\int d^4x\ \Lambda\left[-\frac{1}{\xi}\Box\partial_\nu\left(\frac{\delta}{i\delta J_\nu}\right)-\partial_\mu J^\mu+ie\overline{\eta}\frac{\delta}{i\delta\overline{\eta}}-ie\eta\frac{\delta}{i\delta\eta}\right]Z(J^\mu,\overline{\eta},\eta)=0 \tag{6.311}$$

This can be written as

$$\left[\frac{1}{\xi}\Box\partial_\mu\left(\frac{\delta}{i\delta J_\mu}\right)+\partial_\mu J^\mu-ie\overline{\eta}\left(\frac{\delta}{i\delta\overline{\eta}}\right)+ie\eta\left(\frac{\delta}{i\delta\eta}\right)\right]Z(J,\overline{\eta},\eta)=0 \tag{6.312}$$

6.4.2 Ward-Takahashi identities for the functionals W and Γ

From the point of view of the applications it is more useful the Ward identity for the generating functional of the irreducible Green functions. This problem is simpler than in the case of non-abelian gauge theories, that we just discuss, as the the previous equation is linear in the functional derivatives with respect to the different sources (we notice that if we had chosen a non-linear gauge fixing this would not be true, even in QED). The linearity allow us to write immediately

$$\partial_\mu J^\mu+\left[\frac{1}{\xi}\Box\partial_\mu\left(\frac{\delta}{i\delta J_\mu}\right)-ie\overline{\eta}\frac{\delta}{i\delta\overline{\eta}}+ie\eta\frac{\delta}{i\delta\eta}\right]W(J,\overline{\eta},\eta)=0 \tag{6.313}$$

where W is the generating functional for the connected Green functions,

$$Z(J^\mu,\overline{\eta},\eta)\equiv e^{iW(J^\mu,\overline{\eta},\eta)} \tag{6.314}$$

As we saw the generating functional for the irreducible Green functions is given by,

$$\Gamma(A_\mu,\psi,\overline{\psi})=W(J_\mu,\overline{\eta},\eta)-\int d^4x[J^\mu A_\mu+\overline{\eta}\psi+\overline{\psi}\eta] \tag{6.315}$$

We also have the relations

$$A_\mu=\frac{\delta W}{i\delta J^\mu}\ ;\ \psi=\frac{\delta W}{i\delta\overline{\eta}}\ ;\ \overline{\psi}=-\frac{\delta W}{i\delta\eta} \tag{6.316}$$

and

$$J_\mu=-\frac{\delta\Gamma}{\delta A^\mu}\ ;\ \eta=-\frac{\delta\Gamma}{\delta\overline{\psi}}\ ;\ \overline{\eta}=\frac{\delta\Gamma}{\delta\psi} \tag{6.317}$$

where, as usual, the fermionic derivatives are left derivatives. We can them write

$$\frac{1}{\xi}\Box\partial_\mu A^\mu-\partial_\mu\frac{\delta\Gamma}{\delta A_\mu}-ie\frac{\delta\Gamma}{\delta\psi}\psi-ie\overline{\psi}\frac{\delta\Gamma}{\delta\overline{\psi}}=0 \tag{6.318}$$

This equation is the starting point to generate all the Ward identities in QED. Its application it is much easier than the equivalent expression that was proved using the canonical formalism. The functional methods make this expressions particularly simple.

6.4.3 Example: Ward identity for the QED vertex

To convince ourselves that this equation reproduces the Ward identities that we already know, let us derive the Ward identity for the vertex in QED. We apply $\dfrac{\delta^2}{\delta\psi_\alpha(y)\delta\overline{\psi}_\beta(z)}$ to the master equation. We get then

$$\begin{aligned}\partial_x^\mu &\frac{\delta^3\Gamma}{\delta\psi_\alpha(y)\delta\overline{\psi}_\beta(z)\delta A^\mu(x)}\\ &= -\,i\,e\left[\frac{\delta^2\Gamma}{\delta\psi_\alpha(y)\delta\overline{\psi}_\beta(x)}\delta^4(z-x)-\frac{\delta^2\Gamma}{\delta\psi_\alpha(x)\delta\overline{\psi}_\beta(z)}\delta^4(y-x)\right]\end{aligned} \tag{6.319}$$

This equation means

$$\partial_x^\mu\Gamma_{\mu\beta\alpha}(x,z,y) = -i\,e\left[\Gamma_{\beta\alpha}(x,y)\delta^4(z-x)-\Gamma_{\beta\alpha}(z,x)\delta^4(y-x)\right] \tag{6.320}$$

Taking now the Fourier transform to both sides of the equation, with the momenta defined as the Fig. 6.3, we get,

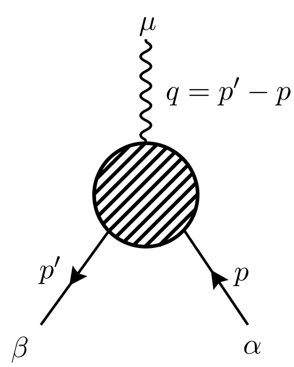


Figure 6.3: Definition of the momenta for the vertex.

$$q^\mu\Gamma_\mu(p',p) = -i\,e[S^{-1}(p)-S^{-1}(p')] \tag{6.321}$$

This is precisely the well known[3] Ward identity, Eq. (4.155).

6.4.4 Ghosts in QED

We said before that the generating functional for QED was given by,

$$Z(J_\mu,\overline{\eta},\eta) = \int\mathcal{D}(A_\mu,\psi,\overline{\psi})e^{i\int d^4x[\mathcal{L}_{QED}+\mathcal{L}_{GF}+J_\mu A^\mu+\overline{\eta}\psi+\overline{\psi}\eta]} \tag{6.322}$$

[3]Note that our convention is that $e = |e|$.

where $\mathcal{L}_{QED}$ is the usual Lagrangian for QED and the gauge fixing term was,

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial \cdot A)^2 \ . \tag{6.323}$$

In fact this is not strictly true. If we use the prescription for the gauge theories, we would get instead,

$$\tilde{Z}(J_\mu, \eta, \overline{\eta}, \zeta, \overline{\zeta}) = \int D(A_\mu, \psi, \overline{\psi}, \omega, \overline{\omega}) e^{i\int d^4x[\mathcal{L}_{eff} + J^\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta + \overline{\omega}\zeta + \overline{\zeta}\omega]} \tag{6.324}$$

In this expression ω e $\overline{\omega}$ are anti-commuting scalar fields known as the *Faddeev-Popov ghosts* as we saw before. Although in physical process they never appear as external states, it is useful to introduce also sources for them to discuss the Ward identities.

In the previous action, the Lagrangian $\mathcal{L}_{\text{eff}}$ is

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QED}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{G}} \tag{6.325}$$

where

$$\mathcal{L}_{\text{G}} = -\overline{\omega} \,\Box\, \omega \tag{6.326}$$

The reason why in QED we can work with the functional Z instead of $\tilde{Z}$ is because the ghosts do not have interactions with the gauge fields and can be integrated out (Gaussian integration) and absorbed in the normalization. Nevertheless, for the Ward identities it is useful to keep them. The effective Lagrangian, $\mathcal{L}_{\text{eff}}$, is invariant under the BRS transformations given by,

$$\begin{cases} \delta\psi = i\, e\omega\theta\psi \\ \delta\overline{\psi} = -i\, e\overline{\psi}\omega\theta \\ \delta A_\mu = \partial_\mu\omega\theta \\ \delta\overline{\omega} = \frac{1}{\xi}(\partial \cdot A)\theta \\ \delta\omega = 0 \end{cases} \tag{6.327}$$

The parameter θ is an anti-commuting (*Grassmann variable*). The BRS transformations on the physical fields are gauge transformations with parameter $\Lambda = \omega\theta$ and therefore $\mathcal{L}_{QED}$ is left invariant. The transformations in the ghosts ω and $\overline{\omega}$ are such that the variation of $\mathcal{L}_{GF}$ cancels that of $\mathcal{L}_G$, just like in the non-abelian case. The invariance of the integration measure and of S_{eff} allows us to write immediately the Ward identities for the generating functionals. The BRS transformations allow us to obtain the Ward identities in a quick way without having to resort to the functional $\tilde{\Gamma}$. This method is based on the fact, as we saw in Theorem 6.7, that the application of the operator δ_{BRS} to any Green function gives zero, that is

$$\delta_{\text{BRS}} \langle 0| \, TA_{\mu_1} \cdots \overline{\omega} \cdots \omega \cdots \psi \cdots \overline{\psi} \cdots |0\rangle = 0 \tag{6.328}$$

Let us show two simple applications of the method in QED.

1) The non-renormalization of the longitudinal photon propagator

This result is equivalent, as we have seen, to the statement that the vacuum polarization is transversal. It is proved easily starting with the Green function, $\langle 0|\, T A_\mu \overline{\omega}\, |0\rangle$, and using

$$\delta_{\text{BRS}} \langle 0|\, T A_\mu \overline{\omega}\, |0\rangle = 0 \tag{6.329}$$

This gives

$$\frac{1}{\xi} \langle 0|\, |T A_\mu \partial^\nu A_\nu\, |0\rangle\, \theta - \langle 0|\, T \partial_\mu \omega \overline{\omega}\, |0\rangle\, \theta = 0 \tag{6.330}$$

After taking the Fourier transform we get

$$\frac{1}{\xi} k^\mu G_{\mu\nu}(k) = -k_\nu \Delta(k) \tag{6.331}$$

where the ghost propagator is the free propagator

$$\Delta(k) = \frac{i}{k^2} \tag{6.332}$$

because the ghosts have no interactions. Multiplying by the inverse propagator of the photon we get

$$\frac{1}{\xi} k^\mu = -i \frac{k_\nu}{k^2} G^{-1\nu\mu}(k) \tag{6.333}$$

Therefore

$$k_\nu G^{-1\nu\mu}(k) = \frac{i}{\xi} k^\mu k^2 = k_\nu G^{-1\nu\mu}_{(0)}(k) \tag{6.334}$$

This shows that the longitudinal part of the photon propagator is equal to the free longitudinal part and therefore does not get any renormalization.

2) Ward Identity for the Vertex

For the vertex we start from

$$\delta_{\text{BRS}} \langle 0|\, T \overline{\omega} \psi \overline{\psi}\, |0\rangle = 0 \tag{6.335}$$

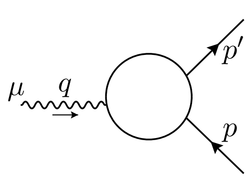
This means

$$\frac{1}{\xi} \langle 0|\, T \partial^\mu A_\mu \psi \overline{\psi}\, |0\rangle = -ie \langle 0|\, T \overline{\omega} \omega \psi \overline{\psi}\, |0\rangle + ie \langle 0|\, T \overline{\omega} \psi \overline{\psi} \omega\, |0\rangle \tag{6.336}$$

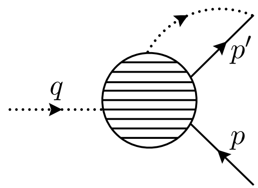
After taking the Fourier transform we get

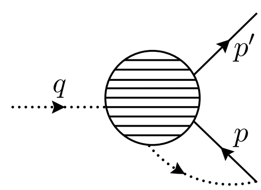
$$\frac{i}{\xi} q^\mu\, i\, T_\mu = i\, T \tag{6.337}$$

where we have defined

$iT_\mu =$ 

$= G_{\mu\nu}(q) S(p') i\Gamma^\nu S(p)$ (6.338)

$iT = ie$ 

$- ie$ 

$$= + ie\Delta(q)S(p) - ie\Delta(q)S(p') \tag{6.339}$$

The last equality results from the fact that ghosts have no interactions in QED in a linear gauge[4]. Putting everything together we get

$$\frac{i}{\xi} q^\mu G_{\mu\nu}(q) S(p') i\Gamma^\nu S(p) = ie\Delta(q)S(p) - ie\Delta(q)S(p') \tag{6.340}$$

Using

$$\frac{1}{\xi} k^\mu G_{\mu\nu}(k) = -k_\nu \Delta(k) \tag{6.341}$$

and multiplying by the inverse of the fermion propagators we get again the well known the Ward identity for the vertex,

$$q_\mu \Gamma^\mu(p', p) = -i\, e \left[S^{-1}(p) - S^{-1}(p') \right] \tag{6.342}$$

6.5 Unitarity and Ward Identities

6.5.1 Optical Theorem

The S matrix, (Heisenberg 1942), can be written in the form

$$S = 1 + iT \tag{6.343}$$

Then its unitarity $SS^\dagger = 1$ implies,

$$2\,\mathrm{Im} T = TT^\dagger \tag{6.344}$$

If we insert this relation between the same initial and final state (elastic scattering) we get

$$\begin{aligned} 2\,\mathrm{Im}\,\langle i|T|i\rangle &= \left\langle i|TT^\dagger|i\right\rangle \\ &= \sum_f |\,\langle f|T|i\rangle\,|^2 \end{aligned} \tag{6.345}$$

[4]Also note that $\Delta(q) = \mathrm{FT}\ \langle 0|\, T\omega\overline{\omega}\,|0\rangle$ and $= \langle 0|\, T\overline{\omega}\omega\psi\overline{\psi}\,|0\rangle = -\langle 0|\, T\omega\overline{\omega}\psi\overline{\psi}\,|0\rangle$, hence the change of signs in Eq. (6.339).

where we have introduced a complete set of states. This relation can still be written in the form,

$$\sigma_{\text{total}} = 2\,\text{Im}\; T^{\text{elastic}}_{\text{forward}} \tag{6.346}$$

known as the optical theorem. What we call here σ_{total} it is not exactly the cross section, because the flux factors are mixing. It is for our purpose the quantity defined by

$$\sigma_{\text{total}} \equiv \sum_f |\,\langle f|T|i\rangle\,|^2 \tag{6.347}$$

Unitarity establishes therefore a relation between the total cross section and the imaginary part of the elastic amplitude in the forward direction (the initial and final state have to be the same).

6.5.2 Cutkosky rules

To show that unitarity is obeyed in a given process we have to know how to calculate the imaginary part of Feynman diagrams. Of course there is always the possibility of doing explicitly the calculations and retrieve the imaginary part, but this only possible for simple diagrams (see below). Therefore it is useful to have rules, known as Cutkosky rules, that give us the imaginary part of any diagram. We will state them now.

Rule 1

The imaginary part of an amplitude is obtained using the expression

$$2\,\text{Im}\; T = -\sum_{\text{cuts}} T \tag{6.348}$$

Rule 2

The cut is obtained by writing the amplitude $iT = \cdots$ and substituting in this expression the propagators of the lines we cut by the following expression,

- *Scalar fields*

$$\Delta(p) \Rightarrow 2\pi\theta(p^o)\delta(p^2 - m^2) \tag{6.349}$$

- *Fermion fields*

$$S(p) \Rightarrow (\not{p} + m)2\pi\theta(p^o)\delta(p^2 - m^2) \tag{6.350}$$

- *Vector gauge fields (in the Feynman gauge)*

$$G_{\mu\nu}(p) \Rightarrow -g_{\mu\nu}2\pi\theta(p^0)\delta(p^2 - m^2) \tag{6.351}$$

In these expression the θ functions ensure the energy flux. The Cutkosky rules are complicated to prove in general (see G. 't Hooft, "Diagrammar", CERN Report 1972) but we are going to show in the two explicit examples how they work.

Example 6.1 *Free propagator*

For the free propagator of a scalar field the amplitude is

$$iT = \frac{i}{p^2 - m^2 + i\varepsilon} \tag{6.352}$$

The imaginary part is obtained using

$$\frac{1}{x + i\varepsilon} = P\left(\frac{1}{x}\right) - i\pi\delta(x) \tag{6.353}$$

Therefore

$$T = P\left(\frac{1}{p^2 - m^2}\right) - i\pi\delta(p^2 - m^2) \tag{6.354}$$

The imaginary part is then

$$2\,ImT = -2\pi\delta(p^2 - m^2) \tag{6.355}$$

Using the Cutkosky rule we get

$$2\,ImT = -2\pi\delta(p^2 - m^2)\theta(p^0) \tag{6.356}$$

which is precisely the same result. The function $\theta(p^0)$ tell us that the flux of energy is from left to right.

Example 6.2 *Self-energy in ϕ^3*

Let us consider the self-energy in the theory given by the Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3 \tag{6.357}$$

The self-energy is given by the diagram in the Fig. 6.4. The corresponding amplitude

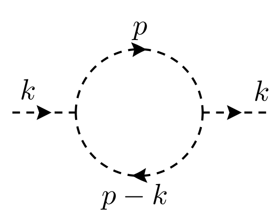


Figure 6.4: Self-energy

is

$$iT = (-i\lambda)^2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} \quad \frac{i}{(p-k)^2 - m^2 + i\varepsilon} \tag{6.358}$$

Let us calculate the imaginary part of T by two methods, first doing the explicit calculation and second using the Cutkosky rule.

i) Explicit Calculation

$$\begin{aligned}
iT =& \lambda^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - m^2 + i\varepsilon)[(p-k)^2 - m^2 + i\varepsilon]} \\
=& \lambda^2 \int \frac{d^4p}{(2\pi)^4} \int_0^1 dx \frac{1}{(p^2 + 2p\cdot P - M^2 + i\varepsilon)^2} \\
=& \lambda^2 \int \frac{d^4p}{(2\pi)^4} \int_0^1 dx \frac{1}{[(p+P)^2 - \Delta]^2}
\end{aligned} \tag{6.359}$$

where

$$\begin{cases} P &= -x\,k \\ \Delta &= P^2 + M^2 = m^2 - k^2 x(1-x) - i\varepsilon \end{cases} \tag{6.360}$$

The amplitude is then

$$iT = \lambda^2 \int \frac{d^4p}{(2\pi)^4} \int_0^1 dx \frac{1}{(p^2 - \Delta)^2} \tag{6.361}$$

The integral is divergent. Using dimensional regularization we get

$$T = \frac{\lambda}{16\pi^2} \mu^{\varepsilon} \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 dx \Delta^{-\left(2-\frac{d}{2}\right)} \tag{6.362}$$

Choosing on-shell renormalization, $T_R(k^2 = m^2) = 0$*, we get*

$$\begin{aligned}
T_R =& T - T(k^2 = m^2) \\
=& \frac{\lambda^2}{16\pi^2} \Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 dx \left[\left(\frac{\Delta(k^2)}{\mu^2}\right)^{-\frac{\varepsilon}{2}} - \left(\frac{\Delta(k^2 = m^2)}{\mu^2}\right)^{-\frac{\varepsilon}{2}} \right] \\
=& \frac{\lambda^2}{16\pi^2} \left(\frac{2}{\varepsilon} - C + O(\varepsilon)\right) \int_0^1 dx \left[1 - 1 - \frac{\varepsilon}{2} \ln \frac{m^2 - k^2 x(1-x) - i\varepsilon}{m^2 - m^2 x(1-x) - i\varepsilon} \right] \\
=& -\frac{\lambda^2}{16\pi^2} \int_0^1 dx \ln\left[\frac{1 - \beta x(1-x) - i\varepsilon}{1 - x(1-x) - i\varepsilon}\right] = -\frac{\lambda^2}{16\pi^2} \left[L(\beta) - L(1)\right]
\end{aligned} \tag{6.363}$$

In this expression $\beta = \frac{k^2}{m^2}$ *and the function* $L(\beta)$ *is given by*

$$L(\beta) \equiv \int_0^1 dx \ln\left[1 - \beta(1-x)x - i\varepsilon\right] \tag{6.364}$$

It satisfies

$$ImL(\beta) = -\pi\sqrt{1 - \frac{4}{\beta}}\ \theta(\beta - 4) \tag{6.365}$$

Therefore

$$ImT = -\frac{\lambda^2}{16\pi^2}\left[ImL(\beta) - ImL(1)\right] \tag{6.366}$$

and we get finally,

$$ImT = \frac{\lambda^2}{16\pi}\sqrt{1 - \frac{4m^2}{k^2}}\ \theta\left(1 - \frac{4m^2}{k^2}\right) \tag{6.367}$$

The θ functions ensures that there is only imaginary part when the intermediate state could also be a final state (production of two particles of mass m).

ii) Using the Cutkosky rules

Using the rules we get

$$\begin{aligned}
2ImT =& -(i\lambda)^2 \int \frac{d^4p}{(2\pi)^4}(2\pi)^2\theta(p^0)\theta(k^0-p^0)\delta(p^2-m^2)\delta((p-k)^2-m^2) \\
=&\lambda^2 \int \frac{d^4p}{(2\pi)^4} d^4p'(2\pi)^2\theta(p^0)\theta(k^0-p^0)\delta(p^2-m^2)\delta(p'^2-m^2)\delta^4(p'-k+p)
\end{aligned} \tag{6.368}$$

Using now the result

$$\int d^4p\,\theta(p^0)\delta(p^2-m^2) = \int d^3p\frac{1}{2p^0} \tag{6.369}$$

We get

$$2ImT = \lambda^2 \int \frac{d^3p}{(2\pi)^3} d^3p' \frac{1}{2p^0}\frac{1}{2p'^0} 2\pi\delta^4(p'-k+p) \tag{6.370}$$

or

$$2ImT = \lambda^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0}\frac{1}{2p'^0} 2\pi\delta(k^0-p^0-p'^0) \tag{6.371}$$

$$k=(\sqrt{s},\vec{0}) \ ; \ p=(\sqrt{|\vec{p}|^2+m^2},\vec{p}) \ ; \ p'=(\sqrt{|\vec{p}'|^2+m^2},-\vec{p}) \tag{6.372}$$

Therefore we get

$$\begin{aligned}
2ImT =&\lambda^2 \int \frac{d^3p}{(2\pi)^3}\frac{1}{4(|\vec{p}|^2+m^2)} 2\pi\delta(\sqrt{s}-2\sqrt{|\vec{p}|^2+m^2}) \\
=&\frac{\lambda^2}{4\pi}\int d|\vec{p}|\frac{|\vec{p}|^2}{|\vec{p}|^2+m^2}\frac{\delta(|\vec{p}|-\sqrt{\frac{s}{4}-m^2})}{\frac{2|\vec{p}|}{\sqrt{|\vec{p}|^2+m^2}}}\theta\left(1-\frac{4m^2}{s}\right) \\
=&\frac{\lambda^2}{8\pi}\sqrt{1-\frac{4m^2}{s}}\ \theta\left(1-\frac{4m^2}{s}\right)
\end{aligned} \tag{6.373}$$

Using $s=k^2$ we get

$$ImT = \frac{\lambda^2}{16\pi}\sqrt{1-\frac{4m^2}{k^2}}\ \theta\left(1-\frac{4m^2}{k^2}\right) \tag{6.374}$$

which is the same result as we got in the explicit calculation.

6.5.3 Example of Unitarity: scalars and fermions

As an example of checking the unitarity let us consider a theory described by the Lagrangian,

$$\mathcal{L} = \overline{i\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + g\overline{\psi}\psi\phi \tag{6.375}$$

We will show unitarity in two cases (cutting fermions lines):

i) Scalar Self-energy

The self-energy of the scalars is given by the diagram in Fig. 6.5, to which corresponds the amplitude,

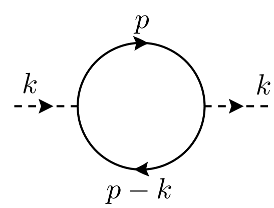


Figure 6.5: Fermion contribution to the scalar self-energy.

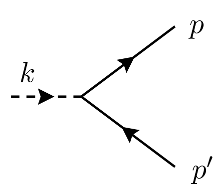
$$iT = g^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m + i\varepsilon}\frac{i}{\not p - \not k - m + i\varepsilon}\right] \tag{6.376}$$

Applying Cutkosky rules we get,

$$\begin{aligned} 2\,\mathrm{Im}T &= -\sum_{\text{cuts}} T \\ &= -\,g^2 \int \frac{d^4p}{(2\pi)^4} Tr[(\not p + m)(\not p - \not k + m)](2\pi)\theta(p^0)\delta(p^2 - m^2) \\ &\qquad (2\pi)\theta(k^0 - p^0)\delta((p-k)^2 - m^2) \end{aligned} \tag{6.377}$$

To show the unitarity we calculate the cross section,

$$\sigma = \sum_f \Bigg| \qquad \Bigg|^2 \tag{6.378}$$



We get

$$\sigma = \sum_f |ig\overline{u}(p)v(p')|^2 = -g^2 \sum_f \mathrm{Tr}[(\not p + m)(-\not p' + m)] \tag{6.379}$$

where we have used $\sum_{\text{spins}} v(p')\overline{v}(p) = -(-\not p' + m)$ and $\sum_{\text{spins}} u(p)\overline{u}(p') = \not p + m$. Therefore

$$\sigma = -g^2 \int d\rho_2 \mathrm{Tr}[(\not p + m)(-\not p' + m)] \tag{6.380}$$

where $d\rho_2$ is the phase space of two particles, that is,

$$\begin{aligned} \int d\rho_2 \equiv& \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{2p^0}\frac{1}{2p'^0}(2\pi)^4\delta^4(k - p - p') \\ =& \int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}(2\pi)\theta(p^0)\delta(p^2 - m^2)(2\pi)\theta(p'^0)\delta(p'^2 - m^2)(2\pi)^4\delta^4(k-p-p') \end{aligned} \tag{6.381}$$

We conclude then that

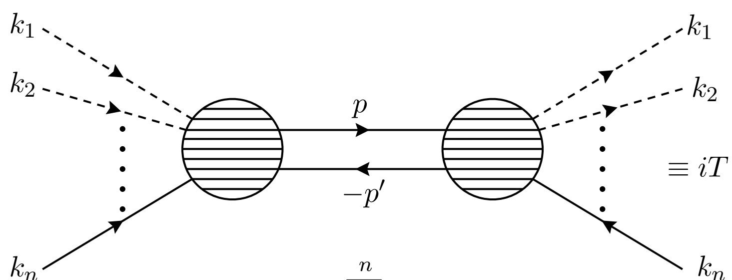
$$\sigma = -\,g^2 \int \frac{d^4p}{(2\pi)^4}(2\pi)\theta(p^0)\delta(p^2-m^2)(2\pi)\theta(k^0-p^0)\delta((p-k)^2-m^2)$$
$$\mathrm{Tr}[(\not{p}+m)(\not{p}-\not{k}+m)] \tag{6.382}$$

Comparing we obtain

$$2\mathrm{Im}T = \sigma \tag{6.383}$$

ii) General case

Let us consider the general case of two internal fermion lines. The amplitude iT is represented by the diagram

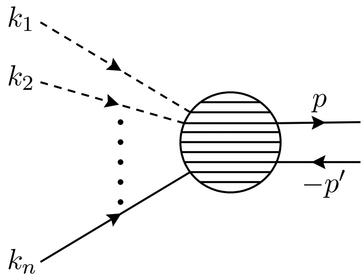


$$p' = \sum_{i=1}^{n} k_i - p \tag{6.384}$$

The amplitude iT is given by

$$iT = -\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\overline{T'}S(p)T'S(-p')\right] \tag{6.385}$$

Where we have defined the amplitude iT' by



$$\equiv \overline{u}(p)iT'v(p') \tag{6.386}$$

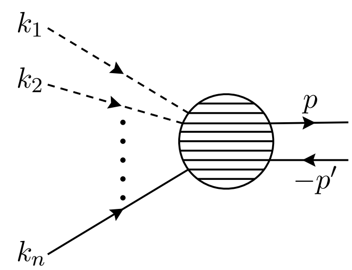
Therefore

$$2\,\mathrm{Im}T = -\int \frac{d^4p}{(2\pi)^4}(2\pi)^2\delta(p^2-m^2)\theta(p^0)\delta(p'^2-m^2)\theta(p'^0)$$

$$\begin{aligned}&\mathrm{Tr}\Big[\overline{T'}(\not p+m)T'(-\not p'+m)\Big]\\ &= -\int d\rho_2 \mathrm{Tr}\Big[\overline{T'}(\not p+m)T'(-\not p'+m)\Big] \qquad (6.387)\end{aligned}$$

On the other hand

$$\sigma = \sum_f \Bigg|$$



$$\Bigg|^2$$

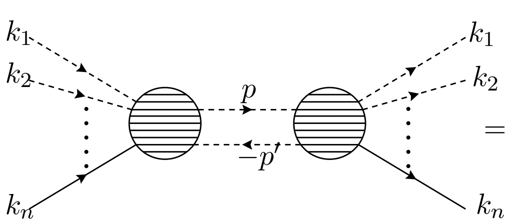
$$\begin{aligned} &= \sum_f |\overline{u}(p)T'v(p')|^2 \\ &= -\int d\rho_2 \mathrm{Tr}\Big[(\not p+m)T'(-\not p'+m)\overline{T'}\Big] \qquad (6.388)\end{aligned}$$

Therefore

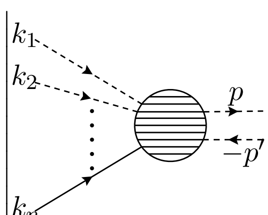
$$\sigma = 2\mathrm{Im}T \qquad (6.389)$$

If the lines to be cut were scalars the result would be the same. In this case there will be no minus sign from the loop but there will be no minus sign from the spins sum. The proof is left as an exercise.

$2\,\mathrm{Im}$



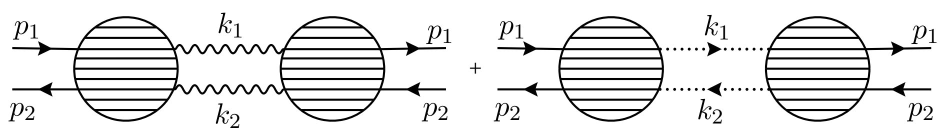
$= \sum_f \Bigg|$

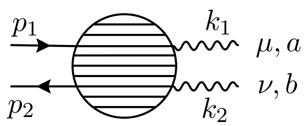


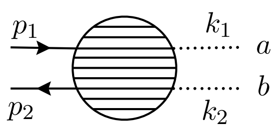
$\Bigg|^2 \qquad (6.390)$

6.5.4 Unitarity and gauge fields

In the previous slides we have shown that unitarity holds for theories with scalar and fermion fields. We are now going to show that the proof of unitarity for gauge theories is more complicated and requires the use of the ward identities. The problem resides in the fact that the gauge fields in internal lines have unphysical polarizations while the final states should have only physical degrees of freedom. This difference would lead to a violation of unitarity in gauge theories. However we will show that the ghosts in the internal lines will compensate for the this and will make the theory unitary as it should. Let us define the following amplitudes

$iT =$ 

$iT^{ab}_{\mu\nu} =$ 

$iT^{ab} =$ 

(6.391)

where

$$k_2 = p_1 + p_2 - k_1 \tag{6.392}$$

Using these definitions we can write the amplitude in the form (the factor 1/2 is a symmetry factor for the gauge fields and the minus sign is for the loop of ghosts)

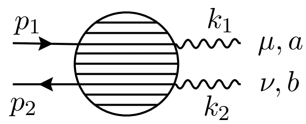
$$iT = \int \frac{d^4k_1}{(2\pi)^4} \left\{ \frac{1}{2} T^{ab}_{\mu\nu} G^{aa'}_{\mu\mu'}(k_1) G^{bb'}_{\nu\nu'}(k_2) T^{*a'b'\mu'\nu'} - T^{ab} \Delta^{aa'}(k_1) \Delta^{bb'}(k_2) T^{*a'b'} \right\} \tag{6.393}$$

Applying the Cutkosky rules we find for the imaginary part

$$\begin{aligned} 2\,\mathrm{Im}T &= \int \frac{d^4k_1}{(2\pi)^4} (2\pi)^2 \theta(k_1^0)\theta(k_2^0)\delta(k_1^2)\delta(k_2^2) \left\{ \frac{1}{2} T^{ab}_{\mu\nu} T^{*ab\mu\nu} - T^{ab} T^{*ab} \right\} \\ &\equiv \int d\rho_2 \left[\frac{1}{2} T^{ab}_{\mu\nu} T^{*ab\mu\nu} - T^{ab} T^{*ab} \right] \end{aligned} \tag{6.394}$$

Now we have to evaluate σ_{total}. As the ghosts are not physical we have

$$\sigma \;=\; \sum \Bigg|$$



$$\Bigg|^2$$

$$= \frac{1}{2}\int d\rho_2 \sum_{Pol} \left| \varepsilon^\mu(k_1)\varepsilon^\nu(k_2) T^{ab}_{\mu\nu} \right|^2 \tag{6.395}$$

where the factor 1/2 comes now from identical particles in the final state. Writing

$$\sum_{\text{Pol}} \varepsilon^\mu(k_1)\varepsilon^{\mu'*}(k_1) = P^{\mu\mu'}(k_1) \tag{6.396}$$

we get

$$\sigma = \int d\rho_2 \frac{1}{2} T^{ab}_{\mu\nu} T^{*ab}_{\mu'\nu'} P^{\mu\mu'}(k_1) P^{\nu\nu'}(k_2) \tag{6.397}$$

We now use the result (see problems)

$$P^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu \eta^\nu + k^\nu \eta^\mu}{k\cdot\eta} \tag{6.398}$$

where η^μ is a four-vector that satisfies $\eta\cdot\varepsilon$ and $\eta^2 = 0$. We get

$$\begin{aligned}
&\frac{1}{2} T^{ab}_{\mu\nu} T^{*ab}_{\mu'\nu'} P^{\mu\mu'}(k_1) P^{\nu\nu'}(k_2) = \\
&\qquad = \frac{1}{2} T^{ab}_{\mu\nu} T^{*ab\mu\nu} - \frac{1}{2}(T^{ab}\cdot k_2)\cdot(T^{*ab}\cdot\eta)\frac{1}{k_2\cdot\eta} \\
&\qquad - \frac{1}{2}(T^{ab}\cdot\eta)\cdot(T^{*ab}\cdot k_2)\frac{1}{k_2\cdot\eta} - \frac{1}{2}(k_1\cdot T^{ab})\cdot(\eta\cdot T^{*ab})\frac{1}{k_1\cdot\eta} \\
&\qquad - \frac{1}{2}(\eta\cdot T^{ab})\cdot(k_1\cdot T^{*ab})\frac{1}{k_1\cdot\eta} + \Bigg[\frac{1}{2}(k_1\cdot T^{ab}\cdot\eta)(\eta\cdot T^{*ab}\cdot k_2) + \\
&\qquad + \frac{1}{2}(k_1\cdot T^{ab}\cdot k_2)(\eta\cdot T^{*ab}\cdot\eta) + \frac{1}{2}(\eta\cdot T^{ab}\cdot\eta)(k_1 T^{*ab}\cdot k_2) \\
&\qquad + \frac{1}{2}(\eta\cdot T^{ab}\cdot k_2)(k_1\cdot T^{*ab}\cdot\eta)\Bigg]\frac{1}{(k_1\cdot\eta)(k_2\cdot\eta)}
\end{aligned} \tag{6.399}$$

Using the following Ward identities (see problems),

$$\begin{aligned} k_1^\mu T^{ab}_{\mu\nu} &= k_{2\nu} T^{ab} \\ k_2^\mu T^{ab}_{\mu\nu} &= k_{1\nu} T^{ab} \end{aligned} \qquad \Longrightarrow \qquad k_1\cdot T^{ab}\cdot k_2 = 0 \tag{6.400}$$

we get

$$\frac{1}{2} T^{ab}_{\mu\nu} T^{*ab}_{\mu'\nu'} P^{\mu\mu'}(k_1) P^{\nu\nu'}(k_2) =$$

$$
\begin{aligned}
=&\frac{1}{2}T^{ab}_{\mu\nu}T^{*ab\mu\nu} - \frac{1}{2}T^{ab}(k_1\cdot T^{*ab}\cdot\eta)\frac{1}{k_2\cdot\eta} \\
&-\frac{1}{2}T^{*ab}(k_1\cdot T^{ab}\cdot\eta)\frac{1}{k_2\cdot\eta} - \frac{1}{2}T^{ab}(\eta\cdot T^{*ab}\cdot k_2)\frac{1}{k_1\cdot\eta} \\
&-\frac{1}{2}(\eta\cdot T^{ab}\cdot k_2)T^{*ab}\frac{1}{k_1\cdot\eta} + \frac{1}{2}T^{ab}T^{*ab} + \frac{1}{2}T^{ab}T^{*ab} \\
=&\frac{1}{2}T^{ab}_{\mu\nu}T^{*ab\mu\nu} - T^{ab}T^{*ab}
\end{aligned}
\tag{6.401}
$$

Therefore after the sum over polarizations is correctly taken in account we obtain,

$$
\sigma = \int d\rho_2 \left[\frac{1}{2}T^{ab}_{\mu\nu}T^{*ab\mu\nu} - T^{ab}T^{*ab}\right] \tag{6.402}
$$

Comparing with the expression for $2\,\mathrm{Im}T$ we get

$$
\sigma = 2\,\mathrm{Im}T \tag{6.403}
$$

as we wanted to show.

It should be clear that the ghosts with their minus sign of the loop played a crucial role in subtracting the extra degrees of freedom. Also the Ward identities were necessary to relate the gauge field amplitudes with the ghost amplitude.

Problems for Chapter 6

6.1 Show that $T(R)$ is related with the Casimir operator of the representation R , $C_2(R)$, through the relation,

$$T(R)r = d(R)C_2(R) \tag{6.404}$$

where r is the dimension of the Group G and $d(R)$ is the dimension of the representation R. The Casimir operator, $C_2(R)$, is defined by

$$\sum_{a,k} T^a_{ik} T^a_{kj} = \delta_{ij} ... C_2(R) \ . \tag{6.405}$$

6.2 Show that a different choice for the auxiliary conditions $\chi^{i\alpha} = 0$ leads to the same result. For this consider an infinitesimal variation

$$\chi^\alpha + \delta\chi^\alpha = 0 \qquad\qquad \alpha = 1, ...m \tag{6.406}$$

Show that one gets

$$\pi_\alpha \delta(\varphi_\alpha)\delta(\chi_a) \det(\{\varphi, \chi\}) \to \pi_\alpha \delta(\varphi_\alpha \delta(\chi_\alpha + \delta\chi_\alpha) \det(\{\varphi, \chi + \delta\chi\}) \ . \tag{6.407}$$

6.3 Show that for infinitesimal transformations

$$\begin{aligned} \delta\vec{E}_a(x) &= -\frac{1}{g}\int_{x_o=y_o} d^3y \{\vec{E}_a(x), \alpha^b(y) C_b(y)\} \\ \delta\vec{A}_a(x) &= -\frac{1}{g}\int_{x_o=y_o} d^3y \{\vec{A}_a(x), \alpha^b(y) C_b(y)\} \end{aligned} \tag{6.408}$$

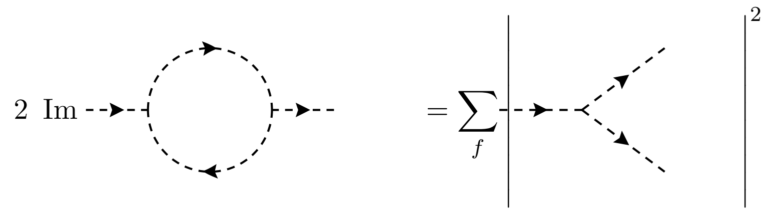
that is, the constraints C_a are the generators for the time independent gauge transformations.

6.4 Show that it is always possible to find a gauge where $A^3_a = 0 \qquad a = 1, ...r$.

6.5 Derive the results of Eq. 6.94.

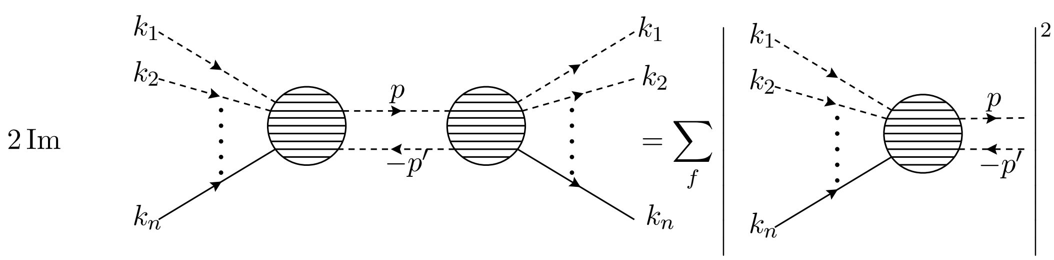
6.6 Show that the imaginary part of the amplitude does not depend on the renormalization scheme. For this evaluate it in MS and $\overline{\text{MS}}$ for the theory described by the Lagrangian of Eq. 6.357 as in Example 6.2.

6.7 Consider the $\frac{\lambda}{3!}\phi^3$ theory of Problem 6.7. Prove unitarity for the self energy of this theory, that is show,



(6.409)

6.8 Consider the theory described by the Lagrangian of Eq. (6.375). Redo the proof for the case where the intermediate states are scalars, that is,



(6.410)

6.9 Show that the integral that results from cutting n internal lines is equal to the phase space integral of n particles. Use this result to make a general proof of the unitarity.

6.10 Show that

$$P^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu \eta^\nu + k^\nu \eta^\mu}{k \cdot \eta} \tag{6.411}$$

where $k^\mu, \varepsilon^\nu(k,1), \varepsilon^\rho(k,2)$ and η^σ are four independent 4-vectors satisfying,

$$\begin{aligned}
&\eta \cdot \varepsilon(k,\sigma) = 0 \qquad \sigma = 1,2 \\
&\varepsilon(k,1) \cdot \varepsilon(k,2) = 0 \\
&k \cdot \varepsilon(k,\sigma) = 0 \qquad \sigma = 1,2 \\
&k^2 = 0 \\
&\eta^2 = 0 \qquad \text{(escolha conveniente)} \\
&\varepsilon^2(k,\sigma) = -1 \qquad \sigma = 1,2
\end{aligned} \tag{6.412}$$

Hint: The most general expression for $P^{\mu\nu}$ is

$$P^{\mu\nu} = a g^{\mu\nu} + b k^\mu k^\nu + c \eta^\mu \eta^\nu + d(k^\mu \eta^\nu + k^\nu \eta^\mu) \ . \tag{6.413}$$

Use the previous relations to find a, b, c, d.

6.11 Prove the Ward identities,

$$\left. \begin{array}{rcl} k_1^\mu T^{ab}_{\mu\nu} &=& k_{2\nu} T^{ab} \\ k_2^\mu T^{ab}_{\mu\nu} &=& k_{1\nu} T^{ab} \end{array} \right. \quad \Longrightarrow \quad k_1 \cdot T^{ab} \cdot k_2 = 0 \tag{6.414}$$

where $T^{ab}_{\mu\nu}$ and T^{ab} are defined in Eq. (6.391).

6.12 Show that the tensor $F^a_{\mu\nu}$ for the Yang-Mills fields satisfy the Bianchi identities,

$$D^{ab}_\mu F^b_{\rho\sigma} + D^{ab}_\rho F^b_{\sigma\mu} + D^{ab}_\sigma F^b_{\mu\rho} = 0 \tag{6.415}$$

or

$$D^{ab}_\mu {}^*F^{\mu\nu\ b} = 0 \tag{6.416}$$

where

$${}^*F^{\mu\nu\ a} = \frac{1}{2}\, \varepsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma} \tag{6.417}$$

6.13 Explain the geometrical meaning of the Bianchi identities.
Hint: See the article of R.P. Feynman in *Les Houches, Session XXIX, 1976*, North Holland, 1977, Pags: 135-140.

6.14 Consider the Yang-Mills (YM) theory without matter fields.
a) Show that the equations of motion can then be written as

$$\left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{E}^a = \rho^a \\ \vec{\nabla} \cdot \vec{B}^a = {}^*\rho^a \\ \vec{\nabla} \times \vec{E}^a = -\frac{\partial \vec{B}^a}{\partial t} + \vec{J}^a \\ \vec{\nabla} \times \vec{B}^a = -\frac{\partial \vec{E}^a}{\partial t} + {}^*\vec{J}^a \end{array} \right. \tag{6.418}$$

Evaluate ρ^a, ${}^*\rho^a$, $\vec{J}^a$ and ${}^*\vec{J}^a$.

b) Show that the 4-currents $j^a_\mu \equiv (\rho^a, \vec{J}^a)$ e ${}^*j^a_\mu \equiv ({}^*\rho^a, {}^*\vec{J}^a)$ are conserved.

6.15 Show that $\text{Tr}\,({}^*F_{\mu\nu} F^{\mu\nu})$ is a 4-divergence. Comment on its inclusion in the action.

6.16 Show that the following *Ansatze* (S. Coleman, Phys. Lett70B (77), 59)

$$\begin{aligned} A^{1a} &= A^{2a} = 0 \\ A^{0a} &= -A^{3a} = x^1 f^a(x^0 + x^3) + x^2 g^a(x^0 + x^3) \end{aligned} \tag{6.419}$$

where f^a and g^a are arbitrary functions, is solution of the YM equations of motion in the absence of matter fields. Discuss this solution.

6.17 Consider the *Ansatze* of Wu-Yang for static solutions of the SU(2) YM equations of motion.

$$A^{0a} = x^a\, \frac{G(r)}{r^2} \qquad\qquad A^{ia} = \varepsilon^{aij}\ x^j \frac{F(r)}{r^2} \tag{6.420}$$

a) Derive the equations that F and G should obey.
b) Show that they are satisfied for $F = -1/g$ and $G =$ constant. Show that this solutions correspond to $\rho^a = {}^*\rho^a = 0$ and $\vec{J}^a = {}^*\vec{J}^a = 0$. ($\rho^a$, ... are defined in Problem 6.14).

c) For these solutions describe the potential, the fields and evaluate the energy.

6.18 Consider QED with a non-linear gauge condition

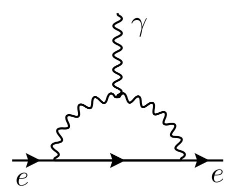
$$F = \partial_\mu A^\mu + \frac{\lambda}{2}\, A_\mu A^\mu \ . \tag{6.421}$$

a) Write the $\mathcal{L}_{eff}$ and show that $s\mathcal{L}_{eff} = 0$, where s is the Slavnov operator.

b) Evaluate the vacuum polarization at *1-loop*. Discuss the renormalization program, giving special attention to the vertices proportional to λ. Consider the theory without fermions (pure gauge).

c) Show the invariance of the renormalized S matrix with respect to the parameter λ.

d) Verify the previous result, showing that the diagram of the figure below, potentially dangerous for the anomalous magnetic moment of the electron, does not give a contribution.



e) Derive the Ward identities for the functionals Z and Γ. Write the generating functional of the Dyson-Schwinger equation for the ghosts, that is

$$\frac{\delta\Gamma}{\delta\overline{\omega}} = \cdots \tag{6.422}$$

f) Evaluate at tree level $\gamma + \gamma \to \gamma + \gamma$. Compare with the result in the linear gauge.

g) Evaluate at tree level the amplitude $T^{\mu\nu}$ for $e^+ + e^- \to \gamma + \gamma$. Verify that $k_{1\mu}T^{\mu\nu} \neq 0$ and $k_{2\mu}T^{\mu\nu} \neq 0$ where k_1 and k_2 are the photons 4-momenta. Use the Ward identities to verify these results. Is there any problem with this result?

6.19 Consider the theory that describes the interactions of the quarks with the gluons, Quantum ChromoDynamics (QCD) given by the following Lagrangian

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \sum_{\alpha=1}^{n} \overline{\psi}^\alpha_i (i\not{D} - m_\alpha)_{ij} \psi^\alpha_j \tag{6.423}$$

where

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \\ (D_\mu)_{ij} &= \delta_{ij}\partial_\mu - ig \left(\frac{\lambda^a}{2}\right)_{ij} A^a_\mu \ . \end{aligned} \tag{6.424}$$

The index $\alpha = 1, 2, \ldots, n$ labels the different quark flavours, (*up*, *down*, $\cdots$, *top*). To quantize the theory consider the gauge condition,

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}\ (\partial_\mu A^{\mu a})^2 \ , \tag{6.425}$$

for which the ghost Lagrangian is

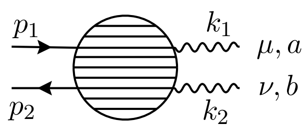
$$\mathcal{L}_G = \partial_\mu \overline{\omega}^a \partial^\mu \omega^a + g f^{abc} \partial^\mu \overline{\omega}^a A_\mu^b \omega^c \ . \tag{6.426}$$

To renormalize the theory we need the following counterterm Lagrangian,

$$\begin{aligned}
\Delta\mathcal{L} \;=\;& -\frac{1}{4}(Z_3-1)\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^2 - (Z_4-1) g f^{abc} \partial_\mu A_\nu^a A^{\mu b} A^{\nu c} \\
& -\frac{1}{4} g^2 (Z_5-1) f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} + \sum_\alpha (Z_2-1) i \overline{\psi}_i^\alpha \gamma^\mu \partial_\mu \psi_i^\alpha \\
& -\sum_\alpha m_\alpha (Z_{m_\alpha}-1) \overline{\psi}_i^\alpha \psi_i^\alpha + (Z_1-1) g \sum_\alpha \overline{\psi}_i^\alpha \gamma^\mu \left(\frac{\lambda^a}{2}\right)_{ij} \psi_j^\alpha A_\mu^a \\
& +(Z_6-1)\partial_\mu \overline{\omega}^a \partial^\mu \omega^a + (Z_7-1) g f^{abc} \partial^\mu \overline{\omega}^a A_\mu^b \omega^c \ .
\end{aligned} \tag{6.427}$$

a) Verify the expression for $\mathcal{L}_G$.
b) Consider the amplitude

$$iT_{\mu\nu}^{ab} \equiv \tag{6.428}$$



Evaluate at tree level $T_{\mu\nu}^{ab}$. Verify that $k_1^\mu T_{\mu\nu}^{ab} \neq 0$.
c) Verify the calculation of the previous item evaluating $k_1^\mu T_{\mu\nu}^{ab}$ through the Ward identities.
d) Supposing that the gluons could be final states, the amplitude for the physical process $q + q \to g + g$ where g is the gluon is given by

$$\mathcal{M} = \varepsilon^\mu(k_1) s^a T_{\mu\nu}^{ab} \varepsilon^\nu(k_2) s^b \ , \tag{6.429}$$

where $\varepsilon^\mu(k_1)$ and s^a are polarization vectors for spin and color, respectively (and also for $\varepsilon^\nu(k_2)$ e s^b). It is known that for a physical process $\mathcal{M}$ should vanish when one makes the substitution $\varepsilon^\mu(k) \to k^\mu$. How is this result compatible with the previous statements?
e) Show that the following relations must hold,

$$\frac{Z_1}{Z_2} = \frac{Z_4}{Z_3} = \frac{Z_7}{Z_6} = \frac{\sqrt{Z_5}}{\sqrt{Z_3}} \tag{6.430}$$

f) Evaluate Z_1, Z_2, Z_3, Z_6 and Z_7, using minimal subtraction and verify explicitly that $Z_1 Z_6 = Z_2 Z_7$.
g) Evaluate the contribution from the fermions to Z_4 e Z_5 and verify that they also obey the above relations.
h) Evaluate the renormalization group functions β, γ_A and γ_F.

Chapter 7

Renormalization Group

7.1 Callan -Symanzik equation

7.1.1 Renormalization scheme with momentum subtraction

In Quantum Field Theory a renormalization scheme has two components. First there is the process, known as regularization, that isolates and controls the infinities that appear in the Feynman diagrams. The regularization is arbitrary, the only requirement is that is should maintain the symmetries of the theory. For theories without gauge fields there are many alternatives. For gauge theories it turns out that the best, and perhaps unique, method is dimensional regularization.

After the regularization we have to specify a systematic method to remove the divergences and to define the parameters of the renormalized theory. We call this process renormalization scheme. There is a great arbitrariness in the choice of the subtraction method that leads to the renormalized theory. The physical results should not depend on this choice. This is the content of the renormalization group: The physical results should be invariant under transformations that only change the renormalization scheme.

We will start by studying the renormalization schemes with momentum subtraction. Depending on the point in the external momenta space that we choose, we can have different forms of this scheme. We will exemplify with the $\lambda\phi^4$ theory.

On–shell renormalization

The on-shell scheme is defined by a Taylor series for the external momenta on-shell. For the self-energy, for instance, we get,

$$\Sigma(p^2) = \Sigma(m^2) + (p^2 - m^2)\Sigma'(m^2) + \widetilde{\Sigma}(p^2) \tag{7.1}$$

With the on-shell conditions,

$$\begin{cases} \widetilde{\Sigma}(m^2) = 0 \\ \left.\dfrac{\partial \widetilde{\Sigma}(p^2)}{\partial p^2}\right|_{p^2=m^2} = 0 \end{cases} \tag{7.2}$$

In terms of the irreducible two point function, $\Gamma_R^{(2)}(p^2)$, defined by,

$$\Gamma_R^2(p) = p^2 - m^2 - \widetilde{\Sigma}(p^2) \tag{7.3}$$

We get,

$$\begin{cases} \Gamma_R^{(2)}(m^2) = 0 \\ \left.\dfrac{\partial \Gamma_R^{(2)}}{\partial p^2}\right|_{p^2=m^2} = 1 \end{cases} \tag{7.4}$$

For $\Gamma_R^{(4)}$ a convenient choice it is,

$$\Gamma_R^{(4)}(p_1, p_2, p_3) = -\lambda \qquad \text{for} \qquad \begin{cases} p_i^2 = m^2 \\ s = t = u = \dfrac{4m^2}{3} \end{cases} \tag{7.5}$$

In this case the parameters m^2 and λ are the physical mass, and except for kinematical factors, the cross section for $s = t = u = \frac{4}{3}m^2$ respectively.

Intermediate renormalization

This scheme corresponds to a Taylor expansion around zero momenta, that is,

$$\Sigma(p^2) = \Sigma(0) + \Sigma'(0)p^2 + \widetilde{\Sigma}(p^2) \tag{7.6}$$

The finite part of $\widetilde{\Sigma}(p^2)$ obeys the conditions,

$$\begin{cases} \widetilde{\Sigma}(0) = 0 \\ \left.\dfrac{\partial \widetilde{\Sigma}}{\partial p^2}\right|_{p^2=0} = 0 \end{cases} \tag{7.7}$$

These conditions translated to $\Gamma_R^{(2)}$ can be written as,

$$\begin{cases} \Gamma_R^{(2)}(0) = -m^2 \\ \dfrac{\partial \Gamma_R^{(2)}}{\partial p^2} = 1 \end{cases} \tag{7.8}$$

We still need a condition for the normalization of the coupling constant λ. This is obtained from $\Gamma_R^{(4)}$ with the following condition,

$$\Gamma_R^{(4)}(p_1, p_2, p_3) = -\lambda \qquad \text{for} \qquad p_1 = p_2 = p_3 = 0 \tag{7.9}$$

In this scheme m^2 is not the physical mass and λ cannot be measured directly experimentally, because the condition $p_i = 0$ does not belong to the physical region. As we will see, we can nevertheless express the physical quantities in terms of these parameters.

General case

The two previous examples are particular cases of a general scheme, where the normalization conditions are functions of several reference momenta, ξ_1, ξ_2... such that

$$\begin{cases} \Gamma_R^{(2)}(\xi_1^2) = -m^2 \\ \left.\dfrac{\partial \Gamma_R^{(2)}}{\partial p^2}\right|_{p^2=\xi_2^2} = 1 \\ \Gamma_R^{(4)}(\xi_3, \xi_4, \xi_5) = -\lambda \end{cases} \tag{7.10}$$

7.1.2 Renormalization Group

Let us now consider two renormalization schemes R and R'. As they both start from the same unrenormalized Lagrangian,

$$\mathcal{L} = \mathcal{L}_R + \Delta\mathcal{L}_R = \mathcal{L}_{R'} + \Delta\mathcal{L}_{R'} \tag{7.11}$$

we should have

$$\phi_R = Z_\phi^{-1/2}(R)\phi_0 \qquad ; \qquad \phi'_R = Z_\phi^{-1/2}(R')\phi_0 \ . \tag{7.12}$$

Therefore we get,

$$\phi'_R = Z_\phi^{-1/2}(R', R)\, \phi_R \tag{7.13}$$

where

$$Z_\phi(R', R) = \frac{Z_\phi(R')}{Z_\phi(R)} \tag{7.14}$$

These relations indicate that the renormalized fields in the two schemes are related by a multiplicative constant. The constant should be finite as both $\phi_{R'}$ as ϕ_R are finite. In a similar way,

$$\begin{aligned} \lambda_{R'} &= Z_\lambda^{-1}(R', R) Z_\phi^2(R', R)\lambda_R \\ m_{R'}^2 &= m_R^2 + \delta m^2(R', R) \end{aligned} \tag{7.15}$$

where

$$\begin{aligned} Z_\lambda(R', R) &= \frac{Z_\lambda(R')}{Z_\lambda(R)} \\ \delta m^2(R', R) &= \delta m^2(R') - \delta m^2(R) \end{aligned} \tag{7.16}$$

are finite quantities The operation that takes the quantities from one renormalization scheme, R, into another scheme, R', can be seen as a transformation from R into R'. The set of these transformation constitutes the *Renormalization Group*.

7.1.3 Callan - Symanzik equation

We are going now to give a mathematical form to this invariance under the renormalization group. The form of the renormalization group (RG) equation depends on the renormalization scheme used. We are going to start by obtaining the equations for the RG in the scheme with momentum subtraction, the so-called Callan-Symanzik equation.

We start by noticing the identity,

$$\frac{\partial}{\partial m_0^2}\left(\frac{i}{p^2-m_0^2+i\varepsilon}\right)=\frac{i}{p^2-m_0^2+i\varepsilon}(-i)\frac{i}{p^2-m_0^2+i\varepsilon} \tag{7.17}$$

This means that the derivative of an unrenormalized Green function with respect to the bare mass, is equivalent to the insertion of a composite operator $\frac{1}{2}\phi^2$ with zero momentum, that is,

$$\frac{\partial\Gamma^{(n)}(p_i)}{\partial m_0^2}=-i\Gamma^{(n)}_{\phi^2}(0,p_i) \tag{7.18}$$

The irreducible renormalized Green functions are given by,

$$\left\{\begin{array}{l}\Gamma^{(n)}_R(p_i;\lambda;m)=Z_\phi^{(n/2)}\Gamma^{(n)}(p_i;\lambda_0;m_0)\\[2ex] \Gamma^{(n)}_{\phi^2R}(p;p_i;\lambda;m)=Z_{\phi^2}^{-1}Z_\phi^{n/2}\Gamma^{(n)}_{\phi^2}(p;p_i;\lambda_0;m_0)\end{array}\right. \tag{7.19}$$

Using this we can write the previous equation as,

$$\frac{\partial}{\partial m_0^2}\left[Z_\phi^{-n/2}\Gamma^{(n)}_R(p_i,\lambda,m)\right]=-iZ_{\phi^2}Z_\phi^{-n/2}\Gamma^{(n)}_{\phi^2R}(0,p_i,\lambda,m) \tag{7.20}$$

and therefore

$$-\frac{n}{2}Z_\phi^{-1}\frac{\partial Z_\phi}{\partial m_0^2}Z_\phi^{-n/2}\Gamma^{(n)}_R+Z_\phi^{-n/2}\frac{\partial}{\partial m_0^2}\Gamma^{(n)}_R=-iZ_{\phi^2}Z_\phi^{-n/2}\Gamma^{(n)}_{\phi^2R}(0,p_i,\lambda,m) \tag{7.21}$$

We therefore get,

$$\left[\frac{\partial}{\partial m_0^2}-\frac{n}{2}\frac{\partial\ln Z_\phi}{\partial m_0^2}\right]\Gamma^{(n)}_R=-i\ Z_{\phi^2}\Gamma^{(n)}_{\phi^2R}$$

$$\left[\frac{\partial m^2}{\partial m_0^2}\frac{\partial m}{\partial m^2}\frac{\partial}{\partial m}+\frac{\partial\lambda}{\partial m_0^2}\frac{\partial}{\partial\lambda}-\frac{n}{2}\frac{\partial\ln Z_\phi}{\partial m_0^2}\right]\Gamma^{(n)}_R=-iZ_{\phi^2}\Gamma^{(n)}_{\phi^2R} \tag{7.22}$$

This can still be written as

$$\left[m\frac{\partial}{\partial m}+\beta\frac{\partial}{\partial\lambda}-n\gamma\right]\Gamma^{(n)}_R=-im^2\alpha\Gamma^{(n)}_{\phi^2R} \tag{7.23}$$

which is the Callan-Symanzik equation for the ϕ^4 theory, where α,β and γ are dimensionless functions. These functions are defined by

$$\beta\ =\ 2m^2\frac{\dfrac{\partial\lambda}{\partial m_0^2}}{\dfrac{\partial m^2}{\partial m_0^2}} \tag{7.24}$$

$$\gamma = m^2 \frac{\dfrac{\partial \ln Z_\phi}{\partial m_0^2}}{\dfrac{\partial m^2}{\partial m_0^2}} \tag{7.25}$$

$$\alpha = 2\frac{Z_{\phi^2}}{\dfrac{\partial m}{\partial m_0^2}} \tag{7.26}$$

The function α is not independent of γ. In fact, if we choose the normalization conditions at $p_i = 0$

$$\begin{cases} \Gamma_R^{(2)}(0,\lambda,m) = -m^2 \\ \\ \Gamma_{\phi^2 R}^{(2)}(0,0,\lambda,m) = i \end{cases} \tag{7.27}$$

We get then,

$$\alpha = 2(\gamma - 1) \tag{7.28}$$

As the quantities $\Gamma_R^{(n)}$ and $\Gamma_{\phi^2 R}^{(n)}$ do not depend on the cut-off, we expect that α, β and γ are cut-off independent. To see that we put $n = 2$ and take a derivative in order to p^2

$$\left[m\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial \lambda} - 2\gamma\right]\frac{\partial}{\partial p^2}\Gamma_R^{(2)}(p,\lambda,m) = -im^2\alpha\frac{\partial}{\partial p^2}\Gamma_{\phi^2 R}^{(2)}(0,p,\lambda,m) \tag{7.29}$$

Setting $p^2 = 0$ and using

$$\left.\frac{\partial \Gamma_R^{(2)}}{\partial p^2}\right|_{p^2=0} = 1 \tag{7.30}$$

We get then

$$\gamma = im^2(\gamma - 1)\left[\frac{\partial}{\partial p^2}\Gamma_{\phi^2 R}^{(2)}(0,p,\lambda,m)\right]_{p^2=0} \tag{7.31}$$

which shows that γ is cut-off independent. Then, as $\alpha = 2(\gamma - 1)$, we must have that α is also independent of the cut-off. As α and γ are cut-off independent, so is β. As α, β e γ are dimensionless and independent of the cut-off they can only depend on the dimensionless coupling constant λ, that is,

$$\begin{aligned} \alpha =& \alpha(\lambda) \\ \beta =& \beta(\lambda) \\ \gamma =& \gamma(\lambda) \end{aligned} \tag{7.32}$$

We will mostly interested in the Minimal Subtraction (MS) scheme (see below), so we will not calculate the functions α, β and γ for the Callan-Symanzik equation in the ϕ^4 theory. We will indicate, however, how they can be easily obtained. Consider, for instance, the function $\beta(\lambda)$. Noticing that

$$\frac{\partial \lambda}{\partial m_0^2}(\lambda_0, \Lambda/m) = \frac{\partial m^2}{\partial m_0^2}\frac{\partial}{\partial m^2}\lambda(\lambda_0, \Lambda/m)$$

$$=\frac{\partial m^2}{\partial m_0^2}\frac{1}{2m}\frac{\partial}{\partial m}\lambda(\lambda_0,\Lambda/m) \tag{7.33}$$

we obtain from its definition,

$$\beta = m\frac{\partial}{\partial m}\lambda(\lambda_0,\Lambda/m) = m\frac{\partial}{\partial m}[\overline{Z}(\lambda_0,\Lambda/m)\lambda_0] = -\lambda_0\Lambda\frac{\partial}{\partial\Lambda}[\overline{Z}(\lambda_0,\Lambda/m)] \tag{7.34}$$

or in other form

$$\beta = -\lambda\frac{\partial}{\partial\ln\Lambda}[\ln\overline{Z}(\lambda_0,\Lambda/m)] \tag{7.35}$$

where, by definition, $\lambda = \overline{Z}\lambda_0$), and therefore $\overline{Z} = Z_\lambda^{-1}Z_\phi^2$.

The one-loop result gives,

$$\begin{aligned} Z_\lambda =& 1+\frac{3\lambda_0}{32\pi^2}\ln\frac{\Lambda^2}{m^2}+O(\lambda_0^2) \\ Z_\phi =& 1+O(\lambda_0^2) \end{aligned} \tag{7.36}$$

Therefore

$$\overline{Z} = 1-\frac{3\lambda_0}{32\pi^2}\ln\frac{\Lambda^2}{m^2}+\cdots \tag{7.37}$$

and to first order,

$$\ln\overline{Z} = -\frac{3\lambda_0}{16\pi^2}\ln\frac{\Lambda}{m}+\cdots \tag{7.38}$$

Therefore, for ϕ^4, we have

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2}+O(\lambda^3)\ . \tag{7.39}$$

7.1.4 Weinberg's theorem and the solution of the RG equations

We now discuss an important theorem due to Weinberg. This theorem deals with the asymptotic behavior of the one-particle irreducible Green functions (1-PI), in the Euclidean region ($p_i^2 < 0$) for values non-exceptional of the momenta (no partial sum vanishes).

Theorem 7.1
If the momenta are not exceptional and if we parameterize them by, $p_i = \sigma k_i$, then the 1-PI Green functions $\Gamma_R^{(n)}$ behave in the deep Euclidean region ($\sigma\to\infty$ and k_i fixed, $p_i^2 < 0$) in the following way:

$$\lim_{\sigma\to\infty}\Gamma^{(n)}(\sigma k_i,\lambda,m) = \sigma^{4-n}[a_0(\ln\sigma)^{b_0}+a_1(\ln\sigma)^{b_1}+\cdots] \tag{7.40}$$

and

$$\lim_{\sigma\to\infty}\Gamma^{(n)}_{\phi^2}(\sigma k_i,\lambda,m) = \sigma^{2-n}[a'_0(\ln\sigma)^{b'_0}+a'_1(\ln\sigma)^{b'_1}+\cdots] \tag{7.41}$$

We will not make the proof of the theorem (see for instance the second volume of Bjorken and Drell) but we just note that the powers of σ are the canonical dimensions, in terms of mass, of the respective Green functions. If the canonical behavior is the one observed asymptotically depends on the sum of the logarithms. If this sum gives a power of σ, for instance, $\sigma^{-\gamma}$, then the asymptotic behavior is modified to $\sigma^{4-n-\gamma}$. The exponent γ is known as the *anomalous dimension*. We will show how to use the Renormalization Group to perform this sum of logarithms and therefore obtain the anomalous dimensions.

7.1.5 Asymptotic solution of the RG equations

From Weinberg's theorem we have that $\Gamma_R^{(n)} \gg \Gamma_{\phi^2 R}^{(n)}$ for any finite order in λ in the deep Euclidean region ($\sigma \to \infty$). If we assume that this remains true, even after summing all the orders in perturbation theory, we can neglect the second term in the Callan-Symanzik equation and we obtain an homogeneous differential equation,

$$\left[m\frac{\partial}{\partial m} + \beta(\lambda)\frac{\partial}{\partial \lambda} - n\gamma(\lambda) \right] \Gamma_{\text{asy}}^{(n)}(p_i, \lambda, m) = 0 \tag{7.42}$$

where $\Gamma_{\text{asy}}^{(n)}$ is the asymptotic form of $\Gamma_R^{(n)}$. The meaning of this equation is that, in this asymptotic region, a change in the mass parameter can always be compensated by appropriate changes in the coupling constant and in the scale of the fields.

To solve this equation we start by defining a dimensionless quantity $\overline{\Gamma}_R^{(n)}$ using dimensional analysis,

$$\Gamma_{\text{asy}}^{(n)}(p_i, \lambda, m) = m^{4-n}\overline{\Gamma}_R^{(n)}(p_i/m, \lambda) \ . \tag{7.43}$$

This dimensionless function, $\overline{\Gamma}_R^{(n)}$, obeys the relation

$$\left(m\frac{\partial}{\partial m} + \sigma\frac{\partial}{\partial \sigma} \right) \overline{\Gamma}_R^{(n)} \left(\sigma\frac{p_i}{m}, \lambda \right) = 0 \ . \tag{7.44}$$

Then we have

$$\left(m\frac{\partial}{\partial m} + \sigma\frac{\partial}{\partial \sigma} \right) m^{n-4}\Gamma_{\text{asy}}^{(n)}(\sigma p_i, \lambda, m) = 0 \tag{7.45}$$

or

$$\left[m\frac{\partial}{\partial m} + \sigma\frac{\partial}{\partial \sigma} + (n-4) \right] \Gamma_{\text{asy}}^{(n)}(\sigma p_i, \lambda, m) = 0 \tag{7.46}$$

Using this equation we can exchange the derivative with respect to the mass with the derivative with respect to the scale in the Callan-Symanzik. We get then

$$\left[\sigma\frac{\partial}{\partial \sigma} - \beta(\lambda)\frac{\partial}{\partial \lambda} + n\gamma(\lambda) + (n-4) \right] \Gamma_{\text{asy}}^{(n)}(\sigma p_i, \lambda, m) = 0 \tag{7.47}$$

To solve this equation we remove the terms without derivatives with the transformation,

$$\Gamma_{\text{asy}}^{(n)}(\sigma p_i, \lambda, m) = \sigma^{4-n} e^{n\int_0^\lambda \frac{\gamma(x)}{\beta(x)}dx} F^{(n)}(\sigma p_i, \lambda, m) \ . \tag{7.48}$$

Substituting in the differential equation we see that those terms disappear and we get a differential equation for $F^{(n)}$

$$\left[\sigma\frac{\partial}{\partial\sigma} - \beta(\lambda)\frac{\partial}{\partial\lambda}\right] F^{(n)}(\sigma p, \lambda, m) = 0 \tag{7.49}$$

Now we introduce $t = \ln\sigma$. We can the write,

$$\left[\frac{\partial}{\partial t} - \beta(\lambda)\frac{\partial}{\partial\lambda}\right] F^{(n)}(e^t p, \lambda, m) = 0 \tag{7.50}$$

To solve this equation we introduce the effective coupling constant $\overline{\lambda}(t,\lambda)$ as solution of the equation,

$$\frac{\partial\overline{\lambda}(t,\lambda)}{\partial t} = \beta(\overline{\lambda}) \tag{7.51}$$

with the boundary condition $\overline{\lambda}(0,\lambda) = \lambda$. To see that this definition will give us the solution we write,

$$t = \int_{\lambda}^{\overline{\lambda}(t,\lambda)} \frac{dx}{\beta(x)} \tag{7.52}$$

and take the derivative with respect to λ. We get,

$$0 = \frac{1}{\beta(\overline{\lambda})}\frac{\partial\overline{\lambda}}{\partial\lambda} - \frac{1}{\beta(\lambda)} \tag{7.53}$$

or

$$\beta(\overline{\lambda}) - \beta(\lambda)\frac{\partial\overline{\lambda}}{\partial\lambda} = 0 \tag{7.54}$$

Using now the definition of $\overline{\lambda}$ we get

$$\left[\frac{\partial}{\partial t} - \beta(\lambda)\frac{\partial}{\partial\lambda}\right] \overline{\lambda}(t,\lambda) = 0 \tag{7.55}$$

The differential operator in the last equation is exactly the same that in the equation for $F^{(n)}(e^t p, \lambda, m)$. Therefore $F^{(n)}$ obeys that equation if it depends on t and λ through the combination $\overline{\lambda}(t,\lambda)$. Then the general solution for $\Gamma^{(n)}_{\text{asy}}$ is

$$\Gamma^{(n)}_{\text{asy}}(\sigma p_i, \lambda, m) = \sigma^{4-n} e^{n\int_0^\lambda \frac{\gamma(x)}{\beta(x)}dx} F^{(n)}(p_i, \overline{\lambda}(t,\lambda), m) \tag{7.56}$$

To have a physical meaning for this result we notice that

$$\begin{aligned} e^{n\int_0^\lambda \frac{\gamma(x)}{\beta(x)}dx} =& e^{n\int_0^{\overline{\lambda}} \frac{\gamma(x)}{\beta(x)}dx} e^{n\int_{\overline{\lambda}}^{\lambda} \frac{\gamma(x)}{\beta(x)}dx} \\ =& e^{n\int_0^{\overline{\lambda}} \frac{\gamma(x)}{\beta(x)}dx} e^{-n\int_{\lambda}^{\overline{\lambda}} \frac{\gamma(x)}{\beta(x)}dx} \\ =& e^{n\int_0^{\overline{\lambda}} \frac{\gamma(x)}{\beta(x)}dx} e^{-n\int_0^t \gamma(\overline{\lambda}(t',\lambda))dt'} \end{aligned} \tag{7.57}$$

Therefore

$$\Gamma^{(n)}_{\text{asy}}(\sigma p_i, \lambda, m) = \sigma^{4-n} e^{-n\int_0^t \gamma(\overline{\lambda}(t',\lambda))dt'} e^{-n\int_0^{\overline{\lambda}} \frac{\gamma(x)}{\beta(x)}dx} F^{(n)}(p_i, \overline{\lambda}(t,\lambda), m) \tag{7.58}$$

If we set $\sigma = 1 (t = 0)$, we get the result that $e^{n\int_0^{\overline{\lambda}} \frac{\gamma}{\beta}dx} F^{(n)}$ is $\Gamma^{(n)}_{\text{asy}}$. Then we get finally the solution for the RG equation,

$$\Gamma^{(n)}_{\text{asy}}(\sigma p_i, \lambda, m) = \sigma^{4-n} e^{-n\int_0^t \gamma(\overline{\lambda}(t',\lambda))dt'} \Gamma^{(n)}_{\text{asy}}(p_i, \overline{\lambda}(t,\lambda), m) \tag{7.59}$$

In this form the solution has a simple interpretation. The effect of making a change of scale in the momenta p_i in the functions, $\Gamma^{(n)}_R$ it is equivalent to substitute the coupling constant λ, by the effective coupling constant $\overline{\lambda}$, except for multiplicative factors. The first factor results simply from the fact that $\Gamma^{(n)}_R$ has canonical mass dimension $4 - n$ in terms of mass. The exponential factor is the anomalous dimension term. It results from summing up all the logarithms in perturbation theory. This factor is controlled by γ, the anomalous dimension. We will see later how to calculate the anomalous dimension in any theory.

7.2 Minimal subtraction (MS) scheme

7.2.1 Renormalization group equations for MS

Let us look now at other forms that the renormalization group equation can have. The statement that the renormalization is multiplicative can be expressed in the form,

$$\Gamma^{(n)}(p_i, \lambda_0, m_0) = Z_\phi^{-n/2} \Gamma^{(n)}_R(p_i, \lambda, m, \mu) \tag{7.60}$$

where μ is the scale used to define the Green functions. The left side of this equation does not depend on μ, but the right-hand side does, both explicitly and implicitly through λ and m. We have then

$$\mu\frac{\partial}{\partial\mu}\left[Z_\phi^{-n/2}\Gamma^{(n)}_R(p_i, \lambda, m, \mu)\right] = 0 \tag{7.61}$$

or

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + \gamma_m m\frac{\partial}{\partial m} - n\gamma\right)\Gamma^{(n)}_R = 0 \tag{7.62}$$

We have defined the RG functions,

$$\begin{aligned}
\beta\left(\lambda, \frac{m}{\mu}\right) &= \mu\frac{\partial\lambda}{\partial\mu} \\
\gamma_m\left(\lambda, \frac{m}{\mu}\right) &= \mu\frac{\partial \ln m}{\partial\mu} \\
\gamma\left(\lambda, \frac{m}{\mu}\right) &= \frac{1}{2}\mu\frac{\partial \ln Z_\phi}{\partial\mu}
\end{aligned} \tag{7.63}$$

This equation has the advantage over the Callan-Symanzik equation of being homogeneous, without approximations. The difficulty comes from the fact that these functions depend on two variables, λ and $\frac{m}{\mu}$, making it difficult to get a solution for the equation. There is, however, a renormalization scheme where the dependence on m/μ disappears and therefore the equation has a simple solution. This scheme is called *Minimal Subtraction* (MS) that we will describe now.

7.2.2 Minimal subtraction scheme (MS)

The minimal subtraction scheme is related to the method of dimensional regularization. The divergences of the integrals appear, in this method, as poles in $\dfrac{1}{\varepsilon}$ where $\varepsilon = 4 - d$. The minimal subtraction scheme consists in choosing the counter-terms to cancel *just these poles*. Let us give the example of the *self-energy* in $\lambda\phi^4$. This corresponds to the diagram of the Fig. 7.1

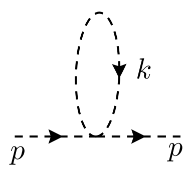


Figure 7.1: Diagram for self-energy in ϕ^4.

We get

$$\begin{aligned} -i\Sigma(p) &= (-i\lambda)\mu^{\varepsilon}\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{i}{p^2-m^2+i\varepsilon} \\ &= -i\lambda\frac{1}{32\pi^2}\mu^{\varepsilon}\frac{\Gamma(1-d/2)}{m^{2-d}}2^{\varepsilon}\pi^{\varepsilon/2} \end{aligned} \tag{7.64}$$

where $\varepsilon = 4 - d$.

$$\begin{aligned} \Sigma(p^2) &= \lambda\frac{1}{32\pi^2}\mu^{\varepsilon}\frac{\Gamma(-1+\varepsilon/2)}{m^{-2+\varepsilon}}\ (2\sqrt{\pi})^{\varepsilon} \\ &= \lambda\frac{m^2}{32\pi^2}\left(\frac{\mu}{m}\right)^{\varepsilon}\Gamma(-1+\varepsilon/2)\ (2\sqrt{\pi})^{\varepsilon} \end{aligned} \tag{7.65}$$

We now use the result (γ is the Euler constant and $\psi(x)$ the logarithm derivative of the Γ function)

$$\Gamma\left(-1+\frac{\varepsilon}{2}\right) = -\left[\frac{2}{\varepsilon}+\overbrace{1-\gamma}^{\psi(2)}+O(\varepsilon)\right] \tag{7.66}$$

and

$$\left(\frac{\mu}{m}\right)^{\varepsilon} = 1+\varepsilon\ln\left(\frac{\mu}{m}\right) \tag{7.67}$$

to get the final result,

$$\Sigma(p^2) = -\frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\varepsilon}+\psi(2)+2\ln(\mu/m)+2\ln 2\sqrt{\pi}+O(\varepsilon)\right] \tag{7.68}$$

Therefore in the minimal subtraction we have to add the counter-term

$$\Delta\mathcal{L}_{\phi^2}^{MS} = -\frac{\lambda m^2}{32\pi^2}\frac{1}{\varepsilon}\phi^2 \tag{7.69}$$

If we had used momentum subtraction at the scale μ, that is, $\Sigma_R(p^2 = \mu^2) = 0$ we would get a different counter-term differing by finite terms.

$$\Delta\mathcal{L}_{\phi^2}^{\mathrm{MOM}} = -\frac{\lambda m^2}{32\pi^2}\left[\frac{1}{\varepsilon} + \frac{1}{2}\psi(2) + \ln(\mu/m) + \ln 2\sqrt{\pi}\right]\phi^2 \tag{7.70}$$

We see that, by definition, the counter-term Lagrangian when expanded in Laurent series in ε only contains the divergent terms.

As usual the counter-term constants are defined by

$$\begin{aligned}\phi_0 =& \sqrt{Z_\phi}\phi\\ m_0 =& Z_m m\\ \lambda_0 =& \mu^\varepsilon Z_\lambda \lambda\end{aligned} \tag{7.71}$$

The renormalization constants, Z_ϕ, Z_m and Z_λ in minimal subtraction should have the form,

$$\begin{aligned}Z_\lambda =& 1 + \sum_{r=1}^{\infty} a_r(\lambda)/\varepsilon^r\\ Z_m =& 1 + \sum_{r=1}^{\infty} b_r(\lambda)/\varepsilon^r\\ Z_\phi =& 1 + \sum_{r=1}^{\infty} c_r(\lambda)/\varepsilon^r\end{aligned} \tag{7.72}$$

Therefore the coefficients of the renormalization group equation are independent of μ and, as they are dimensionless and also independent of m, they should depend only the coupling constant. This simplifies the solution of the renormalization group equation,

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + \gamma_m m\frac{\partial}{\partial m} - n\gamma\right)\Gamma_R^{(n)} = 0 \tag{7.73}$$

Using dimensional analysis we have

$$\left[m\frac{\partial}{\partial m} + (n-4) + \mu\frac{\partial}{\partial\mu} + \sigma\frac{\partial}{\partial\sigma}\right]\Gamma_R^{(n)}(\sigma p, m, \lambda, \mu) = 0 \tag{7.74}$$

and therefore we can write,

$$\left[\sigma\frac{\partial}{\partial\sigma} - \beta\frac{\partial}{\partial\lambda} - (\gamma_m - 1)\, m\frac{\partial}{\partial m} + n\gamma + (n-4)\right]\Gamma_R^{(n)}(\sigma p, m, \lambda, \mu) = 0 \tag{7.75}$$

This equation has the solution,

$$\Gamma_R^{(n)}(\sigma p_i, m, \lambda, \mu) = \sigma^{4-n} e^{-n\int_0^t \gamma(\overline{\lambda}(t'))dt'}\Gamma_R^{(n)}(p_i, \overline{m}(t), \overline{\lambda}(t), \mu) \tag{7.76}$$

where we have introduced the effective mass $\overline{m}(t)$ and the effective coupling constant $\overline{\lambda}(t)$. These are defined by,

$$\begin{cases} \dfrac{d\overline{\lambda}}{dt} = \beta(\overline{\lambda}) \qquad ; \qquad \overline{\lambda}(t=0) = \lambda \\ \dfrac{d\overline{m}(t)}{dt} = \Big[\gamma_m(\lambda) - 1\Big]\overline{m}(t) \qquad ; \qquad \overline{m}(t=0) = m \end{cases} \tag{7.77}$$

The solution of this equation is

$$\begin{aligned} \overline{m}(t) &= m\ e^{\int_0^t [\gamma_m(\overline{\lambda}(t'))-1]dt'} \\ &= m\ e^{-t} e^{\int_0^t \gamma_m(\overline{\lambda}(t'))dt'} \\ &= m\ e^{-t} e^{\int_\lambda^{\overline{\lambda}(t)} dx \frac{\gamma_m(x)}{\beta(x)}} \end{aligned} \tag{7.78}$$

7.2.3 Physical parameters

The parameters defined by the minimal subtraction are not physical parameters. We can however calculate the physical parameters as function of those. As physical parameters we mean an element of the S matrix or the position of the pole of the propagator. For these the following theorem is valid:

Theorem 7.2

Any physical parameter $P(\lambda, m, \mu)$ satisfies the following renormalization group equation:

$$\mathcal{D}P(\lambda, m, \mu) \equiv \left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \gamma_m m\frac{\partial}{\partial m}\right] P(\lambda, m, \mu) = 0 \tag{7.79}$$

Proof: *Let us consider first the propagator $\Delta(p^2)$ that satisfies the renormalization group equation,*

$$[\mathcal{D} + 2\gamma]\Delta(p^2, \lambda, m, \mu) = 0 \tag{7.80}$$

We can write a Laurent series in the neighborhood of the pole$p^2 = m_p^2$

$$\Delta(p^2, \lambda, m, \mu) = \frac{R^2}{p^2 - m_p^2} + \widetilde{\Delta} \tag{7.81}$$

The position of the pole $m_p(\lambda, m, \mu)$ and its residue $R^2(\lambda, m, \mu)$ satisfy renormalization group equations that can be obtained by the application of the operator $(\mathcal{D} + 2\gamma)$ to the previous equation. Equating the residue of the poles we get

$$\mathcal{D}m_p(\lambda, m, \mu) = 0 \tag{7.82}$$

$$[\mathcal{D} + \gamma(\lambda)]R(\lambda, m, \mu) = 0 \tag{7.83}$$

for the physical mass and for the residue at the pole. Now for an element of the S matrix we have ($S_R = R^n\Gamma^{(n)}$)

$$\begin{aligned}\mathcal{D} \lim_{p_i^2 \to m_p^2} R^n\Gamma^{(n)} &= \lim_{p_i^2 \to m_p^2} \mathcal{D}(R^n\Gamma^n) \\ &= \lim_{p_i^2 \to m_p^2} [n\mathcal{D}RR^{n-1}\Gamma^n + R^n\mathcal{D}\Gamma^n] \\ &= \lim_{p_i^2 \to m_p^2} [-n\gamma + n\gamma]R^n\Gamma^n = 0 \end{aligned} \tag{7.84}$$

and this ends the proof.

We will see later how these results can be used to relate the physical parameters with the parameters of the theory.

7.2.4 Renormalization group functions in minimal subtraction

We saw before that we have

$$\begin{cases} \phi_0 &= \sqrt{Z_\phi}\phi \\ m_0 &= Z_m\, m \\ \lambda_0 &= \mu^\varepsilon Z_\lambda \lambda \end{cases} \tag{7.85}$$

and that in MS the renormalization constants have the form,

$$\begin{cases} Z_\lambda &= 1 + \sum_{r=1}^\infty a_r(\lambda)/\varepsilon^r \\ \\ Z_m &= 1 + \sum_{r=1}^\infty b_r(\lambda)/\varepsilon^r \\ \\ Z_\phi &= 1 + \sum_{r=1}^\infty c_r(\lambda)/\varepsilon^r \ . \end{cases} \tag{7.86}$$

Let us now see how to evaluate β, γ_m and γ.

i) Determination of $\beta(\lambda)$

By definition

$$\beta(\lambda) = \mu \frac{\partial \lambda}{\partial \mu} \ . \tag{7.87}$$

This quantity is finite in the limit $\varepsilon \to 0$. This means that before we take the limit $\varepsilon \to 0$ it must be an analytic function of ε. It is then convenient to define

$$\beta(\lambda) = \hat{\beta}(\lambda, \varepsilon = 0) = d_0 \ , \tag{7.88}$$

where

$$\hat{\beta}(\lambda, \varepsilon) = d_0 + d_1\varepsilon + d_2\varepsilon^2 + \cdots \tag{7.89}$$

with coefficients d_r to be determined. Now we use the fact that λ_0 does not depend on the scale μ. Then

$$\begin{aligned} 0 &= \mu\frac{\partial}{\partial\mu}(\mu^\varepsilon Z_\lambda \lambda) \\ &= \varepsilon\mu^\varepsilon Z_\lambda\lambda + \mu^\varepsilon\hat{\beta}(\lambda,\varepsilon)\lambda\frac{\partial Z_\lambda}{\partial\lambda} + \mu^\varepsilon Z_\lambda\hat{\beta}(\lambda,\varepsilon) \ . \end{aligned} \tag{7.90}$$

This can be rewritten as

$$\varepsilon\lambda Z_\lambda + \hat{\beta}(\lambda,\varepsilon)\left(Z_\lambda + \lambda\frac{\partial Z_\lambda}{\partial\lambda}\right) = 0 \ . \tag{7.91}$$

Using the expressions for Z_λ and $\hat{\beta}$ we get

$$\varepsilon\lambda + a_1\lambda + \lambda\sum_{r=1}^{\infty}\frac{a_{r+1}}{\varepsilon^r} + (d_0 + d_1\varepsilon + d_2\varepsilon^2 + \cdots)\left[1 + \sum_{r=1}^{\infty}\frac{1}{\varepsilon^r}\left(a_r + \lambda\frac{da_r}{d\lambda}\right)\right] = 0 \ . \tag{7.92}$$

We conclude then that $d_r = 0$ for $r > 1$ and that

$$\begin{aligned} \varepsilon(\lambda + d_1) + &\left[a_1\lambda + d_0 + d_1\left(a_1 + \lambda\frac{da_1}{d\lambda}\right)\right] + \sum_r \frac{1}{\varepsilon^r}\left[a_{r+1}\lambda + d_0\left(a_r + \lambda\frac{da_r}{d\lambda}\right)\right. \\ &\left. + \, d_1\left(a_{r+1} + \lambda\frac{da_{r+1}}{d\lambda}\right)\right] = 0 \ . \end{aligned} \tag{7.93}$$

Equating equal powers of ε we obtain,

$$\begin{aligned} &\lambda + d_1 = 0 \\ &a_1\lambda + d_0 + d_1\left(a_1 + \lambda\frac{da_1}{d\lambda}\right) = 0 \\ &a_{r+1}\lambda + d_0\left(a_r + \lambda\frac{da_r}{d\lambda}\right) + d_1\left(a_{r+1} + \lambda\frac{da_{r+1}}{d\lambda}\right) = 0 \ . \end{aligned} \tag{7.94}$$

This gives,

$$d_1 = -\lambda \tag{7.95}$$

$$\beta(\lambda) = d_0 = \lambda^2\frac{da_1}{d\lambda} \tag{7.96}$$

$$\lambda^2\frac{d}{d\lambda}(a_{r+1}) = \beta(\lambda)\frac{d}{d\lambda}(\lambda a_r) \ . \tag{7.97}$$

Therefore the $\beta(\lambda)$ function depends only in the coefficient of $\frac{1}{\varepsilon}$ in Z_λ, that it is easily obtained in perturbation theory. Also we see that the residues of the higher order poles can be calculated in terms of the simple pole (lowest order in perturbation theory). For example for $\lambda\phi^4$ one can easily obtain,

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2}\frac{1}{\varepsilon} + \cdots \tag{7.98}$$

Using this we obtain for the β function,

$$\beta(\lambda) = \lambda^2 \frac{da_1}{d\lambda} = \lambda^2 \frac{d}{d\lambda}\left(\frac{3\lambda}{16\pi^2}\right) = \frac{3\lambda^2}{16\pi^2} \ , \tag{7.99}$$

given exactly the same result as we have obtained using the momentum subtraction method.

For gauge theories there is a small modification because we have in this case $g_0 = \mu^{\varepsilon/2} Z_g g$. A trivial calculation gives,

$$d_1 = -\, g/2 \tag{7.100}$$

$$\beta(g) = \frac{1}{2} g^2 \frac{da_1}{dg} \tag{7.101}$$

$$\frac{1}{2} g^2 \frac{da_{r+1}}{dg} = \beta(g) \frac{d}{dg}(g a_r) \ , \tag{7.102}$$

where, as before,

$$Z_g = 1 + \sum_{r=1}^{\infty} a_r(g)/\varepsilon^r \ . \tag{7.103}$$

ii) Determination of $\gamma_m(\lambda)$

We start from $m_0 = Z_m m$. Applying $\mu \frac{\partial}{\partial\mu}$ we get

$$\begin{aligned} 0 &= \mu \frac{\partial Z_m}{\partial \mu} m + Z_m \mu \frac{\partial m}{\partial \mu} \\ &= \hat{\beta}(\lambda, \varepsilon) \frac{\partial Z_m}{\partial \lambda} m + m Z_m \mu \frac{\partial \ln m}{\partial \mu} \end{aligned} \tag{7.104}$$

As $\mu \frac{\partial \ln m}{\partial \mu} = \gamma_m$, we get the equation

$$\left[\hat{\beta}(\lambda, \varepsilon) \frac{\partial}{\partial \lambda} + \gamma_m\right] Z_m = 0 \tag{7.105}$$

which leads to

$$\left(\gamma_m + d_1 \frac{db_1}{d\lambda}\right) + \sum_{r=1}^{\infty} \frac{1}{\varepsilon_r}\left[d_0 \frac{db_r}{d\lambda} + \gamma_m b_r + d_1 \frac{db_{r+1}}{d\lambda}\right] = 0 \tag{7.106}$$

Comparing the powers of ε we get

$$\gamma_m = -d_1 \frac{db_1}{d\lambda} \ , \tag{7.107}$$

$$-\, d_1 \frac{db_{r+1}}{d\lambda} = \beta(\lambda) \frac{db_r}{d\lambda} + \gamma_m b_r \ , \tag{7.108}$$

where

$$d_1 = \begin{cases} -\lambda & \lambda\phi^4 \quad \text{theory} \\ \\ -\, g/2 & \text{gauge theories} \end{cases} \tag{7.109}$$

As in the case of β, we see that γ_m only depends on the residue of the simple pole.

***iii)* Determination $\gamma(\lambda)$**

Here it is easier to start from the definition of $\gamma(\lambda)$

$$\gamma(\lambda) = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_\phi = \frac{1}{2}\frac{1}{Z_\phi}\mu\frac{\partial}{\partial\mu}Z_\phi \ . \tag{7.110}$$

Rearranging we get

$$\left[\hat\beta(\lambda,\varepsilon)\frac{\partial}{\partial\lambda} - 2\gamma(\lambda)\right] Z_\phi = 0 \ . \tag{7.111}$$

Using the expansion of Z_ϕ we obtain,

$$-2\gamma(\lambda) + d_1\frac{dc_1}{d\lambda} + \sum_{r=1}^{\infty}\frac{1}{\varepsilon^r}\left[d_0\frac{dc_r}{d\lambda} - 2\gamma c_r + d_1\frac{dc_{r+1}}{d\lambda}\right] = 0 \ . \tag{7.112}$$

Comparing now the powers of ε we get,

$$\gamma(\lambda) = \frac{1}{2}d_1\frac{dc_1}{d\lambda} \ , \tag{7.113}$$

$$-d_1\frac{dc_{r+1}}{d\lambda} = \beta(\lambda)\frac{dc_r}{d\lambda} - 2\gamma c_r \ , \tag{7.114}$$

where the coefficient d_1 was obtained before. We can conclude by saying that the coefficient of the simple pole in the renormalization constants uniquely determines the the functions β, γ_m and γ as well as the residues of higher order poles.

7.2.5 β and γ properties

We have adopted a particular renormalization scheme. With other scheme we would have another definition of the parameters of the theory and different β, γ_m and γ functions. We are now going to discuss the aspects that are independent of the renormalization scheme used. Let us consider then two different schemes (both mass independent). Then

$$\begin{array}{ll} g' = gF_g(g) & F_g(g) = 1 + O(g^2) \\ Z'_m(g') = Z_m(g)F_m(g) & F_m(g) = 1 + O(g^2) \\ Z'_\phi(g') = Z_\phi(g)F_\phi(g) & F_\phi(g) = 1 + O(g') \end{array} \ . \tag{7.115}$$

The 1 in the functions F expresses the fact that in lowest order (tree level) there is no ambiguity. Using the above relations we can see how are related the functions β, γ_m and γ in the two schemes. For definiteness we consider the case of a gauge theory.

We have

$$\begin{aligned} \beta'(g') &= \mu\frac{\partial}{\partial\mu}g' = \mu\frac{\partial}{\partial\mu}(gF_g(g)) = \beta(g)\left(F_g + g\frac{\partial F_g}{\partial g}\right) \\ \gamma'_m(g') &= \mu\frac{\partial}{\partial\mu}\ln m' = \mu\frac{\partial \ln}{\partial\mu}(F_m^{-1}(g)m) = \gamma_m(g) - \beta(g)\frac{\partial}{\partial g}\ln F_m \end{aligned}$$

$$\gamma'(g') = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z'_\phi(g') = \gamma(g) + \frac{1}{2}\beta(g)\frac{\partial}{\partial g}\ln F_\phi \; . \tag{7.116}$$

The functions β, γ_m and γ will only coincide if the schemes are identical, in which case $F_g = F_m = F_\phi = 1$. However the following properties are scheme independent:

i) The existence of a zero of $\beta(g)$

If $\beta(g_0) = 0$ then $\beta'(g'_0) = 0$ for $g'_0 = g_0 F_g(g_0)$. Notice that, in general, g_0 depends on the scheme, that is $g_0 \neq g'_0$.

ii) The first derivative of $\beta(g)$ at the zero

Let $\beta(g_0) = 0$. Then

$$\begin{aligned}
\frac{\partial\beta'(g'_0)}{\partial g'} &= \left\{\frac{\partial g}{\partial g'}\frac{\partial}{\partial g}\left[\beta(g)\left(F_g + g\frac{\partial F_g}{\partial g}\right)\right]\right\}_{g_0} \\
&= \left[F_g + g\frac{\partial F_g}{\partial g} + g\frac{\partial\beta}{\partial g} + \beta(g)\frac{1}{F_g + g\frac{\partial F_g}{\partial g}}\frac{\partial\left(F_g + g\frac{\partial F_g}{\partial g}\right)}{\partial g}\right]_{g_0} \\
&= \frac{\partial\beta}{\partial g}(g_0) \; .
\end{aligned} \tag{7.117}$$

iii) The first two terms of $\beta(g)$

Let $\beta(g) = b_0 g^3 + b_1 g^5 + O(g^7)$, and

$$F_g(g) = 1 + ag^2 + O(g^4) \cdot \tag{7.118}$$

Then

$$g' = g + ag^3 + O(g^5) \; , \tag{7.119}$$

and

$$g = g' - ag'^3 + O(g^5) \; . \tag{7.120}$$

Therefore

$$\begin{aligned}
\beta'(g') &= \beta(g)\frac{\partial}{\partial g}(gF_g) = (b_0 g^3 + b_1 g^5 + O(g^7))(1 + 3ag^2 + O(g^4)) \\
&= b_0 g^3 + (3ab_0 + b_1)g^5 + O(g^7) \\
&= b_0(g'^3 - 3ag'^5 + O(g'^7) + (3ab_0 + b_1)(g'^5 + O(g'^7)) \\
&= b_0 g'^3 + b_1 g'^5 + O(g'^7) \; .
\end{aligned} \tag{7.121}$$

***iv)* The first term in $\gamma(g)$ and $\gamma_m(g)$**

Let

$$\begin{aligned}\gamma(g) &= cg^2 + O(g^4)\\ \gamma_m(g) &= dg^2 + O(g^4) \ .\end{aligned} \tag{7.122}$$

Then as $\beta(g) = O(g^3)$ it is clear that,

$$\begin{aligned}\gamma'(g') &= cg'^2 + O(g'^4)\\ \gamma'_m(g') &= dg'^2 + O(g'^4) \ .\end{aligned} \tag{7.123}$$

***v)* The value of $\gamma(g_0)$ and $\gamma_m(g_0)$ if $\beta(g_0) = 0$**

This result is obvious. As we will see next, all these results are necessary because they control the physical results and these can not depend on the renormalization scheme used.

7.2.6 Gauge independence of β and γ_m in MS

The renormalization group equation in MS was written for the $\lambda\phi^4$ theory. Let us now consider the modifications that appear in gauge theories. For these we have to introduce a gauge fixing term,

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial \cdot A)^2 \ , \tag{7.124}$$

where we have chosen covariant gauges of the Lorenz type. As there are no corrections to the longitudinal part of the propagator there is no need of a counter-term for this gauge fixing term. Therefore if we define, as usual,

$$A^\mu = Z_A^{-1/2} A_0^\mu \ , \tag{7.125}$$

we get

$$\frac{1}{2\xi}(\partial \cdot A)^2 = \frac{1}{2\xi Z_A}(\partial \cdot A_0)^2 = \frac{1}{2\xi_0}(\partial \cdot A_0)^2 \ . \tag{7.126}$$

This means that the gauge parameter gets renormalized in the following way,

$$\xi_0 = Z_A \xi \ . \tag{7.127}$$

The renormalized irreducible Green functions, in general will depend on ξ, that is,

$$\Gamma_R^{(n)}(g, m, \xi, \mu) = Z_A^{n/2} \Gamma_0^{(n)}(g_0, m_0, \xi_0, \varepsilon) \ . \tag{7.128}$$

The renormalization group equation takes then the form,

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g,\xi)\frac{\partial}{\partial g} + \gamma_m(g,\xi) m\frac{\partial}{\partial m} + \delta(g,\xi)\frac{\partial}{\partial \xi} - \gamma_A(g,\xi)\right] \Gamma_R^{(n)}(g, m, \xi, \mu) = 0 \ , \tag{7.129}$$

where

$$\begin{aligned}
\delta(g,\xi) &= \mu\frac{\partial}{\partial\mu}\xi = \mu\frac{\partial}{\partial\mu}(Z_A^{-1}\xi_0) \\
&= -\xi_0\frac{1}{Z_A^2}\frac{\partial}{\partial\mu}Z_A \\
&= -2\xi\gamma_A(g,\xi)
\end{aligned} \tag{7.130}$$

and we assumed that β, γ_m and γ_A could depend on the parameter ξ. However the dependence on ξ is not arbitrary and should obey certain constraints. To see that let us consider a dimensionless Green function corresponding to gauge invariant operators. Then

$$\frac{\partial}{\partial\xi_0}G_0(g_0,m_0,\xi_0,\varepsilon) = 0 \qquad \text{(gauge independent)} \tag{7.131}$$

and

$$G_0(g_0,m_0,\xi_0,\varepsilon) = G(g,m,\xi,\mu) \ . \qquad \text{(dimensionless)} \tag{7.132}$$

Therefore

$$\frac{\partial}{\partial\xi}G = 0 \ , \tag{7.133}$$

and this gives

$$\mathcal{D}_G G \equiv \left[\frac{\partial}{\partial\xi} + \rho(g,\xi)\frac{\partial}{\partial g} + \sigma(g,\xi)m\frac{\partial}{\partial m}\right]G(g,m,\xi,\mu) = 0 \ , \tag{7.134}$$

where

$$\rho(g,\xi) = \frac{\partial g}{\partial\xi} \qquad ; \qquad \sigma(g,\xi) = \frac{\partial}{\partial\xi}\ln m \ . \tag{7.135}$$

But now G also obeys the renormalization group equation

$$\mathcal{D}G \equiv \left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \gamma_m m\frac{\partial}{\partial m} + \delta\frac{\partial}{\partial\xi}\right]G = 0 \ . \tag{7.136}$$

Using the equation for $\mathcal{D}_G G = 0$ we can substitute the derivative with respect to ξ by derivatives in order to the other parameters, obtaining a renormalization group equation similar to that of theories that are not gauge theories, that is,

$$\left[\mu\frac{\partial}{\partial\mu} + \overline{\beta}\frac{\partial}{\partial g} + \overline{\gamma}_m m\frac{\partial}{\partial m}\right]G = 0 \ , \tag{7.137}$$

where

$$\overline{\beta} \equiv \beta - \rho\delta \qquad\qquad \overline{\gamma}_m = \gamma_m - \sigma\delta \tag{7.138}$$

Let us now evaluate the commutator $[\mathcal{D}_G,\mathcal{D}]G = 0$. We get

$$\left\{\left[\frac{\partial\beta}{\partial\xi} + \beta\frac{\partial\beta}{\partial g} - \beta\frac{\partial\rho}{\partial g} - \delta\frac{\partial\rho}{\partial\xi}\right]\frac{\partial}{\partial g} + \left[\frac{\partial\delta}{\partial\xi} + \rho\frac{\partial\delta}{\partial g}\right]\frac{\partial}{\partial\xi}\right.$$

$$+\left[\frac{\partial\gamma_m}{\partial\xi}+\rho\frac{\partial\gamma_m}{\partial g}-\beta\frac{\partial\sigma}{\partial g}-\delta\frac{\partial\sigma}{\partial\xi}\right] m\frac{\partial}{\partial m}\Bigg\} G=0 \ . \tag{7.139}$$

We introduce now the functions $\overline{\beta}$ and $\overline{\gamma}_m$ and the operator

$$\overline{\mathcal{D}}\equiv\frac{\partial}{\partial\xi}+\rho\frac{\partial}{\partial g} \ , \tag{7.140}$$

to write the previous equation as,

$$\begin{aligned}&\left[(\overline{\mathcal{D}}\delta)\frac{\partial}{\partial\xi}+\left(\overline{\mathcal{D}}\ \overline{\beta}+\overline{\mathcal{D}}(\rho\delta)-\overline{\beta}\frac{\partial\rho}{\partial g}-\delta\overline{\mathcal{D}}\rho\right)\frac{\partial}{\partial g}\right.\\ &\left.+\left(\overline{\mathcal{D}}\overline{\gamma}_m+\overline{\mathcal{D}}(\sigma\delta)-\overline{\beta}-\frac{\partial\sigma}{\partial g}-\delta\overline{\mathcal{D}}\sigma\right)m\frac{\partial}{\partial m}\right]G=0 \ .\end{aligned} \tag{7.141}$$

Multiplying the equation $\mathcal{D}_G G=0$ by $(\overline{\mathcal{D}}\delta)$ we get

$$\left[(\overline{\mathcal{D}}\delta)\frac{\partial}{\partial\xi}+\rho(\overline{\mathcal{D}}\delta)\frac{\partial}{\partial g}+\sigma(\overline{\mathcal{D}}\delta)m\frac{\partial}{\partial m}\right]G=0 \ . \tag{7.142}$$

Comparing both equations we see that

$$\overline{\mathcal{D}}\ \overline{\beta}=\overline{\beta}\frac{\partial\rho}{\partial g} \qquad \text{and} \qquad \overline{\mathcal{D}}\ \overline{\gamma}_m=\overline{\beta}\frac{\partial\sigma}{\partial g} \ . \tag{7.143}$$

These equations ensure that the physical results are gauge independent. In fact $\overline{\beta}=0$ has physical consequences. Then $\overline{\mathcal{D}}\ \overline{\beta}=0$ and $\overline{\mathcal{D}}\ \overline{\gamma}_m=0$ showing the the zeros of $\overline{\beta}$ and the anomalous dimension of the mass, $\overline{\gamma}_m$ are gauge independent. Also, if $\overline{\beta}=0$ we obtain,

$$\begin{aligned}\overline{\mathcal{D}}\left(\frac{\partial\overline{\beta}}{\partial g}\right) &= \frac{\partial}{\partial g}\overline{\mathcal{D}}\ \overline{\beta}+\left[\overline{\mathcal{D}},\frac{\partial}{\partial g}\right]\overline{\beta}\\ &= \frac{\partial}{\partial g}\overline{\mathcal{D}}\ \overline{\beta}-\frac{\partial\rho}{\partial g}\frac{\partial\overline{\beta}}{\partial g}=0 \ .\end{aligned} \tag{7.144}$$

This shows that the first derivative of $\overline{\beta}$ at the zero is gauge independent. Finally as $\rho=O(g^3)$ and $\delta=O(g^2)$ we also get,

$$\overline{\beta}=\beta+O(g^5) \ . \tag{7.145}$$

These results do not depend on the scheme adopted. If we adopt now MS we obtain the following theorem,

Theorem

In the minimal subtraction scheme we have $\rho=\sigma=0$ and therefore

$$\overline{\mathcal{D}}=\frac{\partial}{\partial\xi} \quad ; \quad \overline{\beta}=\beta \quad \textit{and} \quad \overline{\gamma}_m=\gamma_m \tag{7.146}$$

and β and γ_m are gauge independent in all orders.

Dem: We just give the proof for ρ, for σ it is similar.

$$\rho = g\frac{\partial}{\partial\xi}\ln g = -\frac{g}{Z_g}\frac{\partial Z_g}{\partial\xi} \ . \tag{7.147}$$

Then

$$\begin{aligned} 0 &= Z_g\rho + g\frac{\partial}{\partial\xi}\left(1+\frac{a_1}{\varepsilon}+\frac{a_2}{\varepsilon^2}+\cdots\right) \\ &= \rho + \frac{1}{\varepsilon}\left(\rho a_1 + g\frac{\partial a_1}{\partial\xi}\right) + O(1/\varepsilon^2) \ , \end{aligned} \tag{7.148}$$

and we get therefore,

$$\rho = 0 \ . \tag{7.149}$$

7.3 Effective gauge couplings

7.3.1 Fixed points

As we saw in the last section the asymptotic behavior of the irreducible Green functions depends on the asymptotic behavior of the solutions of the equations for the effective coupling constant, $\overline{\lambda}(t)$, and effective mass, $\overline{m}(t)$, which are,

$$\left\{ \begin{array}{lll} \dfrac{d\overline{\lambda}}{dt} = \beta(\overline{\lambda}) & ; & \overline{\lambda}(0) = \lambda \\[2ex] \dfrac{d\overline{m}}{dt} = \left[\gamma_m(\overline{\lambda}) - 1\right]\overline{m}(t) & ; & \overline{m}(0) = m \end{array} \right. \tag{7.150}$$

From these equations we see that variation of the effective coupling and effective mass with the scale are controlled by the functions β and γ_m, respectively. To study the asymptotic behavior of λ we are going to assume that $\beta(\lambda)$ has the form given in Fig. 7.2.

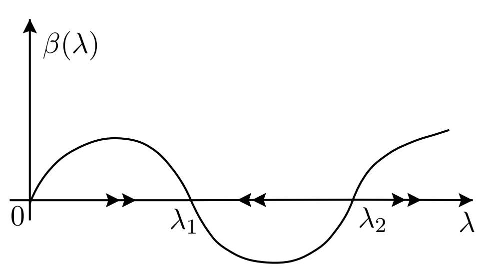


Figure 7.2: $\beta(\lambda)$ as function of λ.

The points $0, \lambda_1$ e λ_2 where $\beta(\lambda)$ vanishes are called *fixed points*. This is because if $\overline{\lambda}$ is at one of these points at $t = 0$ then it will stay there for any momentum scale as $\left(\frac{d\overline{\lambda}}{dt} = 0\right)$. The fixed points can be of two types:

1. *Ultra-Violet (UV) stable fixed point*

Are those in which $\beta'(\lambda) < 0$. It is the case for the point λ_1 in the figure. In this case $\beta(\lambda) > 0$ for $\lambda < \lambda_1$ and $\beta(\lambda) < 0$ for $\lambda > \lambda_1$. Then if for $t = 0 \quad 0 < \lambda < \lambda_1$ then $t \to \infty \quad \overline{\lambda} \to \lambda_1$. On the other hand if $\lambda_1 < \lambda < \lambda_2$ as $t \to \infty$ also $\overline{\lambda} \to \lambda_1$. Therefore in the interval $0 < \lambda < \lambda_2$ the coupling constant is always lead to λ_1 when $t \to \infty$, that is, for large momenta.

2. *Infra-Red (IR) stable fixed point*

   Are those for which $\beta'(\lambda) > 0$. This is the case of points 0 and λ_2 in the figure. We can easily see that as $t \to \infty$ the coupling constant moves away from 0 and λ_2, but it is attracted to them in the limit $t \to 0$.

We can now study the asymptotic behavior of the solutions of the renormalization group equations. Let us suppose that $0 < \lambda < \lambda_2$. Then

$$\lim_{t\to\infty} \overline{\lambda}(t, \lambda) = \lambda_1 \tag{7.151}$$

The way it goes into λ_1 depends on the first derivative of $\beta(\lambda)$. Let us assume that near λ_1 we have

$$\beta(\lambda) = a(\lambda_1 - \lambda) \quad ; \quad a > 0$$

$$\beta'(\lambda_1) = -a < 0 \tag{7.152}$$

Then

$$\overline{\lambda}(t, \lambda) = \lambda_1 + (\lambda - \lambda_1)e^{-at} \tag{7.153}$$

that is, the way it approaches the fixed point is exponential in the variable t. It will be larger if $|\beta'(\lambda_1)| = a$ gets larger. We saw that the solution for the effective mass equation was,

$$\overline{m}(t) = m\, e^{-t} e^{\int_0^t \gamma_m(\overline{\lambda})dt'} \ . \tag{7.154}$$

If $\lim\limits_{t\to\infty} \overline{\lambda} = \lambda_1$ then we have for $t \to \infty$

$$\overline{m} = m e^{-t(1-\gamma_m(\lambda_1))} \ . \tag{7.155}$$

This shows that if $\gamma_m(\lambda_1) < 1$ then $m(t) \to 0$ as $t \to \infty$. In the same approximation

$$\int_0^t \gamma(\overline{\lambda}(t'))dt' \simeq \gamma(\lambda_1)t \ , \tag{7.156}$$

and therefore the asymptotic solution is

$$\lim_{\sigma\to\infty} \Gamma^n(\sigma p_i, m, \lambda, \mu) = \sigma^{4-n[1+\gamma(\lambda_1)]}\Gamma^{(n)}(p_i, \overline{m}, \lambda_1, \mu) \ . \tag{7.157}$$

This shows the the effective dimension of the fields is not 1 but $1 + \gamma(\lambda_1)$. This explains the name of anomalous dimension for $\gamma(\lambda)$.

In general it is difficult to determine the zeros of the β function. This is because for that one would need, in general, results beyond perturbation theory. However $\beta(\lambda), \gamma_m(\lambda)$ and $\gamma(\lambda)$ have a trivial zero at the origin. If it happens that it is a UV stable fixed point, then it means that as the scale gets larger the coupling constant get smaller. In the limit $t \to \infty, \overline{\lambda} \to 0$. For this reason these theories are called *asymptotically free*. It is easy to see that this happens if $\beta'(0) < 0$. In the following we will discuss in which theories this can happen.

7.3.2 β function for theories with scalars, fermions and gauge fields

We will now show that only non-abelian gauge theories can be asymptotically free, that is, only these verify the property $\beta'(0) < 0$.

i) *Theories with scalars*

We have already seen that for the simplest scalar theory, $\lambda\phi^4$, we have

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^4) \tag{7.158}$$

and therefore it is not asymptotically free. Let us now consider a more general theory with scalar fields ϕ_i with couplings

$$\mathcal{L}_I = -\lambda_{ijk\ell}\phi_i\phi_j\phi_k\phi_\ell \tag{7.159}$$

where repeated indices are summed over. Then

$$\beta_{ijk\ell} = \frac{d\overline{\lambda}_{ijk\ell}(t)}{dt} = A(\overline{\lambda}_{i\ell mn}\overline{\lambda}_{kjmn} + \overline{\lambda}_{ijmn}\overline{\lambda}_{k\ell mn} + \overline{\lambda}_{ikmn}\overline{\lambda}_{j\ell mn}) \tag{7.160}$$

with $A > 0$. The theory is not asymptotically free because there are always β functions with positive derivatives. As an example we have

$$\frac{d\overline{\lambda}_{1111}}{dt} = \beta_{1111} = 3A|\overline{\lambda}_{11mn}|^2 > 0 \qquad ; \qquad \forall t \tag{7.161}$$

ii) *Scalar and fermion theories with Yukawa interactions*

The most general interaction term for this theory is

$$\mathcal{L}_I = -\sum_{i,j,k,\ell}\lambda_{ijk\ell}\phi_i\phi_j\phi_k\phi_\ell + \sum_{a,b,k}\overline{\psi}^a(A^k_{ab} + iB^k_{ab}\gamma_5)\overline{\psi}^b\phi_k \ , \tag{7.162}$$

where A and B are real matrices. Now it is no longer possible to show that $\dfrac{d\overline{\lambda}_{iiii}}{dt} > 0$ because of the fermion loop of order A^2 or B^2 with a negative sign. If we define $(g^i)_{ab} \equiv A^i_{ab} + iB^i_{ab}\gamma_5$, we get

$$\begin{aligned} 16\pi^2\frac{dg^i}{dt} &= (\mathrm{Tr}g^i g^{j\dagger})g^j + \mathrm{Tr}(g^{i\dagger}g^j)g^j + M^{ij}g^j \\ &\quad + \frac{1}{2}g^i g^{\dagger j}g^j + \frac{1}{2}g^j g^{\dagger j}g^i + 2g^j g^{\dagger i}g^j \ , \end{aligned} \tag{7.163}$$

where $M^{ij} \equiv \frac{1}{4}\lambda_{ik\ell m}\lambda_{jk\ell m}$. Using this result we can prove the following theorem:

Theorem

The most general theory with fermions and scalars is not asymptotically free because $\frac{d}{dt}\mathrm{Tr}(g^{i\dagger}g^i) > 0$ and therefore it is not possible to have $g_i \to 0$ as $t \to \infty$.

Proof:

$$\begin{aligned} 8\pi^2 \frac{d}{dt} \mathrm{Tr}(g^{i\dagger} g^i) &= 8\pi^2 \frac{d}{dt} \sum_{a,b,i} |g^i_{ab}|^2 \\ &= \mathrm{Tr}(g^i g^{j\dagger}) \mathrm{Tr}(g^{i\dagger} g^j) + \mathrm{Tr}(g^i g^{j\dagger})(\mathrm{Tr} g^i g^{j\dagger}) \\ &\quad + \frac{1}{2}\mathrm{Tr}(g^i g^{i\dagger} g^j g^{j\dagger}) + \frac{1}{2}\mathrm{Tr}(g^{i\dagger} g^i g^{j\dagger} g^j) \\ &\quad + 2\mathrm{Tr}(g^i g^{j\dagger} g^i g^{j\dagger}) + M^{ij}\mathrm{Tr}(g^{i\dagger} g^j) \end{aligned} \tag{7.164}$$

Now the last four terms are positive. Also the first is larger that the second. This gives

$$8\pi^2 \frac{d}{dt} \mathrm{Tr}(g^{i\dagger} g^i) \geq 2\Big[\mathrm{Tr}(g^i g^{j\dagger})\mathrm{Tr}(g^i g^{j\dagger}) + \mathrm{Tr}(g^i g^{j\dagger} g^i g^{j\dagger})\Big] \tag{7.165}$$

The right-hand side is positive as it can be written as

$$8\pi^2 \frac{d}{dt} \mathrm{Tr}(g^{i\dagger} g^i) \geq (g^i_{ab} g^i_{cd} + g^i_{ad} g^i_{cd})(g^{j\dagger}_{ba} g^{j\dagger}_{dc} + g^{j\dagger}_{da} g^{j\dagger}_{bc}) \geq 0 \tag{7.166}$$

as we wanted to show.

iii) *Abelian gauge theories*

Let us consider the case of QED. We have

$$Z_e = Z_1 Z_2^{-1} Z_3^{-1/2} = Z_3^{-1/2} \tag{7.167}$$

The renormalization constant Z_3 can be calculated in the vacuum polarization represented in Fig. 7.3,

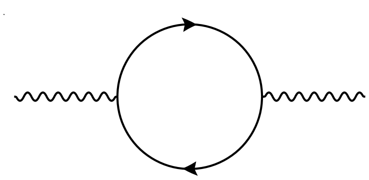


Figure 7.3: Vacuum polarization in QED.

The result is

$$Z_3^{-1/2} = 1 + \frac{e^2}{12\pi^2}\frac{1}{\varepsilon} + \cdots \tag{7.168}$$

Therefore

$$\beta(e) = \frac{1}{2} e^2 \frac{da_1}{de} = \frac{e^3}{12\pi^2} > 0 \ . \tag{7.169}$$

If we had scalar electrodynamics, the renormalization constant Z_3 would be obtained from the diagrams in Fig. 7.4

The result is

$$Z_3^{-1/2} = 1 + \frac{e^2}{48\pi^2}\frac{1}{\varepsilon} \tag{7.170}$$

and this gives $\beta(e) = \frac{e^3}{48\pi^2} > 0$. Therefore the abelian gauge theories are also not asymptotically free.

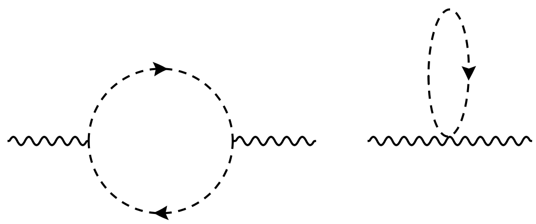


Figure 7.4: Vacuum polarization in scalar electrodynamics.

iv) *Non-abelian gauge theories*

Let us start with the pure gauge theory. The wave function renormalization for the gauge fields is obtained from the one-lop diagrams of Fig. 7.5. In MS we obtain

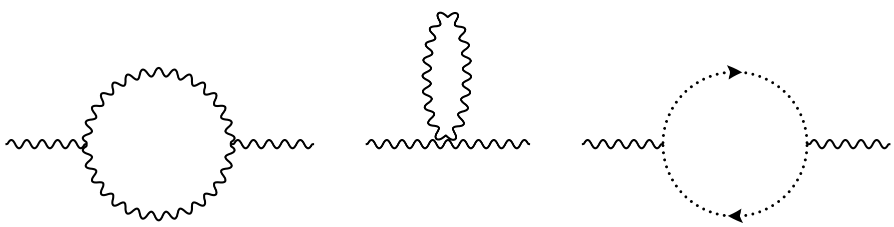


Figure 7.5: Vacuum polarization in a pure non-abelian gauge theory.

$$Z_A = 1 + \frac{g^2}{16\pi^2}\left(\frac{13}{3} - \xi\right) C_2(V)\frac{1}{\varepsilon} \tag{7.171}$$

where $C_2(V)$ is the Casimir operator defined before. In this case it is for the adjoint representation to which belong the gauge fields (vectors).

The renormalization constant for the triple vertex, Z_1, is obtained from the diagrams in Fig. 7.6. We get

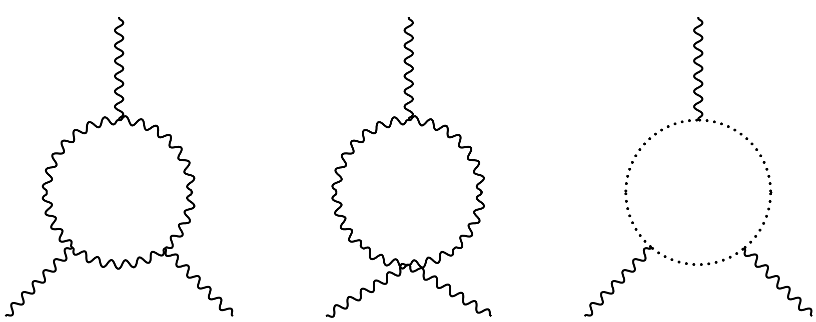


Figure 7.6: Vertex in a pure non-abelian gauge theory.

$$Z_1 = 1 + \frac{g^2}{16\pi^2}\left(\frac{17}{6} - \frac{3\xi}{2}\right) C_2(V)\frac{1}{\varepsilon} + \cdots \tag{7.172}$$

Therefore we get for the renormalization of the coupling constant,

$$Z_g \equiv Z_1 Z_A^{-3/2} = 1 - \frac{g^2}{16\pi^2}\left(\frac{11}{3}C_2(V)\right)\frac{1}{\varepsilon} + \cdots \tag{7.173}$$

Using Z_A and Z_g and the definitions of β and γ we get

$$\beta = -\frac{g^3}{16\pi^2}\frac{11}{3}C_2(V) < 0 \tag{7.174}$$

and

$$\gamma_A = -\frac{g^2}{16\pi^2}\frac{1}{2}\left(\frac{13}{3} - \xi\right)C_2(V) \tag{7.175}$$

Therefore the pure gauge theories, without matter fields, are asymptotically free. Notice that the dependence on the gauge parameter, ξ, has disappeared from β in agreement with a general result that we have shown before.

The inclusion of fermions and scalars minimally coupled it is now trivial. The interaction Lagrangian is dictated by the covariant derivatives,

$$\begin{aligned}\mathcal{L}_{int} &= g\overline{\psi}_i\gamma^\mu\psi_j T^a_{Fij}A^a_\mu \\ &\quad + ig\phi^*_i\overleftrightarrow{\partial}_\mu\phi_j T^a_{Sij}A^{\mu a} \\ &\quad + g^2\phi^*_i T^a_{Sij}T^b_{Sjk}\phi_k A^a_\mu A^{\mu b}\end{aligned} \tag{7.176}$$

where T^a_F and T^a_S are the generators in the representations to which the fermions and scalars belong, respectively. To find the contribution of these articles to the β function we have to calculate their contribution to Z_g. The easiest is to use the results that generalize QED, that is,

$$Z_g = Z_A^{-1/2} \tag{7.177}$$

and calculate the contributions of the fermions and scalars to Z_A. This comes from the diagrams in Fig. 7.7. The result is

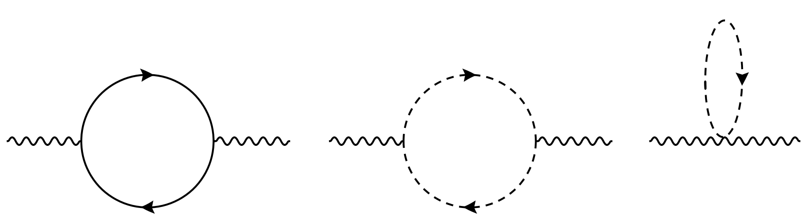


Figure 7.7: Contribution from fermions and scalars to vacuum polarization.

$$Z_g(\text{fermions} + \text{scalars}) = 1 + \frac{g^2}{16\pi^2}\left[\frac{4}{3}T(R_F) + \frac{1}{3}T(R_S)\right]\frac{1}{\varepsilon} + \cdots \tag{7.178}$$

Therefore for fermions,

$$\beta(\text{fermions}) = \frac{g^3}{16\pi^2}\frac{4}{3}T(R_F)\ , \tag{7.179}$$

and for the scalars,

$$\beta(\text{scalars}) = \frac{g^3}{16\pi^2}\frac{1}{3}T(R_S) \tag{7.180}$$

Putting everything together we get

$$\beta = \frac{g^3}{16\pi^2}\left[-\frac{11}{3}C_2(V) + \frac{4}{3}T(R_F) + \frac{1}{3}T(R_s)\right] \tag{7.181}$$

where the quantities $T(R)$ are defined for a given representation by

$$\mathrm{Tr}(T^a T^b) = T(R)\delta^{ab} \tag{7.182}$$

For a theory with Majorana fermions or real scalars, the coefficients in front of $T(R_F)$ and $T(R_S)$ are multiplied by an additional factor of 1/2.

β function for QCD

Let us now consider a simple and important example, QCD ($SU(3)$) with three families of quarks. For $SU(N)$ we have

$$C_2(V) = N \tag{7.183}$$

and as the quarks are in the fundamental representation we have,

$$T(R_F) = \frac{1}{2} \tag{7.184}$$

Then

$$\beta = \frac{g}{16\pi^2}\left[-\frac{33}{3} + \frac{4}{3} \times \frac{1}{2} \times 2N_g\right] \tag{7.185}$$

where $N_g =$ is the number of families or generations. We get then

$$\beta = \frac{g^3}{16\pi^2}\left[-\frac{33 - 4N_g}{3}\right] \tag{7.186}$$

Therefore for $SU(3)$ the theory is asymptotically free if

$$33 - 4N_g > 0 \tag{7.187}$$

which gives

$$N_g < \frac{33}{4} \to N_g \leq 8 \tag{7.188}$$

Therefore there are allowed 8 families of quarks, or 16 triplets of $SU(3)$.

7.3.3 The vacuum of a NAGT as a paramagnetic medium ($\mu > 1$)

There is an interesting argument (Nielsen 1981, Hughs 1981) that allows to understand what is different in the non-abelian gauge theories for them to be asymptotically free. The fact that charge decreases at short distance can be seen as an *anti - shielding* of the vacuum, that is,

$$\varepsilon < 1 \tag{7.189}$$

The problem in understanding this result derives from the fact that we do not known any material with $\varepsilon < 1$. In QED the charge grows at short distances and therefore the vacuum is a normal dielectric with $\varepsilon > 1$. The vacuum must have a permeability given by (we use $c = 1$),

$$\mu\varepsilon = 1 \tag{7.190}$$

Therefore the *anti-screening* corresponds to $\mu > 1$. Therefore the vacuum of a non-abelian gauge theory is a *paramagnetic* and this concept can be better understood.

The magnetic permeability can be obtained from the density of energy of the vacuum in an exterior field,

$$u_0 = \frac{1}{2\mu} B^2_{ext} \tag{7.191}$$

Nielsen and Hughes have shown that $\mu = 1 + \chi$, where the magnetic susceptibility χ is given by

$$\chi \sim (-1)^{2s} q^2 \sum_{s_3} \left(-\frac{1}{3} + \gamma^2 s_3^2 \right) \tag{7.192}$$

where s is the spin, q the charge, γ the gyromagnetic ratio and s_3 the projection of the spin along the external magnetic field. We have therefore for the different types of fields:

- *Scalars*

$$\chi_S \sim -\frac{1}{3} q_S^2 < 0 \qquad \text{(diamagnetic)} \tag{7.193}$$

- *Fermions* ($\gamma_F = 2$)

$$\chi_F \sim (-1) q_F^2 2 \left(-\frac{1}{3} + 1 \right) = -\frac{4}{3} q_F^2 \qquad \text{(diamagnetic)} \tag{7.194}$$

- *Gauge bosons* ($\gamma_V = 2$)

$$\chi_V \sim q_V^2 2 \left(-\frac{1}{3} + 4 \right) = \frac{22}{3} q_V^2 \qquad \text{(paramagnetic)} \tag{7.195}$$

Therefore we get,

$$\chi_{\text{Total}} \sim \frac{22}{3} q_V^2 - \frac{4}{3} q_F^2 - \frac{1}{3} q_S^2 \tag{7.196}$$

Comparing with the β function we can make the correspondence

$$\begin{aligned} q_V^2 &\to \frac{1}{2} C_2(V) \\ q_F^2 &\to T(R_F) \\ q_S^2 &\to T(R_S) \end{aligned} \tag{7.197}$$

This analogy tells us that the vacuum of a non-abelian gauge theory can be understood as a paramagnetic medium.

7.4 Renormalization group applications

We consider the Grand Unified Theory (GUT) with the gauge group $SU(5)$, that is

$$SU(5) \supset SU_c(3) \times SU_L(2) \times U_Y(1) \ . \tag{7.198}$$

The unification takes place at the GUT scale M_X. Using the renormalization group equations and the low energy data on the coupling constants, it is possible to determine the scale M_X as well as other predictions for the theory at the low scale, which we take to be the scale M_Z. For this we need to know how the different coupling constants evolve with the scale.

7.4.1 Scale M_X

We start by writing the covariant derivatives for the unified theory and for the theory with the broken symmetry.

$$SU(5) : D_\mu = \partial_\mu + i g_5 \sum_{a=0}^{23} A_\mu^a \frac{\lambda^a}{2} \tag{7.199}$$

$$\begin{aligned} SU(3) \times SU(2) \times U(1) : D_\mu = \partial_\mu + i g_3 \sum_{\alpha}^{8} G_\mu^a \frac{\lambda}{2}^a \\ + i g_2 \sum_{\alpha}^{3} A_\mu^a \frac{\sigma}{2}^a + i g' \frac{Y}{2} B_\mu \end{aligned} \tag{7.200}$$

At the scale M_X where the unification takes place we have

$$g_5 = g_3 = g_2 = g_1 \tag{7.201}$$

where g_1 is the coupling constant of the abelian subgroup of $SU(5)$. However for the abelian groups there are no constraints in the normalization of the generators, and therefore the the generator λ^0 of that $U(1)$ can be normalized in a different way from the hypercharge. We must have

$$g_1 \lambda^0 = g' Y \tag{7.202}$$

As λ^0 is a generator of $SU(5)$ it is normalized according to

$$T_F(\lambda^a \lambda^b) = 2\delta^{ab} \tag{7.203}$$

that is, for the fundamental representation we must have

$$\lambda^0 = \frac{1}{\sqrt{15}} \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{bmatrix} \tag{7.204}$$

Now, for the fundamental representation, we have

$$5 = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ e^+ \\ \nu_e^c \end{pmatrix}_R \tag{7.205}$$

and the hypercharge[1] can be read directly. We obtain,

$$Y = \begin{bmatrix} -2/3 & & & & \\ & -2/3 & & & \\ & & -2/3 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \tag{7.206}$$

Therefore $Y = -\sqrt{\frac{5}{3}}\lambda^0$ and $g' = -\sqrt{\frac{3}{5}}g_1$. This allows to determine $\sin^2\theta_W$ at the GUT scale M_X,

$$\sin^2\theta_W(M_X) = \frac{g'^2}{g^2 + g'^2} = \frac{\frac{3}{5}g_1}{g_2 + \frac{3}{5}g_1} = \frac{3}{8} \tag{7.207}$$

Also, for future reference, we note that

$$g'^2 = \frac{3}{5}\,g_1^2\ . \tag{7.208}$$

7.4.2 Scale M_Z

Let us look now at what happens at the scale M_Z. The evolution of the coupling constants is governed by the RGE equations for the three gauge groups in the broken phase

$$\frac{dg_i}{dt} = \beta_i \tag{7.209}$$

These β functions are given by

$$\beta_i = \frac{g_i^3}{16\pi^2}\left[-\frac{11}{3}C_2(V) + \sum_j \frac{4}{3}T(R_{F_j}) + \sum_k \frac{1}{3}T(R_{S_k})\right] \tag{7.210}$$

where the sums are over all the fermion and scalar physical states of the theory at a given scale. Given the form of Eq. (7.210), it is usual to define

$$\beta_i \equiv \frac{1}{16\pi^2}\,b_i g_i^3 \tag{7.211}$$

[1]Remember that our convention is such that $Q = T_3 + \dfrac{Y}{2}$.

and therefore the b_i are defined by the bracket in Eq. (7.210). Before we evaluate them let us introduce Eq. (7.211) into Eq. (7.209). We get

$$\frac{dg_i}{dt} = \frac{b_i}{16\pi^2}\, g_i^3 \tag{7.212}$$

Let us solve this equations before we evaluate the beta function coefficients b_i. For that it is usual to introduce the generalization of the fine structure constant, that is, we define

$$\alpha_i \equiv \frac{g_i^2}{4\pi} \tag{7.213}$$

Multiplying both sides of Eq. (7.212) by g_i and doing some trivial algebra we get,

$$\frac{d\alpha_i}{dt} = \frac{b_i}{2\pi}\, \alpha_i^2 \tag{7.214}$$

Rearranging and integrating between some initial (μ_i), and final scale (μ_f), we get

$$\int_i^f \frac{d\alpha_i}{\alpha_i^2} = \frac{b_i}{2\pi} \int_i^f dt \tag{7.215}$$

or[2]

$$\left[-\frac{1}{\alpha_i}\right]_i^f = \frac{b_i}{2\pi}\,(t_f - t_i) = \frac{b_i}{2\pi} \ln\left(\frac{\mu_f}{\mu_i}\right) \tag{7.216}$$

and finally

$$\alpha_i^{-1}(\mu_f) = \alpha_i^{-1}(\mu_i) - \frac{b_i}{4\pi} \ln\left(\frac{\mu_f^2}{\mu_i^2}\right) \tag{7.217}$$

As at the unification scale M_X we have, by definition (see Eq. (7.201)), that

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 \tag{7.218}$$

where α_5 is the $SU(5)$ unified value, and we can write the final solution

$$\alpha_i^{-1}(\mu) = \alpha_5^{-1} + \frac{b_i}{4\pi} \ln\left(\frac{M_X^2}{\mu^2}\right), \quad i = 1, 2, 3 \tag{7.219}$$

We can rewrite these equations in terms of electromagnetic fine structure constant $\alpha(\mu)$ and of the strong coupling equivalent $\alpha_s\mu$), that are measured at the weak scale, to obtain

$$\left\{\begin{aligned} &\alpha_s^{-1}(\mu) = \alpha_5^{-1} + \frac{b_3}{4\pi} \ln\left(\frac{M_X^2}{\mu^2}\right) \\ &\alpha^{-1}(\mu)\sin^2\theta_W(\mu) = \alpha_5^{-1} + \frac{b_2}{4\pi} \ln\left(\frac{M_X^2}{\mu^2}\right) \\ &\frac{3}{5}\cos^2\theta_W(\mu)\alpha^{-1}(\mu) = \alpha_5^{-1} + \frac{b_1}{4\pi} \ln\left(\frac{M_X^2}{\mu^2}\right) \end{aligned}\right. \tag{7.220}$$

[2] Remember that $t = \ln(\mu)$.

From these equations we obtain,

$$\ln \frac{M_X^2}{\mu^2} = \frac{12\pi}{-8b_3 + 3b_2 + 5b_1} \left[\frac{1}{\alpha(\mu)} - \frac{8}{3} \frac{1}{\alpha_s(\mu)} \right] \tag{7.221}$$

That allows to determine M_X, once $\alpha(\mu)$ and $\alpha_s(\mu)$ are known, at a given scale μ, and

$$\sin^2 \theta_W(\mu) = \frac{3(b_2 - b_3)}{5b_1 + 3b_2 - 8b_3} + \frac{5(b_1 - b_2)}{5b_1 + 3b_2 - 8b_3} \frac{\alpha(\mu)}{\alpha_S(\mu)} \tag{7.222}$$

which allows to determine $\sin^2 \theta_W$ at the scale $\mu = M_Z$, once $\alpha(M_Z)$ and $\alpha_s(M_Z)$ are known. Finally we can also solve for the value of α_5^{-1}. We get

$$\alpha_5^{-1} = \alpha^{-1}(\mu) \frac{1}{5b_1 + 3b_2 - 8b_3} \left[-3b_3 + (5b_1 + 3b_2) \frac{\alpha(\mu)}{\alpha_S(\mu)} \right] \tag{7.223}$$

Now we turn to the evaluation of the coefficients b_i first in the Standard Model (SM) and the in the Minimal Supersymmetric Standard Model (MSSM).

Standard Model

In the SM we have the gauge fields, $N_g = 3$ families of leptons, $N_F = 2N_g = 6$ quark flavours and one Higgs. With this information we can find the coefficients b_i for the SM using the definition

$$b_i = -\frac{11}{3} C_2(V_i) + \sum_j \frac{2}{3} T(R_{F_j}) + \sum_k \frac{1}{3} T(R_{S_k}) \tag{7.224}$$

where we have modified Eq. (7.210), as the sum in the fermions is done separately for each chirality. This is important for the SM as the model is described in terms of left and right-handed fermions.

- $SU(3)$

For $SU(3)$, we have $C_2(V_3) = 3$ and the quarks are in the fundamental representation, therefore $T(R_{F_j}) = 1/2$. Then the counting goes as follows,

$$b_3 = \underbrace{-\frac{11}{3} \times 3}_{\text{Gauge}} + N_g \times \left[\underbrace{\frac{2}{3} \times \frac{1}{2} \times (2 + 1 + 1)}_{\text{quarks}} \right] = -7 \tag{7.225}$$

where the meaning of $(2 + 1 + 1)$ is that we count the up and down components of each $(SU(2)_L)$ doublet and then the corresponding right-handed quarks for each generation.

- $SU(2)$

For the $SU(2)$ we get

$$b_2 = \underbrace{-\frac{11}{3}\times 2}_{\text{Gauge}} + N_g \times \left(\underbrace{N_c \times \frac{2}{3}\times\frac{1}{2}}_{\text{quarks}_\text{L}} + \underbrace{\frac{2}{3}\times\frac{1}{2}}_{\text{leptons}} \right) + \underbrace{\frac{1}{3}\times\frac{1}{2}}_{\text{Higgs}} = -\frac{19}{6} \tag{7.226}$$

where $N_c = 3$ is the number of colors.

• $U(1)$

Finally for the $U(1)$ part, with the correct normalization, we have

$$b_1 = \frac{3}{5}\times\left[\frac{2}{3}\times\sum_{f_L,f_R}\left(\frac{Y}{2}\right)^2 + \frac{1}{3}\times\sum_{\text{scalars}}\left(\frac{Y}{2}\right)^2\right] \tag{7.227}$$

and therefore,

$$\begin{aligned} b_1 =& \frac{3}{5}\times\Bigg[N_g \times \Bigg(\underbrace{\frac{2}{3}\times\left(-\frac{1}{2}\right)^2 \times 2}_{\text{Leptons}_\text{L}} + \underbrace{\frac{2}{3}\times(-1)^2}_{\text{Leptons}_\text{R}} + N_c \times \underbrace{\frac{2}{3}\times\left(\frac{1}{6}\right)^2\times 2}_{\text{Quarks}_\text{L}} + N_c \times \underbrace{\frac{2}{3}\times\left(\frac{2}{3}\right)^2}_{\text{Up}-\text{Quarks}_\text{R}} \\ & + N_c \times \underbrace{\frac{2}{3}\times\left(\frac{1}{3}\right)^2}_{\text{Down}-\text{Quarks}_\text{R}} \Bigg) + \underbrace{\frac{1}{3}\times\left(\frac{1}{2}\right)^2\times 2}_{\text{Higgs}} \Bigg] \\ =& \, 4 + \underbrace{\frac{1}{10}}_{\text{Higgs}} = \frac{41}{10} \end{aligned} \tag{7.228}$$

So in summary we have for the SM,

$$b_1 = \frac{41}{10}, \quad b_2 = -\frac{19}{6}, \quad b_3 = -7 \tag{7.229}$$

Now let us look to see what are the results for M_X, $\sin^2\theta_W(M_Z)$ and α_5^{-1}. We will use the current values from the Particle Data Group. These are (without worrying about errors)[3]

$$\alpha^{-1}(M_Z) = 127.916, \quad \alpha_s(M_Z) = 0.118, \quad M_Z = 91.1896 \text{ GeV} \tag{7.230}$$

we get

$$M_X = 6.7\times 10^{14} \text{ GeV}, \quad \sin^2\theta_W(M_Z) = 0.208, \quad \alpha_5^{-1} = 41.48 \tag{7.231}$$

[3]Not only errors but also the difference between different renormalization schemes. Also this discussion is at one-loop level.

At the time that this GUT model was proposed by the first time, the constants were not known so precisely as today. Also the bound on the lifetime of the proton was much lower than today. So at that time the model was completely consistent. However after many years of dedicated experiments for find the decay of the proton, the lower limit was substantially improved and also after LEP the coupling constants are known with greater precision. So today the value for M_X is too low, the same being true for the value of $\sin^2\theta_W(M_Z)$ (the best value today is around $\sin^2\theta_W(M_Z) = 0.230$ [4]).

This can be seen very clearly if we use Eq. (7.217), with $\mu_i = M_Z$ and plot the α_i^{-1} as a function of $\ln(\mu^2/M_Z^2)$. This is shown in Fig. 7.8. We clearly see that the agreement is

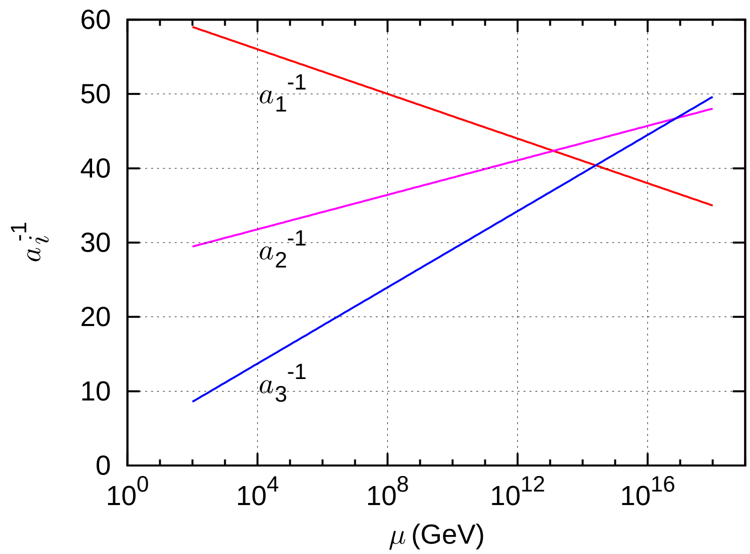


Figure 7.8: Evolution of α_i as function of the scale μ in the Standard Model for a $SU(5)$ minimal GUT theory.

quite poor with today's values.

Minimal Supersymmetric Standard Model

Let us now turn to the MSSM. Below the GUT scale the gauge group is the same as in the SM, but the particle content is larger, more than duplicated in relation to the SM. We summarize in the Table 7.1 the particle content and their quantum numbers under $G = SU_c(3) \otimes SU_L(2) \otimes U_Y(1)$.

With the values in Table 7.1 we can calculate the contribution of the various particles to the b_i coefficients. We will do it in succession for the three groups and for the different supermultiplets.

- $SU(3)$

  – **Gauge Supermultiplet**

[4] Again, without discussing the very small errors and the dependence on the renormalization scheme.

Supermultiplet	$SU_c(3) \otimes SU_L(2) \otimes U_Y(1)$ Quantum Numbers
$\widehat{V}_1 \equiv (\lambda', W_1^{\mu})$	$(1,1,1)$
$\widehat{V}_2 \equiv (\lambda^a, W_2^{\mu a})$	$(1,3,0)$
$V_3 \equiv (\widetilde{g}^b, W_3^{\mu b})$	$(8,1,0)$
$\widehat{L}_i \equiv (\widetilde{L}, L)_i$	$(1,2,-1)$
$\widehat{R}_i \equiv (\widetilde{\ell}_R, \ell_L^c)_i$	$(1,1,2)$
$\widehat{Q}_i \equiv (\widetilde{Q}, Q)_i$	$(3,2,\frac{1}{3})$
$\widehat{D}_i \equiv (\widetilde{d}_R, d_L^c)_i$	$(3,1,\frac{2}{3})$
$\widehat{U}_i \equiv (\widetilde{u}_R, u_L^c)_i$	$(3,1,-\frac{4}{3})$
$\widehat{H}_d \equiv (H_d, \widetilde{H}_d)$	$(1,2,-1)$
$\widehat{H}_u \equiv (H_u, \widetilde{H}_u)$	$(1,2,1)$

Table 7.1: Particle content of the MSSM. Note that $Q = T_3 + Y/2$.

We first do it in general for any gauge group and then apply it to the cases of interest. The gauge multiplet has a gauge boson contributing with

$$b^{\text{gauge boson}} = -\frac{11}{3}\, C_2(V) \tag{7.232}$$

and the left-handed gauginos in the adjoint representation of the gauge group. These therefore contribute

$$b^{\text{gauginos}} = \frac{2}{3}\, C_2(V) \tag{7.233}$$

and therefore

$$b^{\text{gauge SM}} = -3\; C_2(V) \tag{7.234}$$

where SM stands here for *super-multiplet*. Applying now to $SU(3)$ we get

$$b_3^{\text{gauge SM}} = -9 \tag{7.235}$$

– Left-handed Lepton Supermultiplet

$$b_3^{\text{Leptons}_\text{L}\text{ SM}} = 0 \tag{7.236}$$

– Right-handed Lepton Supermultiplet

$$b_3^{\text{Leptons}_\text{R}\text{ SM}} = 0 \tag{7.237}$$

– Left-handed Quark Supermultiplet

$$b_3^{\mathrm{Quarks_L\,SM}} = \underbrace{\frac{2}{3}\times\frac{1}{2}\times 2}_{\mathrm{Quarks_L}} + \underbrace{\frac{1}{3}\times\frac{1}{2}\times 2}_{\mathrm{Squarks_L}} = 1 \tag{7.238}$$

– Right-handed Up-Quark Supermultiplet

$$b_3^{\mathrm{Up-Quark_R\,SM}} = \underbrace{\frac{2}{3}\times\frac{1}{2}}_{\mathrm{Up-Quarks_R}} + \underbrace{\frac{1}{3}\times\frac{1}{2}}_{\mathrm{Up-Squarks_R}} = \frac{1}{2} \tag{7.239}$$

– Right-handed Down-Quark Supermultiplet

$$b_3^{\mathrm{Down-Quark_R\,SM}} = \underbrace{\frac{2}{3}\times\frac{1}{2}}_{\mathrm{Down-Quarks_R}} + \underbrace{\frac{1}{3}\times\frac{1}{2}}_{\mathrm{Down-Squarks_R}} = \frac{1}{2} \tag{7.240}$$

– Up type Higgs Supermultiplet

$$b_3^{\mathrm{Up-Higgs\,SM}} = 0 \tag{7.241}$$

– Down type Higgs Supermultiplet

$$b_3^{\mathrm{Down-Higgs\,SM}} = 0 \tag{7.242}$$

• $SU(2)$

Gauge Supermultiplet

We get

$$b_2^{\mathrm{gauge\ SM}} = -6 \tag{7.243}$$

– Left-handed Lepton Supermultiplet

$$b_2^{\mathrm{Leptons_L\,SM}} = \underbrace{\frac{2}{3}\times\frac{1}{2}}_{\mathrm{Leptons_L}} + \underbrace{\frac{1}{3}\times\frac{1}{2}}_{\mathrm{Sleptons_L}} = \frac{1}{2} \tag{7.244}$$

– Right-handed Lepton Supermultiplet

$$b_2^{\mathrm{Leptons_R\,SM}} = 0 \tag{7.245}$$

– Left-handed Quark Supermultiplet

$$b_2^{\mathrm{Quarks_L\,SM}} = N_c \underbrace{\frac{2}{3} \times \frac{1}{2}}_{\mathrm{Quarks_L}} + N_c \underbrace{\frac{1}{3} \times \frac{1}{2}}_{\mathrm{Squarks_L}} = N_c \frac{1}{2} = \frac{3}{2} \tag{7.246}$$

– Right-handed Up-Quark Supermultiplet

$$b_2^{\mathrm{Up-Quark_R\,SM}} = 0 \tag{7.247}$$

– Right-handed Down-Quark Supermultiplet

$$b_2^{\mathrm{Down-Quark_R\,SM}} = 0 \tag{7.248}$$

– Up type Higgs Supermultiplet

$$b_2^{\mathrm{Up-Higgs\,SM}} = \underbrace{\frac{1}{3} \times \frac{1}{2}}_{\mathrm{Higgs_u}} + \underbrace{\frac{2}{3} \times \frac{1}{2}}_{\mathrm{Higgsino_u}} = \frac{1}{2} \tag{7.249}$$

– Down type Higgs Supermultiplet

$$b_2^{\mathrm{Down-Higgs\,SM}} = \underbrace{\frac{1}{3} \times \frac{1}{2}}_{\mathrm{Higgs_d}} + \underbrace{\frac{2}{3} \times \frac{1}{2}}_{\mathrm{Higgsino_d}} = \frac{1}{2} \tag{7.250}$$

- $U(1)$

  Gauge Supermultiplet

  We get

$$b_1^{\mathrm{gauge\ SM}} = 0 \tag{7.251}$$

– Left-handed Lepton Supermultiplet

$$b_1^{\mathrm{Leptons_L\,SM}} = \frac{3}{5} \times \left[\underbrace{\frac{2}{3} \times \left(-\frac{1}{2}\right)^2 \times 2}_{\mathrm{Leptons_L}} + \underbrace{\frac{1}{3} \times \left(-\frac{1}{2}\right)^2 \times 2}_{\mathrm{Sleptons_L}} \right] = \frac{3}{10} \tag{7.252}$$

– Right-handed Lepton Supermultiplet

$$b_1^{\mathrm{Leptons_R\,SM}} = \frac{3}{5} \times \left[\underbrace{\frac{2}{3} \times (-1)^2}_{\mathrm{Leptons_R}} + \underbrace{\frac{1}{3} \times (-1)^2}_{\mathrm{Sleptons_R}} \right] = \frac{3}{5} \tag{7.253}$$

– Left-handed Quark Supermultiplet

$$b_1^{\mathrm{Quarks_L\,SM}} = \frac{3}{5} \times N_c \times \left[\underbrace{\frac{2}{3} \times \left(\frac{1}{6}\right)^2 \times 2}_{\mathrm{Quarks_L}} + \underbrace{\frac{1}{3} \times \left(\frac{1}{6}\right)^2 \times 2}_{\mathrm{Squarks_L}} \right] = N_c \times \frac{3}{5} \times \frac{1}{18} = \frac{1}{10} \tag{7.254}$$

– Right-handed Up-Quark Supermultiplet

$$b_1^{\mathrm{Up-Quarks_R\,SM}} = \frac{3}{5} \times N_c \times \left[\underbrace{\frac{2}{3} \times \left(\frac{2}{3}\right)^2}_{\mathrm{Up-Quarks_R}} + \underbrace{\frac{1}{3} \times \left(\frac{2}{3}\right)^2}_{\mathrm{Up-Squarks_R}} \right] = N_c \times \frac{3}{5} \times \frac{4}{9} = \frac{4}{5} \tag{7.255}$$

– Right-handed Down-Quark Supermultiplet

$$b_1^{\mathrm{Down-Quarks_R\,SM}} = \frac{3}{5} \times N_c \times \left[\underbrace{\frac{2}{3} \times \left(-\frac{1}{3}\right)^2}_{\mathrm{Down-Quarks_R}} + \underbrace{\frac{1}{3} \times \left(-\frac{1}{3}\right)^2}_{\mathrm{Down-Squarks_R}} \right] = N_c \times \frac{3}{5} \times \frac{1}{9} = \frac{1}{5} \tag{7.256}$$

– Up type Higgs Supermultiplet

$$b_1^{\mathrm{Higgs_u\,SM}} = \frac{3}{5} \times \left[\underbrace{\frac{1}{3} \times \left(\frac{1}{2}\right)^2 \times 2}_{\mathrm{Higgs_u}} + \underbrace{\frac{2}{3} \times \left(\frac{1}{2}\right)^2 \times 2}_{\mathrm{Higgs_u}} \right] = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10} \tag{7.257}$$

– Down type Higgs Supermultiplet

$$b_1^{\text{Higgs}_\text{d}\,\text{SM}} = \frac{3}{5} \times \left[\underbrace{\frac{1}{3} \times \left(-\frac{1}{2}\right)^2 \times 2}_{\text{Higgs}_\text{d}} + \underbrace{\frac{2}{3} \times \left(-\frac{1}{2}\right)^2 \times 2}_{\text{Higgs}_\text{d}} \right] = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10} \tag{7.258}$$

Now we put everything together to obtain fro the MSSM,

$$\begin{aligned} b_1 =& N_g \times \left(\frac{3}{10} + \frac{3}{5} + \frac{1}{10} + \frac{4}{5} + \frac{1}{5} \right) + \frac{3}{10} + \frac{3}{10} = 3 \times 2 + \frac{3}{5} = \frac{33}{5} \\ b_2 =& -6 + N_g \times \left(\frac{1}{2} + \frac{3}{2} \right) + \frac{1}{2} + \frac{1}{2} = 1 \\ b_3 =& -9 + N_g \times \left(1 + \frac{1}{2} + \frac{1}{2} \right) = -3 \end{aligned} \tag{7.259}$$

Now let us look to see what are the results for M_X, $\sin^2\theta_W(M_Z)$ and α_5^{-1} in the MSSM. Using the same inputs as for the SM, Eq. (7.230) [5] , we get

$$M_X = 2.1 \times 10^{16}\ \text{GeV}, \quad \sin^2\theta_W(M_Z) = 0.231, \quad \alpha_5^{-1} = 24.27 \tag{7.260}$$

we immediately see that these values are quite good. This can be seen very clearly if we use Eq. (7.217), with $\mu_i = M_Z$ and plot the α_i^{-1} as a function of $\ln(\mu^2/M_Z^2)$. This is shown in Fig. 7.9 and the agreement is excellent.

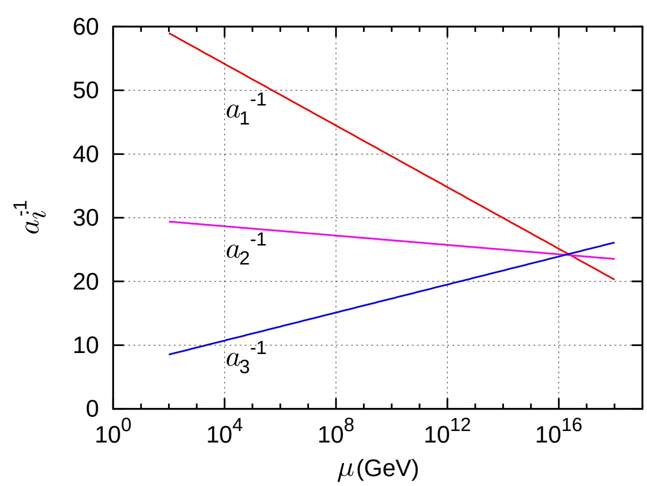


Figure 7.9: Evolution of α_i as function of the scale μ in the MSSM for a $SU(5)$ minimal GUT theory.

[5] Again we do not take into account errors and the difference between different renormalization schemes. Also this discussion is at one-loop level and the effects of the supersymmetric particles not decoupling at the same scale (thresholds) are not taken in account.

We can still go a step further. We know that supersymmetry must be broken above the electroweak scale, so what we have done in Fig. 7.9 is not quite correct because we are running with the MSSM content down to the weak scale. Of course each particle will decouple at its mass, but assuming that their masses are not much different we can assume that there will a scale M_{SUSY}, below which we will have the SM RGEs. We can redo the calculation taking now the evolved SM values at M_{SUSY} as the boundary conditions for the MSSM evolution. In Fig. and Fig. the results are shown for various values of the SUSY scale. We see from these results that if the SUSY scale is much higher than, say

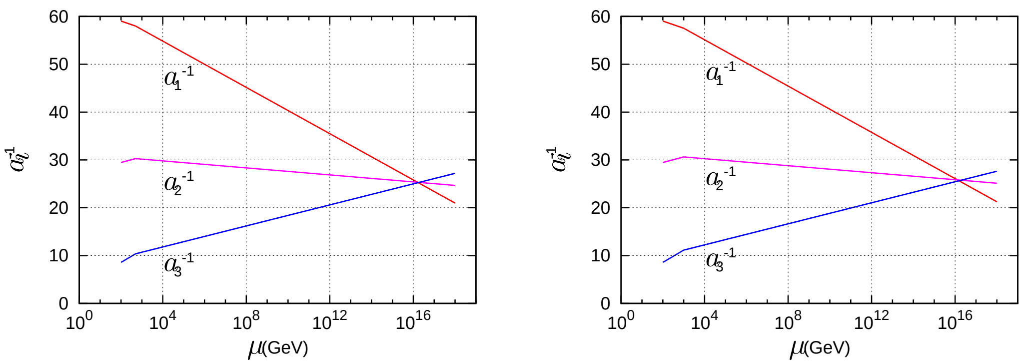


Figure 7.10: Evolution of α_i as function of the scale in the MSSM for a $SU(5)$ minimal GUT theory. On the left panel $M_{\text{SUSY}} = 500$ GeV and on the right panel $M_{\text{SUSY}} = 1000$ GeV.

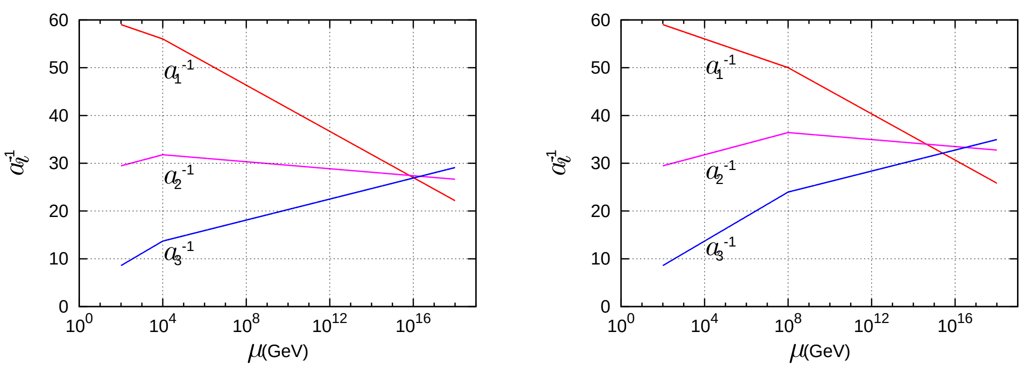
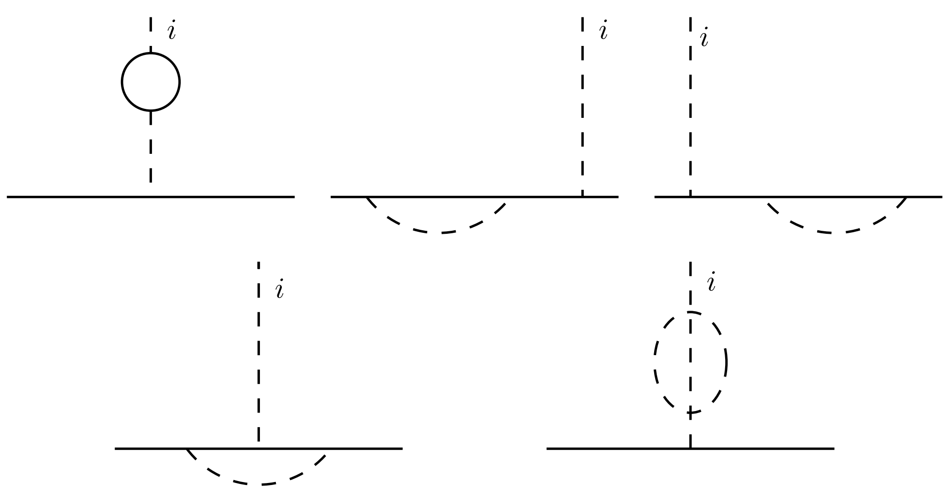


Figure 7.11: Evolution of α_i as function of the scale in the MSSM for a $SU(5)$ minimal GUT theory. On the left panel $M_{\text{SUSY}} = 10^4$ GeV and on the right panel $M_{\text{SUSY}} = 10^8$ GeV.

1 TeV, the good agreement starts do disappear. Before we end we should emphasize that these are one loop results, without many fine details, like thresholds (talking in account that not all the supersymmetric particles decouple at the same scale) and the important two-loop effects.

Problems for Chapter 7

7.1 Verify Eq. (7.163). For this notice that $\beta^i = \frac{dg^i}{dt}$ where β^i is determined from the following diagrams



7.2 Evaluate in minimal subtraction (MS) the renormalization constant Z_3 for QED, Eq. (7.168).

7.3 Evaluate in MS the renormalization constant Z_3 in scalar electrodynamics, Eq. (7.170).

7.4 Consider a non-abelian gauge theory, with symmetry group G and without matter fields. Evaluate the renormalization constants for the gauge field Z_A, and for the triple vertex, Z_1.

7.5 Consider a non-abelian gauge theory in interaction with scalar and fermion fields. Evaluate the contribution of these to Z_A and Z_1. Use these results together with those of Problem 7.4, to determine the β function of the renormalization group for this theory.

7.6 Consider the Standard Model of the electroweak interactions. Considering all the fields in the theory, determine the coefficients b_1, b_2 e b_3 defined in Eq. (7.224).

7.7 Consider now the Minimal Supersymmetric Standard Model (MSSM).Considering all the fields in the theory, determine the coefficients b_1, b_2 e b_3 defined in Eq. (7.224) for this theory.

Appendix A

Path Integral in Quantum Mechanics

A.1 Introduction

The ususal formulation of Quantum Mechanics is given by the Schrödinger equations,

$$i\hbar \frac{\partial}{\partial t} |a(t)\rangle = H |a(t)\rangle \tag{A.1}$$

onde

$$H = \frac{P^2}{2m} + V(Q) \tag{A.2}$$

and

$$[Q, P] = i\hbar \tag{A.3}$$

This formulation it is equivalent to another made using path integrals, due to an idea of Dirac and developed by Feynman [11, 12]. To see this we observe that in quantum mechanics. we know how to answer any question about a system, if we know how to calculate the transition amplitudes,

$$\left\langle b(t') | a(t) \right\rangle = \langle b| \, e^{-iH(t'-t)} \, |a\rangle \tag{A.4}$$

São estas amplitudes de transição que são definidas em termos de integrais de caminho. Conforme a representação escolhida para os estados $|a\rangle$ e $|b\rangle$ as expressões para o integral de caminho vêm diferentes. Assim vamos analisar separadamente os casos das representações no espaço das configurações (coordenadas), no espaço de fase e por meio de estados coerentes (espaço de Bargmann-Fock).

A.2 Configuration space

Introduzimos os estados $|q\rangle$ e $|p\rangle$ tais que

$$
\begin{aligned}
&Q\,|q\rangle = q\,|q\rangle &;\qquad &P\,|p\rangle = p\,|p\rangle \\
&\langle q'|q\rangle = \delta(q'-q) &;\qquad &\langle p'|p\rangle = \delta(p'-p) \\
&\langle q|p\rangle = \langle p|q\rangle^* = \tfrac{1}{\sqrt{2\pi}}e^{ipq} & & \qquad\qquad (A.5)
\end{aligned}
$$

Então

$$
\langle q_f, t_f | q_i, t_i\rangle = \int \mathcal{D}(q) e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dtL(q,\dot{q})} \tag{A.6}
$$

onde $\mathcal{D}$ é uma medida de integração definida pelo limite

$$
\mathcal{D}(q) = \lim_{n\to\infty} \prod_1^{n-1} dq_p \left[\frac{nme^{-i\pi/2}}{2\pi(t_f - t_i)}\right]^{\frac{n}{2}} \tag{A.7}
$$

sendo n o número de intervalos em que se fez a partição do intervalo (t_i, t_f). O limite $n \to \infty$ é bastante complicado e só existe prova matemática para certas classes de potenciais. A Eq. (A.6) permite uma interpretação da mecânica clássica como limite da mecânica quântica. De facto quando $\hbar \to 0$ a maior contribuição para a amplitude vem das trajectórias que minimizam a ac cão, isto é, as trajectórias clássicas. A mecânica quântica é então vista como o estudo das flutuações à volta da trajectória clássica.

A.2.1 Matrix elements of operators

Usando a propriedade dos integrais de caminho

$$
\int \mathcal{D}(q) e^{iS(f,i)} = \int dq(t) \int \mathcal{D}(q) e^{iS(f,t)} \int \mathcal{D}(q) e^{iS(t,i)} \tag{A.8}
$$

onde $t_i < t < t_f$, é fácil mostrar que

$$
\begin{aligned}
\langle q_f, t_f |\, \mathcal{O}(t)\, | q_i, t_i\rangle \;&=\; \int dq' dq'' \int \mathcal{D}(q) e^{iS(q_f,t_f;q'',t)} \\
&\qquad \left\langle q''\right| \mathcal{O} \left|q'\right\rangle \int \mathcal{D}(q) e^{iS(q',t;q_i,t_i)}
\end{aligned} \tag{A.9}
$$

Então se $\mathcal{O}$ for diagonal no espaço das coordenadas, isto é, se

$$
\left\langle q''\right| \mathcal{O} \left|q'\right\rangle = \mathcal{O}(q')\delta(q' - q'') \tag{A.10}
$$

obtemos

$$
\langle q_f, t_f |\, \mathcal{O}\, | q_i, t_i\rangle = \int \mathcal{D}(q)\; e^{iS(f,i)} \mathcal{O}(q(t)) \tag{A.11}
$$

A.2.2 Time ordered product of operators

Seja $\mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n)$ com $t_1 \geq t_2 \geq \cdots \geq t_n$. Então é fácil de mostrar que a ordenação no tempo é automática no integral de caminho, isto é,

$$\langle q_f, t_f | \, \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n) \, | q_i, t_i \rangle = \int \mathcal{D}(q) e^{iS(f,i)} \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n \tag{A.12}$$

Este resultado é particularmente importante, pois permitirá escrever as funções de Green de produtos de operadores ordenados no tempo como simples integrais de caminho de produtos dos equivalentes clássicos desses operadores.

A.2.3 Exact results I: harmonic oscillator

Para alguns potenciais é possível calcular exactamente o limite introduzido em (A.5). Para esses casos o integral de caminho é portanto perfeitamente bem definido. Esses potenciais não são muitos, mas são particularmente importantes. Para o seguimento interessa-nos discutir dois deles. O primeiro é o oscilador harmónico definido pelo potencial

$$V(Q) = m\frac{\omega^2}{2}Q^2 \tag{A.13}$$

Para este caso obtém-se, (os integrais são gaussianos e por isso podem ser explicitamente calculados)

$$\langle f|i\rangle = \left(\frac{m\omega e^{-i\pi/2}}{2\pi \sin\omega t}\right)^{\frac{1}{2}} \exp\left\{ i\frac{m\omega}{2}\left[(q_f^2+q_i^2)\cot\omega t - \frac{2q_f q_i}{\sin\omega t}\right]\right\} \tag{A.14}$$

Este resultado vai ser útil adiante.

A.2.4 Exact results II: external force

Consideremos agora uma força exterior tal que o potencial é dado por

$$V(Q) = -QF(t) \tag{A.15}$$

Neste caso obtemos

$$\langle f|i\rangle_F = \left[\frac{me^{-i\pi/2}}{2\pi(t_f - t_i)}\right]^{\frac{1}{2}} e^{iS(f,i)} \tag{A.16}$$

onde $S(f,i)$ é a acção calculada ao longo da trajectória clássica,

$$S(f,i) \;=\; \frac{m}{2}\frac{(q_f-q_i)^2}{t_f-t_i} + \int_{t_i}^{t_f} dt F(t)\left(q_f\frac{t-t_i}{t_f-t_i} + q_i\frac{t_f-t}{t_f-t_i}\right)$$

$$+\frac{1}{2m}\int_{t_i}^{t_f}\int_{t_i}^{t_f} dt'dt''F(t')G(t',t'')F(t'') \tag{A.17}$$

onde $G(t',t'') = \frac{t't''}{T} - \text{Inf}(t',t'')$ é a função de Green simétrica para o problema $\ddot{q} = F(t)/m$ com as condições na fronteira $G(0,t'') = G(t',0) = 0$.

A.2.5 Perturbation theory

A importância do resultado exacto para a força exterior deve-se ao facto que usando esse resultado podemos formalmente resolver o problema dum potencial qualquer. Para isso notemos que a derivação funcional em realação à fonte $F(t)$ faz baixar $Q(t)$. Mais explicitamente

$$\langle f|\, Q(t)\, |i\rangle_F = \frac{\delta}{i\delta F(t)}\ \langle f|i\rangle_F \tag{A.18}$$

onde $\langle | \rangle_F$ significa calculado na presença da fonte exterior (i.e. para o hamiltoniano $H = P^2/2m - QF(t)$). Então para um potencial arbitrário $V(q)$ temos

$$\begin{aligned}\langle f|i\rangle &= \int \mathcal{D}(q) e^{i\int_{t_i}^{t_f} dt\left[\frac{1}{2}m\dot{q}^2 - V(q)\right]} \\ &= \exp\left\{-i\int_{t_i}^{t_f} dt V\left[\frac{\delta}{i\delta F(t)}\right]\right\}\langle f|i\rangle_F\Big|_{F=0}\end{aligned} \tag{A.19}$$

Esta expressão formal torna-se muito útil quando a exponencial é expandida em série. Então todos os integrais são do tipo gaussiano e podem ser exactamente executados. Obtemos assim a teoria das perturbações. Claro que só terá significado se houver um parâmetro *pequeno* no potencial. É importante notar que enquanto se faça teoria das perturbações não há qualquer problema com a indefinição matemática do integral de caminho, pois todas as integrações são gaussianas.

A.3 Phase space formulation

Para este caso obtemos

$$\langle q_f, t_f | q_i, t_i\rangle = \int \mathcal{D}(p,q) e^{i\int_{t_i}^{t_f} dt\left[p\dot{q} - h(p,q)\right]} \tag{A.20}$$

onde $h(p,q)$ é o hamiltoniano clássico e a medida é dada pelo limite

$$\mathcal{D}(p,q) = \lim_{n\to\infty}\frac{\prod_1^n dp_s \prod_1^{n-1} dq_r}{(2\pi)^n} \tag{A.21}$$

A fase da exponencial é novamente a acção clássica expressa nas variáveis canónicas p e q. Se $h(p,q)$ depender quadraticamente de p como é usual, pode-se fazer a integração gaussiana em p e a expressão reduz-se à do caso anterior, equação (A.4).

A.4 Bargmann-Fock space (coherent states)

Nesta representação usamos funções analíticas de variável complexa para descrevermos os operadores a e $a^\dagger$ ($[a,a^\dagger]=1$). A correspondência é feita do modo seguinte. As funções analíticas geram um despaço de Hilbert com o produto interno definido por

$$\langle g|f\rangle \equiv \int \frac{dzd\overline{z}}{2\pi i}\, e^{-z\overline{z}}\, \overline{g}(z)f(\overline{z}) \tag{A.22}$$

Os operadores a e $a^\dagger$ são representados neste espaço por

$$\begin{aligned} a &\to \frac{\partial}{\partial z} \\ a^\dagger &\to \overline{z} \end{aligned} \tag{A.23}$$

Dado um estado $|f\rangle$, representado pela função $f(\overline{z})$, a acção do operador A em $|f\rangle$ produz outro estado que também pode ser representado por funções analíticas. Se designarmos por $g(\overline{z})$ essa função temos

$$g(\overline{z}) \equiv \langle\overline{z}|\, A\, |f\rangle \equiv \int \frac{d\xi d\overline{\xi}}{2\pi i}\, e^{-\xi\overline{\xi}} A(\overline{z},\xi)f(\overline{\xi}) \tag{A.24}$$

onde $A(\overline{z},\xi)$ é o kernel do operador A. Uma representação explicita para o kernel é fácil de obter. Para isso introduzimos os estados $|n\rangle$, definidos por

$$|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}\,|0\rangle \tag{A.25}$$

É fácil de verificar (ver Problema A.2) que com a definição de produto interno acima introduzida estes estados são ortonormados, isto é $\langle f_m|f_n\rangle = \delta_{mn}$.
Então

$$\begin{aligned} \langle\overline{z}|\, A\, |f\rangle &= \sum_{n,m} \frac{\overline{z}^n}{\sqrt{n!}}\, \langle n|\, A\, |m\rangle\, \langle m|f\rangle \\ &= \sum_{n,m} \frac{\overline{z}^n}{\sqrt{n!}}\, A_{n,m} \int \frac{d\xi d\overline{\xi}}{2\pi i} e^{-\xi\overline{\xi}} \frac{\xi^m}{\sqrt{m!}} f(\overline{\xi}) \\ &= \int \frac{d\xi d\overline{\xi}}{2\pi i} e^{-\xi\overline{\xi}} \left[\sum_{n,m} \frac{\overline{z}^n}{\sqrt{n!}} A_{n,m} \frac{\xi^m}{\sqrt{m!}}\right] f(\overline{\xi}) \end{aligned} \tag{A.26}$$

Portanto

$$A(\overline{z},\xi) \equiv \sum_{n,m} \frac{\overline{z}^n}{\sqrt{n!}} A_{n,m} \frac{\xi^m}{\sqrt{m!}} \tag{A.27}$$

O kernel de qualquer operador é assim obtido desde que se conheçam os seus elementos de matriz na base $|n\rangle$.

Já vimos como se representam estados e operadores. Vamos ver como representar produtos de operadores. Sejam dois operadores A_1 e A_2 e um estado $|f\rangle$. Seja ainda

$$g(\overline{\eta}) = \langle\overline{\eta}|\, A_2 \,|f\rangle = \int \frac{d\xi d\overline{\xi}}{2\pi i} e^{-\xi\overline{\xi}} A_2(\overline{\eta},\xi) f(\overline{\xi}) \tag{A.28}$$

Então

$$\begin{aligned}
\langle\overline{z}|\, A_1 \,|g\rangle &= \int \frac{d\xi d\overline{\xi}}{2\pi i}\ e^{-\xi\overline{\xi}} A_1(\overline{z},\eta) g(\overline{\eta}) \\
&= \int \frac{d\eta d\overline{\eta}}{2\pi i} \frac{d\xi d\overline{\xi}}{2\pi i}\ e^{-\xi\overline{\xi}} e^{-\eta\overline{\eta}}\ A_1(\overline{z},\eta) A_2(\overline{\eta},\xi) f(\overline{\xi}) \\
&= \int \frac{d\xi d\overline{\xi}}{2\pi i}\ e^{-\xi\overline{\xi}}\ \left[\int \frac{d\eta d\overline{\eta}}{2\pi i}\ e^{-\eta\overline{\eta}}\ A_1(\overline{z},\eta) A_2(\overline{\eta},\xi)\right] f(\overline{\xi}) \\
&= \int \frac{d\xi d\overline{\xi}}{2\pi i}\ e^{-\xi\overline{\xi}}\ A_3(\overline{z},\xi) f(\overline{\xi})
\end{aligned} \tag{A.29}$$

Portanto o kernel do operador $A_3 = A_1 A_2$ é obtido por convolução dos kernéis de A_1 e A_2, isto é

$$A_3(\overline{z},\eta) = \int \frac{d\eta d\overline{\eta}}{2\pi i}\ e^{-\eta\overline{\eta}}\ A_1(\overline{z},\eta) A_2(\overline{\eta},\xi) \tag{A.30}$$

A.4.1 Normal form for an operator

Como já sabemos representar estados, operadores e produtos de operadores já temos toos os ingredientes para fazer mecânica quântica neste espaço. Há contudo um outro assunto que é importante tendo em atenção que pretendemos aplicar este formalismo em teoria quântica dos campos. Trata-se da *forma normal* dum operador[1]. O operador A na sua forma normal é definido por

$$A = \sum_{n,m} A^N_{n,m}\ \frac{a^{\dagger n} a^m}{\sqrt{n!}\sqrt{m!}} \tag{A.31}$$

isto é, os operadores de destruição estão à direita dos operadores de criação. O *kernel normal* é definido por

[1] Notar que em teoria quântica dos campos tem que se proceder ao ordenamento normal do hamiltoniano para definir o zero da energia.

$$A^N(\overline{z}, z) \equiv \sum_{n,m} \frac{\overline{z}^n}{\sqrt{n!}} \, A^N_{n,m} \, \frac{z^m}{\sqrt{m!}} \tag{A.32}$$

isto é, é obtido por subtituição directa dos operadores de destruição por z e dos de criação por $\overline{z}$. Para um operador dado na sua forma normal este é o kernel imediato de obter. É contudo diferente do kernel atrás definido. Para ver a relação entre eles notemos a seguinte relação

$$\begin{aligned} f(\overline{z}) &= \sum_n \frac{\overline{z}}{\sqrt{n!}} \, \langle n|f\rangle \\ &= \sum_n \int \frac{d\xi d\overline{\xi}}{2\pi i} \, e^{-\xi\overline{\xi}} \, \frac{\overline{z}^n \xi^n}{n!} f(\overline{\xi}) \\ &= \int \frac{d\xi d\overline{\xi}}{2\pi i} \, e^{-\xi\overline{\xi}} \, e^{\overline{z}\xi} \, f(\overline{\xi}) \end{aligned} \tag{A.33}$$

O kernel $e^{\overline{z}\xi}$ é portanto uma função delta neste espaço. Usando este resultado obtemos

$$\begin{aligned} \langle\overline{z}|\, A \,|f\rangle &= \sum_{n,m} \frac{A^N_{n,m}}{\sqrt{n!}\sqrt{m!}} \, \overline{z}^n \, \frac{d^m}{d\overline{z}^m} \, f(\overline{z}) \\ &= \sum_{n,m} \frac{A^N_{n,m}}{\sqrt{n!}\sqrt{m!}} \int \frac{d\xi d\overline{\xi}}{2\pi i} e^{-\xi\overline{\xi}} \, e^{\overline{z}\xi} \, \overline{z}^n \xi^m f(\overline{\xi}) \\ &= \int \frac{d\xi d\overline{\xi}}{2\pi i} \, e^{-\xi\overline{\xi}} \, e^{\overline{z}\xi} \, A^N(\overline{z}, \xi) f(\overline{\xi}) \end{aligned} \tag{A.34}$$

donde resulta

$$A(\overline{z}, \xi) = e^{\overline{z}\xi} \, A^N(\overline{z}, \xi) \tag{A.35}$$

Esta relação é muito importante pois permite imediatamente escrever o kernel dum operador qualquer uma vez que seja conhecida a sua forma normal. Isto é particularmente útil em teoria quântica dos campos onde o hamiltoniano é dado na sua forma normal.

A.4.2 Evolution operator

Podemos obter agora a expressão para o operador de evolução nesta representação. De acordo com aquilo que acabámos de dizer, para um intervalo infinitesimal, devemos ter para o kernel de U

$$U(\overline{z}, \xi, \Delta t) = e^{\overline{z}\xi} \, e^{-i\Delta t \; h(\overline{z},\xi)} \tag{A.36}$$

onde $h(\overline{z},\xi)$ é o kernel normal obtido por substituição directa dos operadores $a^\dagger$ e a pelas variáveis complexas $\overline{z}$ e ξ. Notar que quando $\Delta t \to 0$ o kernel do operador de evolução se reduz ao kernel da identidade, $e^{\overline{z}\xi}$, que como vimos é a função δ neste espaço.

Para um intervalode tempo finito $t = t_f - t_i$, dividimos o intervalo em n intervalos

$$z_0 \qquad z_1 \qquad z_2 \qquad \cdots \qquad z_{n-1} \qquad z_n \qquad\qquad \Delta t = \frac{t}{n} \tag{A.37}$$

Então

$$\begin{aligned} U(\overline{z}_1, z_0) &\simeq e^{\overline{z}_1 z_0 - i\Delta t h(\overline{z}_1, z_0)} \\ U(\overline{z}_2, z_1) &\simeq e^{\overline{z}_2 z_1 - i\Delta t h(\overline{z}_2, z_1)} \\ \vdots \quad & \vdots \quad \vdots \\ U(\overline{z}_n, z_{n-1}) &\simeq e^{\overline{z}_n z_{n-1} - i\Delta t h(\overline{z}_n, z_{n-1})} \end{aligned} \tag{A.38}$$

Aplicando agora a regra de multiplicação dos kernéis obtemos

$$\begin{aligned} U(\overline{z}_f, t_f; z_i, t_i) = \lim_{n\to\infty} \int \prod_{k=1}^{n-1} \frac{dz_k d\overline{z}_k}{2\pi i} \ \exp \Big[\sum_{k=1}^{n} \overline{z}_k \ z_{k-1} \\ - \sum_{k=1}^{n-1} \overline{z}_k z_k - i \sum_{k=1}^{n} h(\overline{z}_k, z_{k-1}) \Delta t \Big] \end{aligned} \tag{A.39}$$

ou seja

$$U(\overline{z}_f, t_f; z_i, t_i) \equiv \int \mathcal{D}(z,\overline{z}) \ e^{\frac{1}{2} \ (\overline{z}_f z_f + \overline{z}_i z_i) + i \int_{t_i}^{t_f} \left[\frac{1}{2i} (\dot{\overline{z}} z - \overline{z}\dot{z}) - h(\overline{z}, z) \right] dt} \tag{A.40}$$

Nesta expressão $\overline{z}_f(t_f)$ e $z_i(t_i)$ são fixados pelas condições fronteiras mas $\overline{z}_f(t_i)$ e $z_i(t_f)$ são arbitrários. A fase da exponencial é novamente a acção, agora escrita nas variáveis complexas z e $\overline{z}$. Para ver isso basta lembrar que

$$\frac{1}{2}(pdq + qdp) = \frac{1}{2i}(zd\overline{z} - \overline{z}dz) \tag{A.41}$$

A.4.3 Exact results I: harmonic oscillator

Também aqui vamos analisar os casos importantes em que há resultados exactos, nomeadamente o oscilador hamónico e o caso das fontes externas. Comecemos pelo oscilador harmónico. O hamiltoniano é dado por

$$H_0 = \omega \ a^\dagger a \tag{A.42}$$

Trata-se portanto dum caso em que o hamiltoniano é dado na forma normal. Este problema pode ser resolvido exactamente. Temos

$$
\begin{aligned}
U(\overline{z}_f, z_i, t) &= \lim_{n\to\infty} \int \prod_{k=1}^{n-1} \frac{dz_k d\overline{z}_k}{2\pi i} \ \exp\Big[\sum_{k=1}^{n} \overline{z}_k \ z_{k-1} \\
&\qquad\qquad - \sum_{k=1}^{n-1} \overline{z}_k z_k - i\omega \frac{t}{n} \ \sum_{k=1}^{n} \overline{z}_k z_{k-1} \Big] \\
&= \lim_{n\to\infty} \int \prod_{k=1}^{n-1} \frac{dz_k d\overline{z}_k}{2\pi i} \ e^{\left[-\overline{X}AX + \overline{X}B + \overline{B}X\right]}
\end{aligned}
\tag{A.43}
$$

onde

$$
X = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \end{bmatrix} \qquad ; \qquad \overline{X} = (\overline{z}_1, \overline{z}_2, \cdots, \overline{z}_{n-1}) \tag{A.44}
$$

e

$$
B = \begin{bmatrix} z_0 a \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad ; \qquad \overline{B} = (0, 0, \cdots, 0, z_n a) \tag{A.45}
$$

com $z_0 = z_i$ e $z_n = z_f$. A matriz A de dimensão $(n-1)\times(n-1)$ é dada por

$$
\begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots \\
-a & 1 & 0 & \cdots & \cdots & \cdots \\
0 & -a & 1 & 0 & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\cdots & \cdots & 0 & -a & 1 & 0 \\
\cdots & \cdots & \cdots & 0 & -a & 1
\end{bmatrix}
\tag{A.46}
$$

onde se definiu

$$
a \equiv 1 - i\hbar\omega \frac{t}{n} \tag{A.47}
$$

As $(n-1)$ integrações gaussianas podem ser facilmente feitas usando o resultado (ver Problema A.3),

$$
\int \prod \frac{dz_k d\overline{z}_k}{2\pi i} \ e^{-\overline{z}Az + \overline{u}z + \overline{z}u} = (\det A)^{-1} \ e^{\overline{u}A^{-1}u} \tag{A.48}
$$

obtemos então

$$
\begin{aligned}
U_0(\overline{z}_f, z_i; t) &= \lim_{n\to\infty} \left[(\det A)^{-1} \; e^{\overline{B} A^{-1} B} \right] \\
&= \lim_{n\to\infty} \left[(\det A)^{-1} \; e^{\overline{z}_f z_i a^2 (A^{-1})_{n-1,1}} \right] \qquad \text{(A.49)}
\end{aligned}
$$

É fácil de verificar que para a matriz A se tem

$$
(A^{-1})_{k,m} = \begin{cases} a^{k-m} & \text{se } k \geq m \\ \\ 0 & \text{se } k < m \end{cases} \qquad \text{(A.50)}
$$

e portanto

$$
\det A = 1 \qquad \text{(A.51)}
$$

e

$$
A^{-1}_{n-1,1} = (-1)^n (-a)^{n-2} \qquad \text{(A.52)}
$$

donde se conclui que

$$
\lim_{n\to\infty} a^2 (A_{-1})_{n-1,1} = \lim_{n\to\infty} \left(1 - \frac{i\omega t}{n} \right)^n = e^{-i\omega t} \qquad \text{(A.53)}
$$

Obtemos então finalmente

$$
U_0(\overline{z}_f, z_i; t) = \exp\left\{ \overline{z}_f z_i e^{-i\omega t} \right\} \qquad \text{(A.54)}
$$

Podemos verificar que este resultado é da forma e^{iS} onde S é a acção calculada ao longo da trajectória clássica. De facto a estacionaridade do expoente da exponencial dá

$$
\begin{aligned}
&\delta \left\{ \frac{1}{2} (\overline{z}_f z(t_f) + \overline{z}(t_i) z_i) \; + \int_{t_i}^{t_f} \left[\frac{\dot{\overline{z}} z - \overline{z} \dot{z}}{2} - i\omega \overline{z} z \right] dt \right\} \\
&\quad = \frac{1}{2} \overline{z}_f \delta z(t_f) + \frac{1}{2} z_i \delta \overline{z}(t_i) - \frac{1}{2} \overline{z}_f \delta z(t_f) - \frac{1}{2} z_i \delta \overline{z}(t_i) \\
&\qquad + \int_{t_i}^{t_f} \left[\delta z (\dot{\overline{z}} - i\omega \overline{z}) - \delta \overline{z} (\dot{z} + i\omega z) \right] dt \qquad \text{(A.55)}
\end{aligned}
$$

pois $\delta \overline{z}_f = \delta z_i = 0$. As equações de movimento são portanto

$$
\begin{cases} \dot{\overline{z}} - i\omega \overline{z} = 0 \\ \\ \dot{z} + i\omega z = 0 \end{cases} \qquad \text{com} \qquad \begin{cases} \overline{z}(t_f) = z_f \\ \\ z(t_i) = z_i \end{cases} \qquad \text{(A.56)}
$$

que têm como solução

$$\begin{cases} z(t) = z_i \; e^{i\omega(t_i - t)} \\ \\ \overline{z}(t) = \overline{z}_f \; e^{i\omega t - t_f)} \end{cases} \tag{A.57}$$

Substituindo estas soluções no expoente obtemos

$$\begin{aligned} \frac{1}{2}\left[\overline{z}_f z(t_f) + z_i \overline{z}(t_i)\right] \quad &+ \quad \int_{t_i}^{t_f} \left[\frac{1}{2}(\dot{\overline{z}} z - \overline{z}\dot{z}) - i\omega \overline{z} z\right] dt \\ &= \quad \overline{z}_f z_i \; e^{i\omega(t_i - t_f)} \\ &= \quad \overline{z}_f z_i \; e^{-i\omega t} \end{aligned} \tag{A.58}$$

para $t \equiv t_f - t_i$, como queríamos mostrar.

Um outro resultado importante do oscilador harmónico é que a evolução dum estado sob a acção de $H_0 = \omega a^\dagger a$ é particularmente simples neste espaço das funções de variável complexa. Seja $f(\overline{z})$ a representação do estado $|f\rangle$. A evolução debaixo de H_0 é dada por

$$\begin{aligned} U_0(t) f(\overline{z}) \quad &= \quad \int \frac{d\xi d\overline{\xi}}{2\pi i} \; e^{-\xi\overline{\xi}} \; e^{\overline{z}\xi e^{-i\omega t}} \; f(\overline{\xi}) \\ &= \quad f(\overline{z} e^{-i\omega t}) \end{aligned} \tag{A.59}$$

isto é, é reduzida à multiplicação por $e^{-i\omega t}$

$$\overline{z} \to \overline{z} \; e^{-i\omega t} \tag{A.60}$$

Isto é importante para descrever a matriz S, em que os estados assimptóticos evoluem de acordo com o hamiltoniano livre.

A.4.4 Exact results II: external force

Seja o hamiltoniano

$$H = \omega a^\dagger a - f(t) a^\dagger - \overline{f}(t) a \tag{A.61}$$

Este hamiltoniano também conduz a um resultado exacto. Usando os mesmos métodos que foram utilizados para o oscilador harmónico pode-se mostrar que neste caso também temos

$$U(\overline{z}_f, z_i; t) = e^{iS(f,i)} \tag{A.62}$$

onde $S(f,i)$ é a acção calculada ao longo das trajectórias clássicas (ver Problema A.1).

A.5 Fermion systems

Vamos generalizar os resultados anteriores ao caso de sistemas de fermiões. Começamos com sistemas com dois níveis com os operadores $a^\dagger$ e a tais que

$$\left\{a^\dagger, a\right\} = 1 \qquad ; \qquad a^2 = a^{2\dagger} = 0 \tag{A.63}$$

Para efectuar a construção anterior vamos tentar representar estes operadores num espaço de Hilbert de funções analíticas. Isto é possível se considerarmos funções (de facto polinómios) com coeficientes complexos em duas variáveis que anticomutam η e $\overline{\eta}$, designadas por variáveis de Grassmann e que obedecem a

$$\eta\overline{\eta} + \overline{\eta}\eta = 0 \qquad ; \qquad \overline{\eta}^2 = \eta^2 = 0 \tag{A.64}$$

Então qualquer função $P(\eta, \overline{\eta})$ terá a forma

$$P(\eta, \overline{\eta}) = p_0 + p_1\overline{\eta} + \widetilde{p}_1\eta + p_{12}\eta\overline{\eta} \tag{A.65}$$

A.5.1 Derivatives

Neste espaço a derivação é definida por (as derivadas são esquerdas)

$$\begin{aligned} \frac{\partial P}{\partial \eta} &= \widetilde{p}_1 + p_{12}\overline{\eta} \\ \frac{\partial P}{\partial \overline{\eta}} &= p_1 - p_{12}\eta \end{aligned} \tag{A.66}$$

De entre todas as funções nas variáveis η e $\overline{\eta}$ definimos o subconjunto das *funções analíticas* tais que

$$\frac{\partial}{\partial \eta} f = 0 \tag{A.67}$$

isto é as funções analíticas têm a forma

$$f = f_0 + f_1\overline{\eta} \tag{A.68}$$

A.5.2 Dot product

No espaço das funções analíticas define-se o produto interno

$$(g, f) = \overline{g}_0\ f_0 + \overline{g}_1\ f_1 \tag{A.69}$$

este produto interno pode ser representado por um integral desde que definamos a integração convenientemente (ver equação (A.74)) .

A.5.3 Integration

A integração nas variáveis de Grassmann é definida pelas relações

$$\begin{aligned}\int d\eta\ \eta = \int d\overline{\eta}\ \overline{\eta} = 1 \\ \int d\eta\ 1 = \int d\overline{\eta}\ 1 = 0\end{aligned} \tag{A.70}$$

Notar que a integração assim definida é semelhante à derivação. De facto[2]

$$\begin{aligned}&\int d\eta\ P = \partial P \qquad ; \qquad \int d\overline{\eta}\ P = \overline{\partial} P \\ &\int d\overline{\eta} d\eta\ P = \overline{\partial}\partial P\end{aligned} \tag{A.72}$$

Devido à forma da equação A.62 é claro que se tem

$$\overline{\partial}^2 = \partial^2 = 0 \tag{A.73}$$

e que portanto o integral duma derivada é zero. Consideremos agora a mudança de variáveis nos integrais. Seja

$$\begin{pmatrix}\eta \\ \overline{\eta}\end{pmatrix} = A \begin{pmatrix}\xi \\ \overline{\xi}\end{pmatrix} \tag{A.74}$$

Então obtemos

$$\begin{aligned}\eta\overline{\eta} &= (A_{11}\xi + A_{12}\overline{\xi})(A_{21}\xi + A_{22}\overline{\xi}) \\ &= (A_{11}A_{22} - A_{12}A_{21})\xi\overline{\xi} \\ &= \det A\ \xi\overline{\xi}\end{aligned} \tag{A.75}$$

Pelo que

$$\int d\overline{\eta} d\eta P(\eta, \overline{\eta}) = \int d\overline{\xi} d\xi (\det A)^{-1} Q(\xi, \overline{\xi}) \tag{A.76}$$

onde $Q(\xi, \overline{\xi})$ é o polinómio que se obtém de $P(\eta, \overline{\eta})$ por substituição de η e $\overline{\eta}$ por ξ e $\overline{\xi}$.

[2] Estamos a usar a notação compacta

$$\partial \equiv \frac{\partial}{\partial \eta} \qquad ; \qquad \overline{\partial} \equiv \frac{\partial}{\partial \overline{\eta}} \tag{A.71}$$

Finalmente notemos que se definirmos a conjugação complexa de f por

$$\overline{f} = \overline{f}_0 + \overline{f}_1 \eta \tag{A.77}$$

então podemos encontrar uma representação integral para o produto interno dada por

$$(g, f) \equiv \int d\overline{\eta} d\eta \; e^{-\overline{\eta}\eta} \; \overline{g} \; f \tag{A.78}$$

Para vermos isso calculemos o integral. Obtemos

$$\begin{aligned}
&\int d\overline{\eta} d\eta \; e^{-\overline{\eta}\eta} \; \overline{g} \; f \\
&\quad = \int d\overline{\eta} d\eta (1 - \overline{\eta}\eta)(\overline{g}_0 + \overline{g}_1 \eta)(f_0 + f_1 \overline{\eta}) \\
&\quad = \overline{g}_0 f_0 + \overline{g}_1 f_1 \\
&\quad = (g, f)
\end{aligned} \tag{A.79}$$

A.5.4 Representation of operators

Os operadores a e $a^\dagger$ podem ser representados por

$$\begin{aligned}
a &\to \overline{\partial} \\
a^\dagger &\to \overline{\eta}
\end{aligned} \tag{A.80}$$

É fácil de ver que com estas definições temos $a^2 = a^{\dagger 2} = 0$ e $aa^\dagger + a^\dagger a = 1$.

Consideremos agora os estados $|0\rangle$ e $|1\rangle = a^\dagger |0\rangle$ a que correspondem as funções 1 e $\overline{\eta}$. Então podemos encontrar o kernel de qualquer operador

$$A = \sum_{n,m} |n\rangle \, A_{n,m} \, \langle m| \tag{A.81}$$

De facto

$$\begin{aligned}
(Af)\overline{\eta} &= \sum_{n,m} \overline{\eta}^n A_{n,m} \langle m|f\rangle \\
&= \int d\overline{\xi} d\xi \; e^{\overline{\xi}\xi} \; \sum_{n,m} \overline{\eta}^n \; A_{n,m} \; \xi^m \; f(\overline{\xi}) \\
&= \int d\overline{\xi} d\xi \; e^{-\overline{\xi}\xi} \; A(\overline{\eta}, \xi) \; f(\overline{\xi})
\end{aligned} \tag{A.82}$$

onde

$$A(\overline{\eta},\xi) \equiv \sum_{n,m} \overline{\eta}^n \; A_{n,m} \; \xi^m \qquad ; \qquad n,m = 0,1 \tag{A.83}$$

Para o produto de operadores é fácil de ver que temos como anteriormente

$$A_1 A_2(\overline{\eta},\eta) = \int d\overline{\xi} d\xi \; e^{-\overline{\xi}\xi} \; A_1(\overline{\eta},\xi) \; A_2(\overline{\xi},\eta) \tag{A.84}$$

A.5.5 Normal form for operators

Seja um operador definido por

$$A = \sum_{n,m} |n\rangle \, A_{n,m} \, \langle m| = \sum_{n,m} a^{\dagger n} \, |0\rangle \, \langle 0| \, a^m \; A_{n,m} \tag{A.85}$$

O projector do estado base é

$$|0\rangle \, \langle 0| =: e^{-a^\dagger a} : \tag{A.86}$$

logo

$$\begin{aligned} A &= \sum_{n,m} A_{n,m} : a^{\dagger n} \; e^{-a^\dagger a} \; a^m : \\ &\equiv \sum_{n,m} A^N_{n,m} \; a^{\dagger n} a^m \end{aligned} \tag{A.87}$$

O *kernel normal* é então definido pela substituição $a^\dagger \to \overline{\eta}$ e $a \to \eta$, isto é

$$A^N(\overline{\eta},\eta) = \sum_{n,m} A^N_{n,m} \; \overline{\eta}^n \eta^m \tag{A.88}$$

O kernel da identidade é $e^{\overline{\eta}\eta}$, isto é

$$f(\overline{\eta}) = \int d\overline{\xi} d\xi \; e^{-\overline{\xi}\xi + \overline{\eta}\xi} \; f(\overline{\xi}) \tag{A.89}$$

o que permite obter a relação entre o kernel usual e kernel normal. De facto

$$\begin{aligned} \left[a^{\dagger n} a^m f\right] \overline{\eta} &= \overline{\eta}^n \; \frac{\partial^m}{\partial \overline{\eta}^m} f(\overline{\eta}) \\ &= \int d\overline{\xi} d\xi \; e^{-\overline{\xi}\xi} \; e^{\overline{\eta}\xi} \; \overline{\eta}^n \xi^m \; f(\overline{\xi}) \end{aligned} \tag{A.90}$$

o que permite escrever a relação procurada

$$A(\overline{\eta},\eta) = e^{\overline{\eta}\eta}\ A^N(\overline{\eta},\eta) \tag{A.91}$$

Finalmente seguindo um raciocínio análogo ao do sistema de bosões é fácil obter o kernel do operador de evolução

$$U(\overline{\eta}_f,t_f;\eta_i,t_i) = \int \mathcal{D}(\overline{\eta},\eta)\ e^{\frac{1}{2}(\overline{\eta}_f\eta_f+\overline{\eta}_i\eta_i)}\ e^{i\int_{t_i}^{t_f} dt\left[\frac{1}{2i}(\overline{\eta}\dot{\eta}-\dot{\overline{\eta}})-h(\overline{\eta},\eta)\right]} \tag{A.92}$$

Problems for Appendix A

A.1 Show the result expressed in Eq. (A.62). Show also that

$$
\begin{aligned}
iS(f,i) \;\; = \;\; & \overline{z}_f \; e^{-i\omega(t_f-t_i)} \; z_i + i\int_{t_i}^{t_f} dt \left[\overline{z}_f \; e^{-i\omega(t_f-t)} \; f(t) + \overline{f}(t) \; e^{-i\omega(t-t_i)} \; z_i\right] \\
& - \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \; \overline{f}(t) \; e^{-i\omega(t-t')} \; f(t')\theta(t-t') \qquad \text{(A.93)}
\end{aligned}
$$

This result it is useful in many applications (see, for instance, Eq. (B.9)).

A.2 Show that the representative of the states $|n\rangle$ and $|m\rangle$, $\frac{\overline{z}^n}{\sqrt{n!}}$ and $\frac{\overline{z}^m}{\sqrt{m!}}$, respectively,are orthonormalized, thyat is, $\langle f_n|f_m\rangle = \delta_{n,m}$.

A.3 Show that for gaussian integrals we have

$$
\int \prod \frac{dz_k d\overline{z}_k}{2\pi i} \; e^{-\overline{z}Az+\overline{u}z+\overline{z}u} = (\det A)^{-1} \; e^{\overline{u}A^{-1}u} \qquad \text{(A.94)}
$$

Notice that the exponent is the staionary point.

A.4 Show that for gaussian integrals of Grassmann variables we have

$$
\begin{aligned}
& \int \prod_1^n d\overline{\eta}_k d\eta_k \; e^{\sum \overline{\eta}_k A_{k\ell}\eta_\ell + \sum(\overline{\eta}_k\xi_k+\overline{\xi}_k\eta_k)} \\
& \qquad = \det A \; e^{\sum \overline{\xi}_k (A^{-1})_{k\ell}\xi_\ell} \qquad \text{(A.95)}
\end{aligned}
$$

Compare with the result of Problem A.3.

Appendix B

Path Integral in Quantum Field Theory

B.1 Path integral quantization

Vamos aqui generalizar os resultados do apêndice A para o caso de sistemas com um número infinito de graus de liberdade que são os que interessam em teoria quântica dos campos. Para evitar complicações com índices e com problemas decorrentes da invariância de gauge vamos estudar o caso do campo escalar cuja acção clássica em presença duma fonte exterior é

$$S(\phi, J) = S_0(\phi, J) + \int d^4x \; V(x) \tag{B.1}$$

onde

$$S_0(\phi, J) = \int d^4x \; \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2} \; m^2\phi^2 + J\phi\right] \tag{B.2}$$

é a acção do campo escalar livre acoplada a uma fonte exterior. Vamos primeiro estudar este caso, isto é, supor que $V = 0$. O caso geral é fácil, de obter a partir deste, como veremos mais à frente. O hamiltoniano é dado por

$$H = \int d^3x \left[\frac{1}{2}\pi_{\rm op}^2 + \frac{1}{2}\left(\nabla\phi_{\rm op}\right)^2 + \frac{1}{2}m^2\phi_{\rm op}^2 - J\phi_{\rm op}\right] \tag{B.3}$$

e podemos introduzir os operadores $a(k)$ e $a^\dagger(k)$ tais que num certo instante

$$\phi_{\rm op} = \int \tilde{d}k \left[a(k) \; e^{i\vec{k}\cdot\vec{x}} + a^\dagger(k) \; e^{-i\vec{k}\cdot\vec{x}}\right] \tag{B.4}$$

e

$$\pi_{\rm op} = -i \int \tilde{d}k \left[a(k) \; e^{i\vec{k}\cdot\vec{x}} - a^\dagger(k) \; e^{-i\vec{k}\cdot\vec{x}}\right] \omega(k) \tag{B.5}$$

então

$$H = \int \tilde{d}k \left[\omega(k) a^\dagger(k) a(k) - f(t,\vec{k}) a^\dagger(k) - \overline{f}(t,\vec{k}) a(k)\right] \tag{B.6}$$

onde introduzimos a transformada de fourier espacial da fonte,

$$f(t,\vec{k}) = \int d^3x\ e^{-i\vec{k}\cdot\vec{x}}\ j(t,\vec{x}) \tag{B.7}$$

e onde definimos

$$\tilde{d}k \equiv \frac{d^3k}{(2\pi)^3 2\omega_k} = \frac{d^4k}{(2\pi)^4}\ 2\pi\delta(k^2-m^2)\theta(k^0) \tag{B.8}$$

usando os resultados do problema A.1 podemos escrever imediatamente o kernel do operador de evolução

$$\begin{aligned} U(\overline{z}_f, t_f; z_i, t_i) \quad = \quad & \exp\Bigg\{ \int \tilde{d}k \Bigg[\overline{z}_f(k)\ e^{-i\omega(k)(t_f-t_i)}\ z_i(k) \\ & + i \int_{t_i}^{t_f} dt \left[\overline{z}_f(k)\ e^{-i\omega(k)(t_f-t)}\ f(t,\vec{k}) + \overline{f}(t,\vec{k})\ e^{-i\omega(k)(t-t_i)}\ z_i(k) \right] \\ & - \frac{1}{2} \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt'\ \overline{f}(t,\vec{k})\ e^{-i\omega(k)(t-t')}\ f(t',\vec{k}) \Bigg] \Bigg\} \end{aligned} \tag{B.9}$$

A matriz S é então definida como o limite

$$\lim_{-t_i, t_f \to \infty} e^{it_f H_0}\ U(t_f, t_i)\ e^{-it_i H_0} \tag{B.10}$$

onde H_0 é obtido a partir de H fazendo $J = 0$. Na representação que estamos a usar a acção de e^{-itH_0} é uma simples multiplicação (ver eq. A.60).

$$\overline{z} \to \overline{z}\ e^{-i\omega t} \tag{B.11}$$

Portanto o kernel da matriz S é

$$\begin{aligned} S(\overline{z}_f, z_i) \quad = \quad & \lim_{-t_i, t_f \to \infty} \exp\left[\int \tilde{d}k \overline{z}_f(k) z_i(k)\right]\ \exp\Bigg\{ \int \tilde{d}k \Bigg[\\ & i \int_{t_i}^{t_f} \left[\overline{z}_f(k)\ e^{i\omega(k)t}\ f(t,\vec{k}) + \overline{f}(t,\vec{k})\ e^{-i\omega(k)t} z_i(k) \right] \\ & - \frac{1}{2} \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt dt' \overline{f}(t,\vec{k})\ e^{-i\omega(k)(t-t')}\ f(t',\vec{k}) \Bigg] \Bigg\} \end{aligned} \tag{B.12}$$

O primeiro factor é aquilo que é necessário para passar do kernel usual para o kernel normal. O restante pode ser interpretado se definirmos

$$\phi_{\rm as} \equiv \int \tilde{d}k \left[z_i(k)\ e^{-ik\cdot x} + \overline{z}_f(k)\ e^{ik\cdot x} \right] \tag{B.13}$$

Como $\overline{z}_f$ não é o complexo conjugado de z_i então $\phi_{\rm as}$ é dado em termos de condições na fronteira com frequências positivas para $t \to -\infty$ e frequências negativas para $t \to \infty$. Estas são precisamente as condições na fronteira de Feynman. Com estas convenções e notações obtemos para o primeiro termo

$$\begin{aligned}
&\int \tilde{d}k \int_{t_i}^{t_f} dt \left[\overline{z}_f(k)\ e^{i\omega(k)t}\ f(t,\vec{k}) + \overline{f}(t,\vec{k})\ e^{-i\omega(k)t}\ z_i(k) \right] \\
&= \int d^4x \int \tilde{d}k J(x) \left[\overline{z}_f e^{i\omega(k)t - i\vec{k}\cdot\vec{x}} + z_i(k)\ e^{-i\omega(k)t + i\vec{k}\cdot\vec{x}} \right] \\
&= \int d^4x J(x) \phi_{\rm as}(x)
\end{aligned} \tag{B.14}$$

e para o segundo

$$\begin{aligned}
&\int \tilde{d}k \int dt \int dt'\ \overline{f}(t,\vec{k})\ e^{-i\omega(k)(t-t')}\ f(t',\vec{k}) \\
&= \int d^4x d^4x' J(x) J(x') \int \tilde{d}k\ e^{-i\omega(k)(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')} \\
&= \int d^4x d^4x'\ J(x) G_F(x-x') J(x')
\end{aligned} \tag{B.15}$$

pois

$$\begin{aligned}
&\int \tilde{d}k\ e^{-i\omega(k)(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')} \\
&\qquad = \int \tilde{d}k\ e^{-i\omega(k)(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')}\ \theta(t-t') \\
&\qquad\quad + \int \tilde{d}k\ e^{i\omega(k)(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')}\ \theta(t'-t) \\
&\qquad = i \int d^4k\ e^{-ik\cdot(x-x')}\ \frac{1}{k^2 - m^2 + i\varepsilon} \\
&\qquad = G_F(x-x')
\end{aligned} \tag{B.16}$$

Notar que as condições na fronteira mistas conduzem ao propagador de Feynman. Podemos portanto finalmente escrever o *kernel normal* da matriz S na presença da fonte J,

$$S^N(\overline{z}_f, z_i)\Big|_J = e^{i\int d^4x\ j(x)\phi_{\rm as}(x)}\ e^{-\frac{1}{2}\int d^4x d^4x'\ J(x) G_F^0(x-x') J(x')} \tag{B.17}$$

Para se obter o operador S substituimos ϕ_{as} por ϕ_{op} e fazemos o ordenamento normal, isto é

$$S_0(J) =: e^{i\int d^4x \ J(x)\phi_{\text{op}}(x)} : \ e^{-\frac{1}{2}\int d^4x d^4x' \ J(x)G_F^0(x-x')J(x')} \tag{B.18}$$

Como o funcional gerador das funções de green é $\langle 0|S_0(J)|0\rangle$ obtemos imediatamente

$$Z_0(J) = e^{-\frac{1}{2}\int d^4x d^4x' J(x)G_F^0(x-x')J(x')} \tag{B.19}$$

Este resultado permite resolver o problema de qualquer potencial $V(x)$. De facto é fácil de mostrar que no caso geral os kernéis estão relacionadoss por

$$S^N = \exp\left[-i\int d^4x V\left(\frac{\delta}{i\delta J(x)}\right)\right] \ S^N(J)\big|_{J=0} \tag{B.20}$$

e o operador S é

$$S =: e^{\int d^4x J(x)\phi_{\text{op}}(x)} : \ \exp\left[-i\int d^4x \ V\left(\frac{\delta}{i\delta J(x)}\right)\right] \ Z_0(J)\big|_{J=0} \tag{B.21}$$

ou seja

$$Z(J) = \exp\left[-i\int d^4x \ V\left(\frac{\delta}{i\delta J(x)}\right)\right] \ Z_0(J) \tag{B.22}$$

com

$$Z_0(J) = e^{-\frac{1}{2}\int d^4x d^4x' \ J(x)G_F^0(x-x')J(x')} \tag{B.23}$$

Estas expressões permitem calcular qualquer função de green com as regras usuais da teoria das perturbações. A quantificação usando os integrais de caminho conduziu aos mesmos resultados (em teoria das perturbações) que a quantificação canónica. As expressões para os funcionais geradores embora dêem resultados perturbativos duma forma imediata não são as mais úteis quando estamos interessados em encontrar resultados válidos para além da teoria das perturbações. Para esses casos (identidades de Ward, etc) é mais útil ter uma expressão formal em termos dum integral de caminho. É isso que vamos agora estudar.

B.2 Path integral for generating functionals

O ponto de partida é a expressão para o kernel da matriz S,

$$\begin{aligned} S(\overline{z}_f, z_i) \quad = \quad & \lim_{-t_i, t_f \to \infty} \int \mathcal{D}(\overline{z}, z) \ e^{\frac{1}{2}\int \tilde{d}k[\overline{z}(k,t_f)z(k,t_f)+\overline{z}(k,t_i)z(k,t_i)]} \\ & \exp\left\{ i\int_{t_i}^{t_f} dt \int \tilde{d}k \left[\frac{1}{2i}(\dot{\overline{z}}(k,t)z(k,t) - \overline{z}(k,t)\dot{z}(k,t))\right.\right. \end{aligned}$$

$$-\omega(k)\overline{z}(k,t)z(k,t) - V(\overline{z},z)\Big]\Big\} \tag{B.24}$$

com as condições na fronteira[1]

$$\begin{cases} \overline{z}(k,t_f) = \overline{z}_f(k)\ e^{i\omega t_f} \\ \\ z(k,t_i) = z_i(k)\ e^{-i\omega t_i} \end{cases} \tag{B.25}$$

Em vez das variáveis $z(k,t)$ e $\overline{z}(k,t)$ vamos introduzir os campos clássicos $\phi(\vec{x},t)$ e $\pi(\vec{x},t)$ definidos por

$$\phi(\vec{x},t) = \int \tilde{d}k \left[z(k,t)\ e^{i\vec{k}\cdot\vec{x}} + \overline{z}(k,t)\ e^{-i\vec{k}\cdot\vec{x}} \right] \tag{B.26}$$

e

$$\pi(\vec{x},t) = -i\int \tilde{d}k\ \omega(k) \left[z(k,t)\ e^{i\vec{k}\cdot\vec{x}} - \overline{z}(k,t)\ e^{-i\vec{k}\cdot\vec{x}} \right] \tag{B.27}$$

Estas fórmulas são obviamente sugeridas pelas relações entre $\phi_{\rm op}$, $\pi_{\rm op}$ e $a(k)$, $a^\dagger(k)$ expressas nas equações B.4 e B.5, só que aqui não se trata de operadores mas sim de campos clássicos. Comecemos por escrever a acção em termos das novas variáveis,

$$\begin{aligned} &\int_{t_i}^{t_f} dt \int \tilde{d}k \left[\frac{1}{2i}(\dot{\overline{z}}(k,t)z(k,t) - \overline{z}(k,t)\dot{z}(k,t)) \right. \\ &\qquad\qquad \left. -\omega(k)\overline{z}(k,t)z(k,t) - V(\overline{z},z) \right] \\ &= \int d^3x \int_{t_i}^{t_f} dt \left[\frac{1}{2}(\pi\partial_0\phi - \partial_0\pi\phi) - \frac{1}{2}\pi^2 \right. \\ &\qquad\qquad \left. -\frac{1}{2}(\partial_k\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi) \right] \end{aligned} \tag{B.28}$$

Introduzimos agora novas variáveis $\phi_1(\vec{x},t)$ e $\pi_1(\vec{x},t)$ definidas do modo seguinte

$$\begin{cases} \phi(\vec{x},t) \equiv \phi_{\rm as}(\vec{x},t) + \phi_1(\vec{x},t) \\ \\ \pi(\vec{x},t) \equiv \partial_0\phi(\vec{x},t) + \phi_1(\vec{x},t) \end{cases} \tag{B.29}$$

onde

$$\phi_{\rm as,in}(\vec{x},t) = \int \tilde{d}k \left[\overline{z}_{\rm in}(k)\ e^{ik\cdot x} + z_{\rm in}(k)\ e^{-ik\cdot x} \right] \tag{B.30}$$

e

[1] Não há restrições em $\overline{z}(k,t_i)$ e $z(k,t_f)$.

$$\phi_{\rm as,out}(\vec{x},t) = \int \tilde{d}k \left[\overline{z}_{\rm out}(k)\ e^{ik\cdot x} + z_{\rm out}(k)\ e^{-ik\cdot x}\right] \tag{B.31}$$

onde $in \equiv t \to -\infty$ e $out \equiv t \to +\infty$ com

$$\left\{ \begin{array}{l} \overline{z}_{\rm out}(k) \equiv \overline{z}_f(k) \\ \\ z_{\rm in}(k) \equiv z_i(k) \end{array} \right. \tag{B.32}$$

O campo $\phi_{\rm as}$ tem portanto as condições fronteira apropriadas para o problema e satisfaz à equação de Klein-Gordon

$$(\Box + m^2)\phi_{\rm as} = 0 \tag{B.33}$$

Escrevemos a acção nas novas variáveis

$$\begin{aligned}
&\int d^3x \int_{t_i}^{t_f} dt \left[\frac{1}{2}(\pi\partial_0\phi - \partial_0\pi\phi) - \frac{1}{2}\pi^2 - \frac{1}{2}(\partial_k\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi)\right] \\
&\quad = \int d^3x \left[-\frac{1}{2}\pi\phi\right]_{t_i}^{t_f} \\
&\quad + \int d^3x \int_{t_i}^{t_f} dt \left[\pi\partial_0\phi - \frac{1}{2}(\pi_1^2 + 2\pi\partial_0\phi - (\partial_0\phi)^2) - \frac{1}{2}(\partial_k\phi_{\rm as})^2 - \frac{1}{2}(\partial_k\phi_1)^2\right. \\
&\qquad\qquad \left. -\partial_k\phi_{\rm as}\partial_k\phi_1 - \frac{1}{2}m^2\phi_{\rm as}^2 - \frac{1}{2}m^2\phi_1^2 - m^2\phi_{\rm as}\phi_1 - V(\phi)\right] \\
&= \quad \int d^3x \left[-\frac{1}{2}\pi\phi\right]_{t_i}^{t_f} \\
&\quad + \int d^3x \int_{t_i}^{t_f} dt \left[-\frac{1}{2}\pi_1^2 + \frac{1}{2}(\partial_0\phi_{\rm as})^2 + \partial_0\phi_{\rm as}\partial_0\phi_1 + \frac{1}{2}(\partial_0\phi_1)^2 - \frac{1}{2}(\partial_k\phi_{\rm as})^2\right. \\
&\qquad\qquad \left. -\frac{1}{2}(\partial_k\phi_1)^2 - \partial_k\phi_{\rm as}\partial_k\phi_1 - \frac{1}{2}m^2\phi_{\rm as}^2 - \frac{1}{2}m^2\phi_1^2 - m^2\phi_{\rm as}\phi_1 - V(\phi)\right] \\
&= \quad \int d^3x \left[\frac{1}{2}\partial_0\phi_{\rm as}\phi_{\rm as} + \partial_0\phi_{\rm as}\phi_1 - \frac{1}{2}\pi\phi\right]_{t_i}^{t_f} \\
&\quad + \int d^3x \int_{t_i}^{t_f} dt \left[-\frac{1}{2}\pi_1^2 + \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 - V(\phi)\right] \\
&= \quad \int d^3x \left[\partial_0\phi_{\rm as}\phi - \frac{1}{2}\partial_0\phi_{\rm as}\phi_{\rm as} - \frac{1}{2}\pi\phi\right]_{t_i}^{t_f} \\
&\quad + \int d^3x \int_{t_i}^{t_f} dt \left[-\frac{1}{2}\pi_1^2 + \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 - V(\phi)\right]
\end{aligned} \tag{B.34}$$

Vemos que no segundo termo as variáveis ϕ_1 e π_1 estão separadas e π_1 aparece quadraticamente. Isto permitirá eliminar π_1 como veremos no seguimento. Analisemos contudo primeiro o termo que tem as condições na fronteira. Usando as definições de $\phi_{\rm as}$, ϕ e π podemos escrever

$$
\begin{aligned}
& i\int d^3x \left[\partial_0\phi_{\rm as}\phi - \frac{1}{2}\partial_0\phi_{\rm as}\phi_{\rm as} - \frac{1}{2}\pi\phi\right]_{t_i}^{t_f} \\
&= \int \tilde{d}k \left\{ \overline{z}_f(k) z_i(k) - \frac{1}{2}\left[\overline{z}(k,t_f)z(k,t_f) + \overline{z}(k,t_i)z(k,t_i)\right] \right. \\
&\qquad \left. -\frac{1}{4}\left[z(k,t_f) - z_i(k)\ e^{-i\omega t_f}\right]^2 - \frac{1}{4}\left[\overline{z}(k,t_i) - \overline{z}_f(k)\ e^{i\omega t_i}\right]^2 \right\}
\end{aligned}
\tag{B.35}
$$

Nesta expressão o primeiro termo dá a passagem do kernel usual para o kernel normal, o segundo cancela exactamente o termo na fronteira na definição inicial de $S(\overline{z}_f, z_i)$ e os últimos têm que ser estudados em detalhe. Reunindo tudo até este ponto a expressão do *kernel normal* da matriz S é

$$
\begin{aligned}
S^N(\phi_{\rm as}) \quad = \quad & \lim_{-t_i,t_f\to\infty} \int \mathcal{D}(\phi,\pi) \exp\left\{ -\frac{1}{4}\int \tilde{d}k \left[\left(z(k,t_f) - z_i(k)\ e^{-i\omega t_f}\right)^2 \right.\right. \\
& \qquad \left.\left. + \left(\overline{z}(k,t_i) - z_f(k)\ e^{i\omega t_i}\right)^2 \right] \right\} \\
& \exp\left\{ \int d^3x \int_{t_i}^{t_f} dt \left[-\pi_1^2 + \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 - V(\phi) \right] \right\}
\end{aligned}
\tag{B.36}
$$

Esta expressão já está próxima do resultado final. Falta só mostrar que os termos dentro da primeira exponencial tendem para zero quando $-t_i, t_f \to \infty$. Esta é a parte mais delicada do argumento. Vamos expô-lo por passos:

i) Funções rapidamente decrescentes

Queremos que $I(t) = \int d^3x\ \pi_1^2(\vec{x},t)$ seja integrável. Dizemos então que funções como $\pi_1(\vec{x},t)$ são rapidamente decrescentes (RD) quando $|t| \to \infty$.

ii) Informação sobre $\overline{z}_{1,\rm out}(k,t)$ e $z_{1,\rm in}(k,t)$

Da definição $\phi = \phi_{\rm as} + \phi_1$ resultam as definições

$$
\left\{
\begin{aligned}
z(k,t) &= z_i(k)\ e^{-i\omega t} + z_1(k,t) \\
\overline{z}(k,t) &= \overline{z}_f(k)\ e^{i\omega t} + \overline{z}_1(k,t)
\end{aligned}
\right.
\tag{B.37}
$$

As condições na fronteira dizem-nos que $\overline{z}_{1,\rm out}(k,t)$ e $z_{1,\rm in}(k,t)$ são funções RD quando $t \to +\infty$ e $t \to -\infty$ respectivamente, mas não nos dizem nada sobre $\overline{z}_{1,\rm in}$ e $z_{1,\rm out}$, que são precisamente os limites que precisamos.

iii) Informação sobre os limites $z_{1,\rm out}$ e $\overline{z}_{1,\rm in}$

Informação sobre os limites $z_{1,\rm out}$ e $\overline{z}_{1,\rm in}$ obtém-se a partir do seguinte raciocínio,

$$
\begin{aligned}
\pi_1 &= \pi - \partial_0\phi \\
&= \int \tilde{d}k \Big\{ \left[i\omega(k)\overline{z}(k,t) - \partial_0\overline{z}(k,t) \right] \ e^{-i\vec{k}\cdot\vec{x}} \\
&\qquad\qquad - \left[i\omega(k) z(k,t) + \partial_0 z(k,t) \right] \ e^{i\vec{k}\cdot\vec{x}} \Big\} \\
&\equiv -\int \tilde{d}k \left[\overline{z}_2(k,t) \ e^{-i\vec{k}\cdot\vec{x}} + z_2(k,t) \ e^{i\vec{k}\cdot\vec{x}} \right]
\end{aligned}
\tag{B.38}
$$

Para que π_1 seja do tipo RD quando $|t| \to \infty$ também teremos que ter $z_2(k,t)$ e $\overline{z}_2(k,t)$ RD nesses limites. Vejamos qual a informação contida neste resultado.

- $t \to +\infty$

Obtemos então que a função

$$
z_2(k,t) \equiv \partial_0 z(k,t) + i\omega(k) z(k,t)
\tag{B.39}
$$

é RD quando $t \to +\infty$. A informação sobre $\overline{z}_2(k,t)$ não trás nada de novo já que está contida nas condições fronteiras. De facto

$$
\begin{aligned}
\lim_{t\to+\infty} \overline{z}_2(k,t) &\equiv \lim_{t\to+\infty} \left[\partial_0\overline{z}(k,t) - i\omega(k)\overline{z}(k,t) \right] \\
&= i\omega(k)\overline{z}_f(k) \ e^{i\omega t} + \partial_0 \overline{z}_{1,\text{out}}(k,t) \\
&\qquad -i\omega(k)\overline{z}_f(k) \ e^{i\omega t} - i\omega(k)\overline{z}_{1,\text{out}}(k,t) \\
&= \text{RD} \qquad t \to +\infty
\end{aligned}
\tag{B.40}
$$

- $t \to -\infty$

A informação contida nas condições na fronteira é

$$
\overline{z}_2(k,t) \equiv \partial_0\overline{z}(k,t) - i\omega(k)\overline{z}(k,t) = \text{RD} \qquad t \to -\infty
\tag{B.41}
$$

iv) Demonstração que $z_{1,\text{out}}$ *e* $\overline{z}_{1,\text{in}}$ *são* RD

Da definição

$$
\phi(\vec{x},t) = \phi_{\text{as}} + \phi_1
\tag{B.42}
$$

resulta

$$
\begin{cases}
\phi(\vec{x},t) = \phi_{\text{as,in}}(\vec{x},t) + \phi_{1,\text{in}}(\vec{x},t) & t \to -\infty \\
\\
\phi(\vec{x},t) = \phi_{\text{as,out}}(\vec{x},t) + \phi_{1,\text{out}}(\vec{x},t) & t \to +\infty
\end{cases}
\tag{B.43}
$$

ou seja

$$\begin{cases} \overline{z}(k,t) = \overline{z}_{\rm in}(k)\ e^{i\omega t} + \overline{z}_{1,\rm in}(k,t) & t \to -\infty \\ \\ z(k,t) = z_{\rm out}(k)\ e^{-i\omega t} + z_{1,\rm out}(k,t) & t \to +\infty \end{cases} \tag{B.44}$$

Mas usando os resultados anteriores

$$\begin{aligned} \partial_0 \overline{z}(k,t) \ &- \ i\omega(k)\overline{z}(k,t) = \mathrm{RD} \qquad t \to -\infty \\ &= \ i\omega(k)\overline{z}_{\rm in}(k)\ e^{i\omega t} + \partial_0 \overline{z}_{1,\rm in}(k,t) \\ &\quad -i\omega(k)\overline{z}_{\rm in}(k)\ e^{i\omega t} - i\omega(k)\overline{z}_{1,\rm in}(k,t) \end{aligned} \tag{B.45}$$

ou seja

$$\overline{z}_{1,\rm in}(k,t) = \mathrm{RD} \qquad t \to -\infty \tag{B.46}$$

e igualmente

$$z_{1,\rm out}(k,t) = \mathrm{RD} \qquad t \to +\infty \tag{B.47}$$

Isto quer dizer que

$$\begin{cases} \phi_{1,\rm in} = \mathrm{RD} & t \to -\infty \\ \\ \phi_{1,\rm out} = \mathrm{RD} & t \to +\infty \end{cases} \tag{B.48}$$

isto é, assimptoticamente

$$\phi = \phi_{\rm as} + \mathrm{RD} \tag{B.49}$$

v) Resultado final

Estamos agora em condições de atacar o nosso problema. Temos

$$\begin{aligned} &\lim_{t_i \to -\infty} \int \tilde{d}k\ \left[\overline{z}(k,t_i) - z_f(k)\ e^{i\omega t_i} \right]^2 \\ =\ &\lim_{t_i \to -\infty} \int \tilde{d}k\ \left[(\overline{z}_{\rm in}(k) - \overline{z}_f(k))\ e^{i\omega t_i} + \overline{z}_{1,\rm in}(k,t_i) \right]^2 \\ =\ &\lim_{t_i \to -\infty} \int \tilde{d}k\ \Big[(\overline{z}_{\rm in}(k) - \overline{z}_f(k))^2\ \ e^{2i\omega t_i} + 2\,(\overline{z}_{\rm in}(k) - \overline{z}_f(k))\ \ e^{i\omega t_i}\overline{z}_{1,\rm in}(k,t_i) \\ &\qquad + \overline{z}^2_{1,\rm in}(k,t_i) \Big] \end{aligned}$$

$$= \quad 0 \tag{B.50}$$

Para o outro termo obter-se-ia o mesmo resultado. Chegamos portanto ao resultado

$$\begin{aligned} S^N(\phi_{\rm as}) &= \int \mathcal{D}(\phi,\pi)\ \exp\left\{-\frac{i}{2}\int d^4x\ \pi_1^2\right\} \\ &\qquad \times \exp\left\{i\int d^4x \left[\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 - V(\phi)\right]\right\} \end{aligned} \tag{B.51}$$

onde a integração é feita sobre os campos $\phi = \phi_{\rm as} + \phi_1$ com as condições fronteiras apropriadas. Fazendo a integração sobre π_1 obtemos (a menos duma normalização)

$$\begin{aligned} S^N(\phi_{\rm as}) &= \int \mathcal{D}(\phi)\ e^{i\int d^4x\left[\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 - V(\phi)\right]} \\ &= \int_{\phi=\phi_{\rm as}+\phi_1} \mathcal{D}(\phi)\ e^{i\int d^4x[\mathcal{L}(\phi_1)-(V(\phi)-V(\phi_1))]} \end{aligned} \tag{B.52}$$

Na presença de fontes exteriores obtemos

$$S^N(\phi_{\rm as}, J) = \int_{\phi=\phi_{\rm as}+\phi_1} \mathcal{D}(\phi)\ e^{i\int d^4x[\mathcal{L}(\phi_1)-(V(\phi)-V(\phi_1))+J\phi]} \tag{B.53}$$

Normalmente não estamos interessados na matriz S mas no funcional gerador das funções de Green. Por definição

$$Z(J) \equiv S(\phi_{\rm as}, J)\Big|_{\phi_{\rm as}=0} \tag{B.54}$$

Obtemos portanto a expressão fundamental

$$Z(J) = \int \mathcal{D}(\phi)\ e^{i\int d^4x[\mathcal{L}(\phi)+J\phi]} \tag{B.55}$$

B.3 Fermion systems

O uso de variáveis de Grassmann permite escrever expressões de integrais de caminho para a matriz S e para o funcional gerador das funções de Green Z para este caso. Não vamos aqui repetir os cálculos que fizémos para os sistemas de bosões, mas antes apresentar somente os resultados deixando as demonstrações para os problemas.

O ponto de partida é a definição do funcional gerador das funções de Green em presença das fontes exteriores fermiónicas. Este é dado por[2]

[2] Comparar com a definição do caso bosónico, equação 5.15.

$$Z[\eta,\overline{\eta}] = \langle 0|\, T \exp\left[i \int d^4x \left(\overline{\eta}(x)\psi(x) + \overline{\psi}(x)\eta(x) \right) \right] |0\rangle \tag{B.56}$$

Então as funções de Green

$$G^{2n}(x_1,\ldots,y_n) \equiv \langle 0|\, T\psi(x_1)\cdots\psi(x_n)\overline{\psi}(y_1)\cdots\overline{\psi}(y_n)\,|0\rangle \tag{B.57}$$

são dadas por

$$G^{2n}(x_1,\ldots,y_n) = \frac{\delta^{2n} Z}{i\delta\eta(y_n)\cdots i\delta\eta(y_1) i\delta\overline{\eta}(x_n)\cdots i\delta\overline{\eta}(x_1)} \tag{B.58}$$

onde as derivadas são esquerdas, isto é

$$\begin{aligned} \frac{\delta}{\delta\overline{\eta}(x)} \int d^4y \overline{\eta}(y)\psi(y) &= \psi(x) \\ \frac{\delta}{\delta\eta(x)} \int d^4y \overline{\psi}(y)\eta(y) &= -\overline{\psi}(x) \end{aligned} \tag{B.59}$$

e por convenção a ordem da derivação é a indicada, isto é

$$\frac{\delta}{i\delta\eta(y_n)} \cdots \frac{\delta}{i\delta\overline{\eta}(x_1)} \tag{B.60}$$

Consideramos agora o lagrangeano de Dirac livre

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi \tag{B.61}$$

Pode-se mostrar (ver problema B.2) que o funcional gerador é neste caso dado por

$$Z_0[\eta,\overline{\eta}] = e^{-\int d^4x d^4y\ \overline{\eta}(x) S_F^0(x-y)\eta(y)} \tag{B.62}$$

onde $S_F^0(x-y)$ é o propagador de Feynman para a teoria de Dirac livre, dado por

$$S_F^0(x-y) = \int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}\ \frac{i}{p\!\!\!/ - m + i\varepsilon} \tag{B.63}$$

Seguindo métodos semelhantes ao do caso bosónico podemos também mostrar que este funcional gerador pode ser representado pelo integral de caminho,

$$Z_0[\eta,\overline{\eta}] = \int \mathcal{D}(\psi,\overline{\psi})\ e^{i\int d^4x\ \left[\mathcal{L}(x) + \overline{\eta}\psi + \overline{\psi}\eta\right]} \tag{B.64}$$

Tendo o funcional gerador para a teoria livre podemos formalmente escrever o funcional gerador para qualquer teoria fermiónica com interacções. Um exemplo é dado no Problema B.4.

Problems Appendix B

B.1 Mostre que as funções de Green

$$G^{2n}(x_1, \ldots, y_n) \equiv \langle 0| T\psi(x_1)\cdots\psi(x_n)\overline{\psi}(y_1)\cdots\overline{\psi}(y_n)|0\rangle \tag{B.65}$$

são dadas por

$$G^{2n}(x_1, \ldots, y_n) = \frac{\delta^{2n} Z}{i\delta\eta(y_n)\cdots i\delta\eta(y_1) i\delta\overline{\eta}(x_n)\cdots i\delta\overline{\eta}(x_1)} \tag{B.66}$$

B.2 Mostre que o funcional gerador das funções de Green para a teoria de Dirac livre é dado por

$$Z_0[\eta, \overline{\eta}] = e^{-\int d^4x d^4y\ \overline{\eta}(x) S_F^0(x-y)\eta(y)} \tag{B.67}$$

B.3 Mostre que o funcional gerador das funções de Green para a teoria de Dirac livre se pode representar pelo seguinte integral de caminho

$$Z_0[\eta, \overline{\eta}] = \int \mathcal{D}(\psi, \overline{\psi})\ e^{i\int d^4x\ \left[\mathcal{L}(x) + \overline{\eta}\psi + \overline{\psi}\eta\right]} \tag{B.68}$$

B.4 Considere o lagrangiano seguinte,

$$\begin{aligned}\mathcal{L}(x) &= \overline{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m\phi^2 \\ &\quad -g\overline{\psi}(x)\psi(x)\phi(x)\end{aligned} \tag{B.69}$$

que descreve a interacção dum campo de Dirac com um campo escalar.

a) Mostre que

$$Z[\eta,\overline{\eta},J] = \exp\left\{-ig\int d^4x \left(\frac{\delta}{i\delta\overline{\eta}}\right)\left(\frac{\delta}{i\delta\eta}\right)\left(\frac{\delta}{i\delta J}\right)\right\}\ Z_0[\eta,\overline{\eta},J] \tag{B.70}$$

onde

$$Z_0[\eta,\overline{\eta},J] = e^{-\int d^4x d^4y \left[\overline{\eta}(x)S_F^0(x-y)\eta(y)+\frac{1}{2}J(x)\Delta_F(x-y)J(y)\right]} \tag{B.71}$$

e Δ_F é o propagador livre do campo escalar.

b) Mostre que $Z[\eta,\overline{\eta},J]$ se pode exprimir por meio do integral de caminho

$$Z[\eta,\overline{\eta},J] = \int \mathcal{D}(\psi,\overline{\psi},\phi)\ e^{i\int d^4x\left[\mathcal{L}(x)+J\phi+\overline{\eta}\psi+\overline{\psi}\eta\right]} \tag{B.72}$$

Appendix C

Useful techniques for renormalization

C.1 μ parameter

The reason for the μ parameter introduced in section C.10.1 is the following. In dimension $d = 4 - \epsilon$, the fields A_μ and ψ have dimensions given by the kinetic terms in the action,

$$\int d^d x \left[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + i\,\overline{\psi}\gamma \cdot \partial\psi \right] \tag{C.1}$$

We have therefore

$$\begin{aligned} 0 &= -d + 2 + 2[A_\mu] \quad \Rightarrow [A_\mu] &= \tfrac{1}{2}(d-2) = 1 - \tfrac{\epsilon}{2} \\ 0 &= -d + 1 + 2[\psi] \quad \Rightarrow [\psi] &= \tfrac{1}{2}(d-1) = \tfrac{3}{2} - \tfrac{\epsilon}{2} \end{aligned} \tag{C.2}$$

Using these dimensions in the interaction term

$$S_I = \int d^d x\ e\overline{\psi}\gamma_\mu \psi A^\mu \tag{C.3}$$

we get

$$\begin{aligned} [S_I] &= -d + [e] + 2[\psi] + [A] \\ &= -4 + \epsilon + [e] + 3 - \epsilon + 1 - \frac{\epsilon}{2} \\ &= [e] - \frac{\epsilon}{2} \end{aligned} \tag{C.4}$$

Therefore, if we want the action to be dimensionless (remember that we use the system where $\hbar = c = 1$), we have to set

$$[e] = \frac{\epsilon}{2} \tag{C.5}$$

We see then that in dimensions $d \neq 4$ the coupling constant has dimensions. As it is more convenient to work with a dimensionless coupling constant we introduce a parameter μ with dimensions of a mass and in $d \neq 4$ we will make the substitution

$$e \to e\mu^{\frac{\epsilon}{2}} \qquad (\epsilon = 4 - d) \tag{C.6}$$

while keeping e dimensionless.

C.2 Feynman parameterization

The most general form for a *1–loop* is [1]

$$\hat{T}_n^{\mu_1 \cdots \mu_p} \equiv \int \frac{d^d k}{(2\pi)^d} \, \frac{k^{\mu_1} \cdots k^{\mu_p}}{D_0 D_1 \cdots D_{n-1}} \tag{C.7}$$

where

$$D_i = (k + r_i)^2 - m_i^2 + i\epsilon \tag{C.8}$$

and the momenta r_i are related with the external momenta (all taken to be incoming) through the relations,

$$\begin{aligned} r_j &= \sum_{i=1}^{j} p_i \quad ; \quad j = 1, \ldots, n-1 \\ r_0 &= \sum_{i=1}^{n} p_i = 0 \end{aligned} \tag{C.9}$$

as indicated in Fig. (C.1). In these expressions there appear in the denominators products

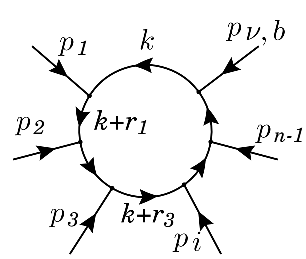


Figure C.1: Conventions for the momenta in the loop.

of the denominators of the propagators of the particles in the loop. It is convenient to

[1] We introduce here the notation $\hat{T}$ to distinguish from a more standard notation that will be explained in subsection C.9.

combine these products in just one common denominator. This is achieved by a technique due to Feynman. Let us exemplify with two denominators.

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} \tag{C.10}$$

The proof is trivial. In fact

$$\int dx \, \frac{1}{[ax + b(1-x)]^2} = \frac{x}{b\,[(a-b)x + b]} \tag{C.11}$$

and therefore Eq. (C.10) immediately follows. Taking successive derivatives with respect to a and b we get

$$\frac{1}{a^p\, b^q} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 dx \, \frac{x^{p-1}(1-x)^{q-1}}{[ax + b(1-x)]^{p+q}} \tag{C.12}$$

and using induction we obtain a general formula

$$\begin{aligned}\frac{1}{a_1 a_2 \cdots a_n} =& \Gamma(n) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \\ & \int_0^{1-x_1-\cdots-x_{n-2}} \frac{dx_{n-1}}{[a_1 x_1 + a_2 x_2 + \cdots + a_n(1 - x_1 - \cdots - x_{n-1})]^n}\end{aligned} \tag{C.13}$$

Complement C.1

Let us take a closer look at Eq. (C.13) and derive it in a different way that will make more clear the range of variation of the Feynman parameters. We follow closely the argument of Gross [13].

We start with the definition of the Γ function,

$$\Gamma(\alpha) = \int_o^\infty dt \ t^{\alpha-1} e^{-t} \tag{C.14}$$

Making a change of variables we also get

$$\frac{\Gamma(\alpha)}{a^\alpha} = \int_0^\infty dt \ t^{\alpha-1} e^{-t\,a} \tag{C.15}$$

We consider first the case of two denominators using Eq. (C.15) with $\alpha = 1$. We get

$$\frac{1}{a\,b} = \int_0^\infty \int_0^\infty dt_1\, dt_2\, e^{-(t_1\, a + t_2\, b)} \tag{C.16}$$

Now we introduce 1 in the form

$$1 = \int_0^\infty dt \ \delta(t - t_1 - t_2) \tag{C.17}$$

in Eq. (C.16) to get

$$\frac{1}{a\,b} = \int_0^\infty \int_0^\infty \int_0^\infty dt\, dt_1\, dt_2\, \delta(t - t_1 - t_2)\, e^{-(t_1 a + t_2 b)} \tag{C.18}$$

To continue we scale the variables $t_1 = t\,x_1$ and $t_2 = t\,x_2$. We then get

$$\frac{1}{a\,b} = \int_0^\infty \int_0^\infty dx_1\, dx_2\, \delta(1 - x_1 - x_2) \int_0^\infty dt\, t\, e^{-t(x_1 a + x_2 b)} \tag{C.19}$$

Now we use the definition in Eq. (C.15) to obtain

$$
\begin{aligned}
\frac{1}{a\,b} =& \Gamma(2) \int_0^\infty \int_0^\infty dx_1\, dx_2\, \delta(1-x_1-x_2) \frac{1}{[x_1 a + x_2 b]^2} \\
=& \int_0^1 dx_1\, \frac{1}{[x_1 a + (1-x_1 b]^2}
\end{aligned}
\tag{C.20}
$$

in agreement with Eq. (C.10). The nice thing about this procedure is that it can generalized easily to obtain

$$
\frac{1}{a_1 a_2 \cdots a_n} =\Gamma(n) \int_0^\infty dx_1 \cdots \int_0^\infty dx_n \frac{\delta(1-x_1-\cdots-x_n)}{[a_1 x_1 + a_2 x_2 + \cdots + a_n x_n]^n} \tag{C.21}
$$

$$
=\Gamma(n) \int_0^1 dx_1 \cdots \int_0^{1-x_1\cdots x_{n-1}} dx_{n-1} \frac{1}{[a_1 x_1 + a_2 x_2 + \cdots + a_n(1-x_1\cdots x_{n-1})]^n}
$$

where the limits in the last equation can be understood by the fact that the delta function defines an hyperplane that constrains the variables. For instance consider the case of $n=3$. One gets the condition that defines a plane in the 3 dimensional space,

$$
1-x_1-x_2-x_3=0\ , \tag{C.22}
$$

as can be seen in Fig. C.2. As the x_i are positive, we immediately see that they obey, for the case

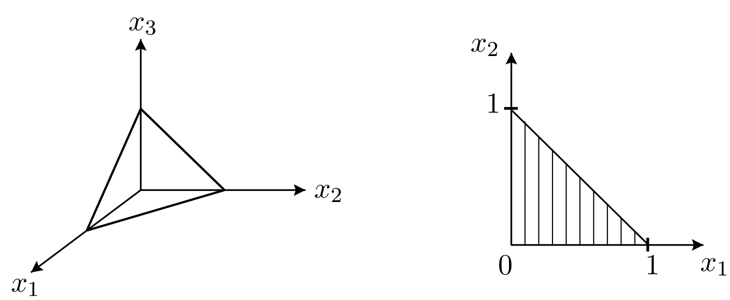


Figure C.2: Graphical representation of the constraint of Eq. (C.22) on the Feynman parameters. On the right panel the projection on the $x_1 x_2$ plane.

of n denominators,

$$
x_1 < 1,\ x_2 < 1-x_1,\ x_3 < 1-x_1-x_2,\ \cdots,\ x_{n-1} < 1-x_1-\cdots-x_{n-2} \tag{C.23}
$$

Before closing the section let us give an example that will be useful in the self-energy case. Consider the situation with the kinematics described in Fig. (C.3).

We get

$$
I \quad = \quad \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{\left[(k+p)^2 - m_1^2 + i\epsilon\right] \left[k^2 - m_2^2 + i\epsilon\right]}
$$

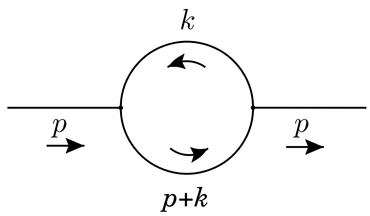


Figure C.3: Kinematics for the self-energy in ϕ^3.

$$
\begin{aligned}
&= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2 + 2p\cdot k\, x + p^2\, x - m_1^2\, x - m_2^2\,(1-x) + i\epsilon\right]^2} \\
&= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2 + 2P\cdot k - M^2 + i\epsilon\right]^2} \\
&= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[(k+P)^2 - P^2 - M^2 + i\epsilon\right]^2}
\end{aligned}
\tag{C.24}
$$

where in the last line we have completed the square in the term with the loop momenta k. The quantities P and M^2 are, in this case, defined by

$$P = xp \tag{C.25}$$

and

$$M^2 = -x\, p^2 + m_1^2\, x + m_2^2\,(1-x) \tag{C.26}$$

They depend on the masses, external momenta and Feynman parameters, but not in the loop momenta. Now changing variables $k \to k - P$ we get rid of the linear terms in k and finally obtain

$$I = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2 - C + i\epsilon\right]^2} \tag{C.27}$$

where C is independent of the loop momenta k and it is given by

$$C = P^2 + M^2 \tag{C.28}$$

Notice that the $i\epsilon$ factors will add correctly and can all be put as in Eq. (C.27).

C.3 Wick Rotation

From the example of the last section we can conclude that all the scalar integrals can be reduced to the form

$$I_{r,m} = \int \frac{d^dk}{(2\pi)^d} \frac{k^{2^r}}{\left[k^2 - C + i\epsilon\right]^m} \tag{C.29}$$

It is also easy to realize that also all the tensor integrals can be obtained from the scalar integrals. For instance

$$\begin{aligned}
&\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{[k^2 - C + i\epsilon]^m} = 0 \\
&\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{[k^2 - C + i\epsilon]^m} = \frac{1}{d} g^{\mu\nu} \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{[k^2 - C + i\epsilon]^m}
\end{aligned} \tag{C.30}$$

and so on. Therefore the integrals $I_{r,m}$ are the important quantities to evaluate. We will consider that $C > 0$. The case $C < 0$ can be done by analytical continuation of the final formula for $C > 0$.

To evaluate the integral $I_{r,m}$ we will use integration in the complex plane of the variable k^0 as described in Fig. C.4. We can then write

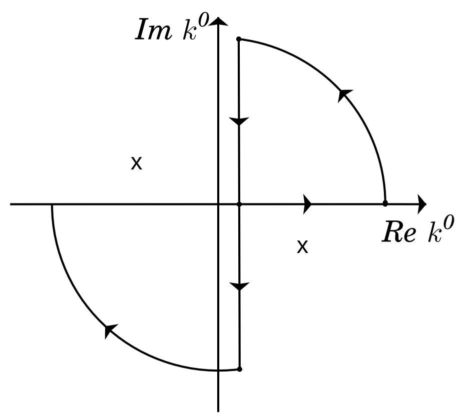


Figure C.4: Integration contour path for the Wick rotation.

$$I_{r,m} = \int \frac{d^{d-1} k}{(2\pi)^d} \int dk^0 \frac{k^{2^r}}{\left[k_0^2 - |\vec{k}|^2 - C + i\epsilon\right]^m} \tag{C.31}$$

The function under the integral has poles for

$$k^0 = \pm \left(\sqrt{|\vec{k}|^2 + C} - i\epsilon \right) \tag{C.32}$$

as shown in Fig. C.4. Using the properties of functions of complex variables (Cauchy theorem) we can deform the contour, changing the integration from the real to the imaginary axis plus the two arcs at infinity. This can be done because in deforming the contour we do not cross any pole. Notice the importance of the $i\epsilon$ prescription to be able to do this. The contribution from the arcs at infinity vanishes in dimension sufficiently low for the integral to converge, as we assume in dimensional regularization (see the details below in Complement C.2). This means that

$$\int_{-\infty}^{+\infty} dk^0 + \int_{+i\infty}^{-i\infty} dk^0 = 0 \Longrightarrow \int_{-\infty}^{+\infty} dk^0 = \int_{-i\infty}^{+i\infty} dk^0 \tag{C.33}$$

We can then change the integration along the real axis into an integration along the imaginary axis in the plane of the complex variable k^0. If we write

$$k^0 = ik_E^0 \qquad \text{com} \qquad \int_{-\infty}^{+\infty} dk^0 \to i \int_{-\infty}^{+\infty} dk_E^0 \tag{C.34}$$

and $k^2 = (k^0)^2 - |\vec{k}|^2 = -(k_E^0)^2 - |\vec{k}|^2 \equiv -k_E^2$, where $k_E = (k_E^0, \vec{k})$ is an euclidean vector. By this we mean that we calculate the scalar product using the euclidean metric diag$(+,+,+,+)$,

$$k_E^2 = (k_E^0)^2 + |\vec{k}|^2 \tag{C.35}$$

We can them write

$$I_{r,m} = i(-1)^{r-m} \int \frac{d^d k_E}{(2\pi)^d} \, \frac{k_E^{2r}}{\left[k_E^2 + C\right]^m} \tag{C.36}$$

where we do not need the $i\epsilon$ because the denominator is strictly positive ($C > 0$). This procedure is known as *Wick Rotation*. We note that the Feynman prescription for the propagators that originated the $i\epsilon$ rule for the denominators is crucial for the Wick rotation to be possible.

Complement C.2

In the argument that allowed for the Wick rotation it was claimed that the integrals over the circles at infinite vanish. Let us be more careful on this point. We just start with the simplest integral,

$$I_{0,m} = \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{\left[k^2 - C + i\epsilon\right]^m} \tag{C.37}$$

We begin by using the following representation for the denominator,

$$\frac{1}{[k^2 - C + i\epsilon]} = (-i) \int_0^\infty dz \, e^{-i\,z(C - k^2 - i\epsilon)} \tag{C.38}$$

which can verified by direct integration noticing the crucial role of the $i\epsilon$ prescription. This representation is related to the Schwinger proper time method [14]. Now we differentiate both sides with respect to C to obtain,

$$\frac{1}{\left[k^2 - C + i\epsilon\right]^m} = \frac{(-i)^m}{\Gamma(m)} \int_0^\infty dz \; z^{m-1} e^{-i\,z(C - k^2 - i\epsilon)} \tag{C.39}$$

Now introduce this in Eq. (C.37) and separate the integral in k^0. We get,

$$\begin{aligned}
I_{0,m} &= \int \frac{d^{d-1}k}{(2\pi)^d} \int dk^0 \; \frac{1}{\left[k^2 - C + i\epsilon\right]^m} \\
&= \int \frac{d^{d-1}k}{(2\pi)^d} \int dk^0 \; \frac{(-i)^m}{\Gamma(m)} \int_0^\infty dz \; z^{m-1} e^{-i\,z(C - k^2 - i\epsilon)} \\
&= \int \frac{d^{d-1}k}{(2\pi)^d} \frac{(-i)^m}{\Gamma(m)} \int_0^\infty dz \; z^{m-1} \int dk^0 \; e^{-i\,z(C - (k^0)^2 + \vec{k}\cdot\vec{k} - i\epsilon)}
\end{aligned} \tag{C.40}$$

We now go to the plane of complex $k^0 = |k^0|(\cos\theta + i\sin\theta)$. Therefore

$$(k^0)^2 = |k^0|^2\,(\cos 2\theta + i\sin 2\theta) \tag{C.41}$$

and the integral in k^0 will be

$$\int dk^0 \; e^{-i\,z(C-(k^0)^2+\vec{k}\cdot\vec{k}-i\epsilon)} = e^{-i\,z(C+\vec{k}\cdot\vec{k}-i\epsilon)} \int dk^0 e^{-z|k^0|^2\sin 2\theta} e^{-iz|k^0|^2\cos 2\theta} \tag{C.42}$$

and it will vanish in the circle at infinity for any value of θ. This shows that for $I_{0,m}$ we can perform the Wick rotation. This is also true for the general case of $I_{r,m}$ as the exponential vanishes faster than any power. This concludes the proof that we are allowed to perform the Wick rotation that lead to Eq. (C.36). We also note that the integration on the circles also vanish for finite values of $|k^0|$, as they are equal and with opposite signs.

C.4 Scalar integrals in dimensional regularization

We have seen in the last section that the scalar integrals to be calculated with dimensional regularization had the general form of Eq. (C.36). We are now going to find a general formula for $I_{r,m}$. We begin by writing

$$\int d^d k_E = \int d\overline{k}\; \overline{k}^{\;d-1}\, d\Omega_{d-1} \tag{C.43}$$

where $\overline{k} = \sqrt{(k_E^0)^2 + |\vec{k}|^2}$ is the length of the vector k_E in the euclidean space in d dimensions and $d\Omega_{d-1}$ is the solid angle that generalizes spherical coordinates in that euclidean space. The angles are defined by

$$k_E = \overline{k}(\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \ldots, \sin\theta_1\cdots\sin\theta_{d-1}) \tag{C.44}$$

We can then write

$$\int d\Omega_{d-1} = \int_0^{\pi} \sin\theta_1^{d-2}\, d\theta_1 \cdots \int_0^{2\pi} d\theta_{d-1} \tag{C.45}$$

Using now

$$\int_0^{\pi} \sin\theta^m\, d\theta = \sqrt{\pi}\; \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})} \tag{C.46}$$

where $\Gamma(z)$ is the gamma function (see section C.6) we get

$$\int d\Omega_{d-1} = 2\,\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{C.47}$$

The integration in $\overline{k}$ is done using the result

$$\int_0^{\infty} dx\; \frac{x^p}{(x^2+C)^m} = \frac{\Gamma\left(\frac{p+1}{2}\right) C^{\frac{1}{2}(p-2m+1)}\Gamma\left(-\frac{p}{2}+m-\frac{1}{2}\right)}{2\,\Gamma(m)}\;. \tag{C.48}$$

and we finally get

$$I_{r,m} = iC^{r-m+\frac{d}{2}} \frac{(-1)^{r-m}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(r+\frac{d}{2})}{\Gamma(\frac{d}{2})} \frac{\Gamma(m-r-\frac{d}{2})}{\Gamma(m)} \tag{C.49}$$

Before ending the section we note that the integral representation for $I_{r,m}$, Eq. (C.29), is valid only for $d < 2(m-r)$ to ensure convergence when $\overline{k} \to \infty$. However the final form in Eq. (C.49) can be analytically continued for all values of d except for those where the function $\Gamma(m-r-d/2)$ has poles, that is for (see section C.6),

$$m - r - \frac{d}{2} \neq 0, -1, -2, \ldots \tag{C.50}$$

For the application in dimensional regularization it is convenient to rewrite Eq. (C.49) using the relation $d = 4 - \epsilon$. We get

$$I_{r,m} = i\frac{(-1)^{r-m}}{(4\pi)^2} \left(\frac{4\pi}{C}\right)^{\frac{\epsilon}{2}} C^{2+r-m} \frac{\Gamma(2+r-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})} \frac{\Gamma(m-r-2+\frac{\epsilon}{2})}{\Gamma(m)} \tag{C.51}$$

C.5 Tensor integrals in dimensional regularization

We are frequently faced with the task of evaluating the tensor integrals of the form of Eq. (C.7),

$$\hat{T}_n^{\mu_1\cdots\mu_p} \equiv \int \frac{d^dk}{(2\pi)^d} \frac{k^{\mu_1}\cdots k^{\mu_p}}{D_0 D_1 \cdots D_{n-1}} \tag{C.52}$$

The first step is to reduce to one common denominator using the Feynman parameterization technique. The result is,

$$\begin{aligned} \hat{T}_n^{\mu_1\cdots\mu_p} &= \Gamma(n) \int_0^1 dx_1 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1} \int \frac{d^dk}{(2\pi)^d} \frac{k^{\mu_1}\cdots k^{\mu_p}}{[k^2+2k\cdot P - M^2 + i\epsilon]^n} \\ &= \Gamma(n) \int_0^1 dx_1 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1}\, I_n^{\mu_1\cdots\mu_p} \end{aligned} \tag{C.53}$$

where we have defined

$$I_n^{\mu_1\cdots\mu_p} \equiv \int \frac{d^dk}{(2\pi)^d} \frac{k^{\mu_1}\cdots k^{\mu_p}}{[k^2+2k\cdot P - M^2 + i\epsilon]^n} \tag{C.54}$$

that we call, from now on, the tensor integral. In principle all these integrals can be written in terms of scalar integrals. It is however convenient to have a general formula for them. We start with the result,

$$\begin{aligned} I_{0,n} =& \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2+2k\cdot P - M^2 + i\epsilon]^n} \\ =& \frac{i}{(4\pi)^{d/2}} (-1)^n \frac{\Gamma(n-d/2)}{\Gamma(n)} \left(\frac{1}{C}\right)^{n-d/2} \end{aligned} \tag{C.55}$$

where we used the result in Eq. (C.49) and use the definition of the Γ function,

$$\left(\frac{1}{C}\right)^z = \frac{1}{\Gamma(z)}\int_0^\infty dt\, t^{z-1}e^{-tC} \tag{C.56}$$

to write

$$\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{[k^2+2k\cdot P-M^2+i\epsilon]^n} = \frac{i}{(4\pi)^{d/2}}(-1)^n\frac{1}{\Gamma(n)}\int_0^\infty dt\, t^{n-1-d/2}e^{-tC} \tag{C.57}$$

Now we use

$$\frac{\partial}{\partial P^\mu}\,\frac{1}{[k^2+2k\cdot P-M^2+i\epsilon]^n} = -n\,\frac{2k_\mu}{[k^2+2k\cdot P-M^2+i\epsilon]^{n+1}} \tag{C.58}$$

to show that

$$\frac{k^{\mu_1}\cdots k^{\mu_p}}{[k^2+2k\cdot P-M^2+i\epsilon]^n} = \frac{(-1)^p}{2^p}\frac{\Gamma(n-p)}{\Gamma(n)}\frac{\partial}{\partial P_{\mu_1}}\cdots\frac{\partial}{\partial P_{\mu_p}}\frac{1}{[k^2+2k\cdot P-M^2+i\epsilon]^{n-p}} \tag{C.59}$$

We then use Eq. (C.57) to write

$$\begin{aligned}\int \frac{d^dk}{(2\pi)^d}\frac{1}{[k^2+2k\cdot P-M^2+i\epsilon]^{n-p}} =& \frac{i}{(4\pi)^{d/2}}(-1)^{n-p}\frac{1}{\Gamma(n-p)}\int_0^\infty dt\, t^{n-p-1-d/2}e^{-tC}\\ =& \frac{i}{16\pi^2}(-1)^{n-p}\frac{(4\pi)^{\epsilon/2}}{\Gamma(n-p)}\int_0^\infty dt\, t^{n-p-3+\epsilon/2}e^{-tC}\end{aligned} \tag{C.60}$$

Inserting Eq. (C.59) and Eq. (C.60) into Eq. (C.54) we finally get the result

$$I_n^{\mu_1\cdots\mu_p} = \frac{i}{16\pi^2}\frac{(4\pi)^{\epsilon/2}}{\Gamma(n)}(-1)^n\int_0^\infty \frac{dt}{(2t)^p}\, t^{n-3+\epsilon/2}\,\frac{\partial}{\partial P_{\mu_1}}\cdots\frac{\partial}{\partial P_{\mu_p}}\,e^{-t\,C} \tag{C.61}$$

where $C = P^2 + M^2$. After doing the derivatives the remaining integrals can be done using the properties of the Γ function (see section C.6). Notice that P, M^2 and therefore also C depend not only in the Feynman parameters but also in the exterior momenta. The advantage of having a general formula is that it can be programmed [15] and all the integrals can then be obtained automatically.

Complement C.3

The steps that lead to Eq. (C.59) and Eq. (C.60) might pose some questions when $n \le p$, as for this case the Gamma function has poles. The other question is how are these results related to those of section C.4? We will just give an example that illustrates this relation and shows that the final result in Eq. (C.61) is correct.

Consider, in the notation of Eq. (C.54), the integral

$$I_2^{\mu\nu} \equiv \int \frac{d^dk}{(2\pi)^d}\,\frac{k^\mu k^\nu}{[k^2+2k\cdot P-M^2+i\epsilon]^2} \tag{C.62}$$

that is $n = p = 2$. With the method of section C.4 we complete the square and shift the integration momentum $k \to k - P$. Then

$$I_2^{\mu\nu} = \int \frac{d^d k}{(2\pi)^d} \, \frac{k^\mu k^\nu}{\left[k^2 - C + i\epsilon\right]^2} + \int \frac{d^d k}{(2\pi)^d} \, \frac{P^\mu P^\nu}{\left[k^2 - C + i\epsilon\right]^2} \tag{C.63}$$

where we have used the fact that the odd terms in k vanish. We obtain therefore,

$$I_2^{\mu\nu} = \frac{1}{d} g^{\mu\nu} I_{1,2} + P^\mu P^\nu I_{0,2} \tag{C.64}$$

Now we use Eq. (C.51) and the properties of the Γ function (see section C.6) to obtain

$$I_{0,2} = \frac{i}{16\pi^2} \, \left[\Delta_\epsilon - \ln C + O(\epsilon)\right], \; \frac{1}{d} I_{1,2} = \frac{i}{16\pi^2} \frac{C}{2} \left[\Delta_\epsilon + 1 - \ln C + O(\epsilon)\right] \tag{C.65}$$

where

$$\Delta_\epsilon = \frac{2}{\epsilon} - \gamma + \ln 4\pi \tag{C.66}$$

Putting everything together we finally obtain,

$$I_2^{\mu\nu} = \frac{i}{16\pi^2} \frac{1}{2} \left[C g^{\mu\nu} (\Delta_\epsilon + 1 - \ln C) + 2(\Delta_\epsilon - \ln C) P^\mu P^\nu\right] + O(\epsilon) \tag{C.67}$$

We now use Eq. (C.61) that for our case reads

$$I_2^{\mu\nu} = \frac{i}{16\pi^2} \frac{(4\pi)^{\epsilon/2}}{\Gamma(2)} \int_0^\infty \frac{dt}{(2t)^2} \, t^{-1+\epsilon/2} \, \frac{\partial}{\partial P_\mu} \frac{\partial}{\partial P_\nu} \, e^{-t\,C} \tag{C.68}$$

Now

$$\frac{\partial}{\partial P_\mu} \frac{\partial}{\partial P_\nu} \, e^{-t\,C} = \left[(-2t) g^{\mu\nu} + (-2t)^2 P^\mu P^\nu\right] e^{-t\,C} \tag{C.69}$$

and therefore

$$\begin{aligned} I_2^{\mu\nu} =& \frac{i}{16\pi^2} \, (4\pi)^{\epsilon/2} \left[-\frac{1}{2} g^{\mu\nu} \int_0^\infty dt \; t^{-2+\epsilon/2} e^{-t\,C} + P^\mu P^\nu \int_0^\infty dt \; t^{-1+\epsilon/2} e^{-t\,C} \right] \\ =& \frac{i}{16\pi^2} \, (4\pi)^{\epsilon/2} \left[-\frac{1}{2} g^{\mu\nu} C^{1-\epsilon/2} \Gamma(-1 + \frac{\epsilon}{2}) + P^\mu P^\nu C^{-\epsilon/2} \Gamma(\frac{\epsilon}{2}) \right] \\ =& \frac{i}{16\pi^2} \, \frac{1}{2} \left[C g^{\mu\nu} (\Delta_\epsilon + 1 - \ln C) + 2(\Delta_\epsilon - \ln C) P^\mu P^\nu\right] + O(\epsilon) \end{aligned} \tag{C.70}$$

where we have used the definition of the Γ function, Eq. (C.72). This coincides exactly with what we have obtained before in Eq. (C.67).

C.6 Γ function and useful relations

The Γ function is defined by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{C.71}$$

or equivalently

$$\int_0^\infty t^{z-1} e^{-\mu t} dt = \mu^{-z}\Gamma(z) \tag{C.72}$$

The function $\Gamma(z)$ has the following important properties

$$\begin{aligned}\Gamma(z+1) &= z\Gamma(z) \\ \Gamma(n+1) &= n!\end{aligned} \tag{C.73}$$

Another related function is the logarithmic derivative of the Γ function, with the properties,

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) \tag{C.74}$$

$$\psi(1) = -\gamma \tag{C.75}$$

$$\psi(z+1) = \psi(z) + \frac{1}{z} \tag{C.76}$$

where γ is the Euler constant. The function $\Gamma(z)$ has poles for $z = 0, -1, -2, \cdots$. Near the pole $z = -m$ we have ($\epsilon \to 0$)

$$\Gamma(-m+\epsilon) = \frac{(-1)^m}{m!}\frac{1}{\epsilon} + \frac{(-1)^m}{m!}\psi(m+1) + O(\epsilon) \tag{C.77}$$

From this we conclude that when $\epsilon \to 0$

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} + \psi(1) + O(\epsilon) \qquad \Gamma(-n+\frac{\epsilon}{2}) = \frac{(-1)^n}{n!}\left[\frac{2}{\epsilon} + \psi(n+1)\right] \tag{C.78}$$

For positive integers the function $\Gamma(z)$ has no poles. But as we have to expand everything up to order ϵ, before making $\epsilon \to 0$, we need the expansion near the positive integers. Using the definition in Eq. (C.74) we get for a general n, up to order ϵ

$$\Gamma(n+\epsilon) = \Gamma(n) + \Gamma(n)\psi(n)\,\epsilon \tag{C.79}$$

giving, in particular,

$$\Gamma(1+\frac{\epsilon}{2}) = 1 - \gamma\,\frac{\epsilon}{2} + O(\epsilon^2) \tag{C.80}$$

Using these results we can expand our integrals in powers of ϵ and separate the divergent and finite parts. For instance for the one of the integrals of the self-energy,

$$\begin{aligned}I_{0,2} &= \frac{i}{(4\pi)^2}\left(\frac{4\pi}{C}\right)^{\frac{\epsilon}{2}}\,\Gamma(\frac{\epsilon}{2}) \\ &= \frac{i}{16\pi^2}\left[\frac{2}{\epsilon} - \gamma + \ln 4\pi - \ln C + O(\epsilon)\right]\end{aligned}$$

$$= \frac{i}{16\pi^2} \left[\Delta_\epsilon - \ln C + O(\epsilon)\right] \tag{C.81}$$

where we have introduced the notation

$$\Delta_\epsilon = \frac{2}{\epsilon} - \gamma + \ln 4\pi \tag{C.82}$$

for a combination that will appear in all expressions. In a similar way,

$$\begin{aligned} I_{1,2} =& \frac{i}{(4\pi)^2}\,(-1)\left(\frac{4\pi}{C}\right)^{\frac{\epsilon}{2}} C\,\frac{\Gamma(3-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})}\frac{\Gamma(-1+\frac{\epsilon}{2})}{\Gamma(2)} \\ =& \frac{i}{(4\pi)^2}\,2C\left[\Delta_\epsilon + \frac{1}{2} - \ln C\right] + O(\epsilon) \end{aligned} \tag{C.83}$$

C.7 Explicit formulæ for the *1–loop* integrals

Although we have presented in the previous sections the general formulæ for all the integrals that appear in *1–loop*, Eqs. (C.51) and (C.61), in practice it is convenient to have expressions for the most important cases with the expansion on the ϵ already done. The results presented below were generated with the `Mathematica` package `OneLoop` [15] from the general expressions. In these results the integration on the Feynman parameters has still to be done (see Eq. (C.53)). This is in general a difficult problem and we will present in section C.9 an alternative way of expressing these integrals more convenient for a numerical evaluation.

C.7.1 *Tadpole* integrals

With the definitions of Eqs. (C.51) and (C.61) we get

$$\begin{aligned} I_{0,1} &= \frac{i}{16\pi^2}\,C(1+\Delta_\epsilon - \ln C) \\ I_1^{\mu} &= 0 \\ I_1^{\mu\nu} &= \frac{i}{16\pi^2}\,\frac{1}{8}\,C^2\,g^{\mu\nu}(3+2\Delta_\epsilon - 2\ln C) \end{aligned} \tag{C.84}$$

where for the *tadpole* integrals

$$P = 0 \quad ; \quad C = m^2 \tag{C.85}$$

because there are no Feynman parameters and there is only one mass. In this case the above results are final.

C.7.2 *Self–Energy* integrals

For the integrals with two denominators we get,

$$
\begin{aligned}
I_{0,2} &= \frac{i}{16\pi^2}\,(\Delta_\epsilon - \ln C) \\
I_2^{\mu} &= \frac{i}{16\pi^2}\,(-\Delta_\epsilon + \ln C)P^\mu \\
I_2^{\mu\nu} &= \frac{i}{16\pi^2}\,\frac{1}{2}\,\Big[Cg^{\mu\nu}(1+\Delta_\epsilon - \ln C) + 2(\Delta_\epsilon - \ln C)P^\mu P^\nu\Big] \\
I_2^{\mu\nu\alpha} &= \frac{i}{16\pi^2}\,\frac{1}{2}\,\Big[-Cg^{\mu\nu}(1+\Delta_\epsilon - \ln C)P^\alpha - Cg^{\nu\alpha}(1+\Delta_\epsilon - \ln C)P^\mu \\
&\quad - Cg^{\mu\alpha}(1+\Delta_\epsilon - \ln C)P^\nu - 2(\Delta_\epsilon - \ln C)P^\alpha P^\mu P^\nu\Big]
\end{aligned}
\tag{C.86}
$$

where, with the notation and conventions of Fig. (C.1), we have

$$
P^\mu = x\,r_1^\mu \quad ; \quad C = x^2\,r_1^2 + (1-x)\,m_2^2 + x\,m_1^2 - x\,r_1^2 \tag{C.87}
$$

C.7.3 *Triangle* integrals

For the integrals with three denominators we get,

$$
\begin{aligned}
I_{0,3} &= \frac{i}{16\pi^2}\,\frac{-1}{2C} \\
I_3^{\mu} &= \frac{i}{16\pi^2}\,\frac{1}{2C}\,P^\mu \\
I_3^{\mu\nu} &= \frac{i}{16\pi^2}\,\frac{1}{4C}\,\Big[Cg^{\mu\nu}(\Delta_\epsilon - \ln C) - 2P^\mu P^\nu\Big] \\
I_3^{\mu\nu\alpha} &= \frac{i}{16\pi^2}\,\frac{1}{4C}\,\Big[Cg^{\mu\nu}(-\Delta_\epsilon + \ln C)P^\alpha + Cg^{\nu\alpha}(-\Delta_\epsilon + \ln C)P^\mu \\
&\quad + Cg^{\mu\alpha}(-\Delta_\epsilon + \ln C)P^\nu + 2P^\alpha P^\mu P^\nu\Big] \\
I_3^{\mu\nu\alpha\beta} &= \frac{i}{16\pi^2}\,\frac{1}{8C}\,\Big[C^2\,(1+\Delta_\epsilon - \ln C)\Big(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} + g^{\alpha\beta}g^{\mu\nu}\Big) \\
&\quad + 2C\,(\Delta_\epsilon - \ln C)\Big(g^{\mu\nu}P^\alpha P^\beta + g^{\nu\beta}P^\alpha P^\mu + g^{\nu\alpha}P^\beta P^\mu + g^{\mu\alpha}P^\beta P^\nu \\
&\quad + g^{\mu\beta}P^\alpha P^\nu + g^{\alpha\beta}P^\mu P^\nu\Big) - 4P^\alpha P^\beta P^\mu P^\nu\Big]
\end{aligned}
\tag{C.88}
$$

where

$$
\begin{aligned}
P^\mu &= x_1\, r_1^\mu + x_2\, r_2^\mu \\
C &= x_1^2\, r_1^2 + x_2^2\, r_2^2 + 2 x_1\, x_2\, r_1 \cdot r_2 + x_1\, m_1^2 + x_2\, m_2^2 \\
&\quad + (1 - x_1 - x_2)\, m_3^2 - x_1\, r_1^2 - x_2\, r_2^2
\end{aligned} \tag{C.89}
$$

C.7.4 *Box* integrals

$$
\begin{aligned}
I_{0,4} &= \frac{i}{16\pi^2}\,\frac{1}{6C^2} \\
I_4^\mu &= \frac{i}{16\pi^2}\,\frac{-1}{6C^2}\, P^\mu \\
I_4^{\mu\nu} &= \frac{i}{16\pi^2}\,\frac{-1}{12C^2}\left[C g^{\mu\nu} - 2P^\mu P^\nu\right] \\
I_4^{\mu\nu\alpha} &= \frac{i}{16\pi^2}\,\frac{1}{12C^2}\left[C\left(g^{\mu\nu} P^\alpha + g^{\nu\alpha} P^\mu + g^{\mu\alpha} P^\nu\right) - 2P^\alpha P^\mu P^\nu\right] \\
I_4^{\mu\nu\alpha\beta} &= \frac{i}{16\pi^2}\,\frac{1}{24C^2}\left[C^2\left(\Delta_\epsilon - \ln C\right)\left(g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} + g^{\alpha\beta} g^{\mu\nu}\right)\right. \\
&\quad - 2C\left(g^{\mu\nu} P^\alpha P^\beta + g^{\nu\beta} P^\alpha P^\mu + g^{\nu\alpha} P^\beta P^\mu + g^{\mu\alpha} P^\beta P^\nu\right. \\
&\quad \left.\left. + g^{\mu\beta} P^\alpha P^\nu + g^{\alpha\beta} P^\mu P^\nu\right) + 4P^\alpha P^\beta P^\mu P^\nu\right]
\end{aligned} \tag{C.90}
$$

where

$$
\begin{aligned}
P^\mu &= x_1\, r_1^\mu + x_2\, r_2^\mu + x_3\, r_3^\mu \\
C &= x_1^2\, r_1^2 + x_2^2\, r_2^2 + x_3^2\, r_3^2 + 2x_1\, x_2\, r_1 \cdot r_2 + 2x_1\, x_3\, r_1 \cdot r_3 + 2x_2\, x_3\, r_2 \cdot r_3 \\
&\quad + x_1\, m_1^2 + x_2\, m_2^2 + x_3\, m_3^2 + (1 - x_1 - x_2 - x_3)\, m_4^2 \\
&\quad - x_1\, r_1^2 - x_2\, r_2^2 - x_3\, r_3^2
\end{aligned} \tag{C.91}
$$

C.8 Divergent part of *1–loop* integrals

When we want to study the renormalization of a given theory it is often convenient to have expressions for the divergent part of the one-loop integrals, with the integration on the Feynman parameters already done. We present here the results for the most important cases. These divergent parts were calculated with the help of the package `OneLoop` [15]. The results are for the functions $\hat{T}_n^{\mu,\mu_2,\cdots\mu_n}$ defined in Eq. (C.52).

C.8.1 *Tadpole* integrals

$$\begin{aligned}
\mathrm{Div}\left[\hat{T}_1\right] &= \frac{i}{16\pi^2}\,\Delta_\epsilon\, m^2 \\
\mathrm{Div}\left[\hat{T}_1^\mu\right] &= 0 \\
\mathrm{Div}\left[\hat{T}_1^{\mu\nu}\right] &= \frac{i}{16\pi^2}\,\frac{1}{4}\,\Delta_\epsilon\, m^4\, g^{\mu\nu}
\end{aligned} \tag{C.92}$$

C.8.2 *Self–Energy* integrals

$$\begin{aligned}
\mathrm{Div}\left[\hat{T}_2\right] &= \frac{i}{16\pi^2}\,\Delta_\epsilon \\
\mathrm{Div}\left[\hat{T}_2^\mu\right] &= \frac{i}{16\pi^2}\left(-\frac{1}{2}\right)\Delta_\epsilon\, r_1^\mu \\
\mathrm{Div}\left[\hat{T}_2^{\mu\nu}\right] &= \frac{i}{16\pi^2}\,\frac{1}{12}\,\Delta_\epsilon\left[(3m_1^2+3m_2^2-r_1^2)g^{\mu\nu}+4r_1^\mu r_1^\nu\right] \\
\mathrm{Div}\left[\hat{T}_2^{\mu\nu\alpha}\right] &= \frac{i}{16\pi^2}\left(-\frac{1}{24}\right)\Delta_\epsilon\left[(4m_1^2+2m_2^2-r_1^2)\,(g^{\mu\nu}r_1^\alpha+g^{\nu\alpha}r_1^\mu+g^{\mu\alpha}r_1^\nu)\right. \\
&\qquad\qquad \left.+\,6\,r_1^\alpha r_1^\mu r_1^\nu\right]
\end{aligned} \tag{C.93}$$

C.8.3 *Triangle* integrals

$$\begin{aligned}
\mathrm{Div}\left[\hat{T}_3\right] &= 0 \\
\mathrm{Div}\left[\hat{T}_3^\mu\right] &= 0 \\
\mathrm{Div}\left[\hat{T}_3^{\mu\nu}\right] &= \frac{i}{16\pi^2}\,\frac{1}{4}\,\Delta_\epsilon\, g^{\mu\nu} \\
\mathrm{Div}\left[\hat{T}_3^{\mu\nu\alpha}\right] &= \frac{i}{16\pi^2}\left(-\frac{1}{12}\right)\Delta_\epsilon\left[g^{\mu\nu}(r_1^\alpha+r_2^\alpha)+g^{\nu\alpha}(r_1^\mu+r_2^\mu)+g^{\mu\alpha}(r_1^\nu+r_2^\nu)\right] \\
\mathrm{Div}\left[\hat{T}_3^{\mu\nu\alpha\beta}\right] &= \frac{i}{16\pi^2}\,\frac{1}{48}\,\Delta_\epsilon\left[(2m_1^2+2m_2^2+2m_3^2)\left(g^{\mu\alpha}g^{\nu\beta}+g^{\alpha\beta}g^{\mu\nu}+g^{\mu\beta}g^{\nu\alpha}\right)\right. \\
&\quad +g^{\alpha\beta}\left[2r_1^\mu r_1^\nu+r_1^\mu r_2^\nu+(r_1\leftrightarrow r_2)\right]+g^{\mu\beta}\left[2r_1^\alpha r_1^\nu+r_1^\alpha r_2^\nu+(r_1\leftrightarrow r_2)\right] \\
&\quad +g^{\nu\beta}\left[2r_1^\alpha r_1^\mu+r_1^\alpha r_2^\mu+(r_1\leftrightarrow r_2)\right]+g^{\mu\nu}\left[2r_1^\alpha r_1^\beta+r_1^\alpha r_2^\beta+(r_1\leftrightarrow r_2)\right] \\
&\quad +g^{\mu\alpha}\left[2r_1^\beta r_1^\nu+r_1^\beta r_2^\nu+(r_1\leftrightarrow r_2)\right]+g^{\nu\alpha}\left[2r_1^\beta r_1^\mu+r_1^\beta r_2^\mu+(r_1\leftrightarrow r_2)\right]
\end{aligned}$$

$$+\left(-r_1^2+r_1\cdot r_2-r_2^2\right)\left(g^{\mu\alpha}g^{\nu\beta}+g^{\alpha\beta}g^{\mu\nu}+g^{\mu\beta}g^{\nu\alpha}\right)\Big] \tag{C.94}$$

C.8.4 *Box* integrals

$$\begin{aligned}\text{Div}\left[\hat{T}_4\right] &= \text{Div}\left[\hat{T}_4^{\mu}\right]=\text{Div}\left[\hat{T}_4^{\mu\nu}\right]=\text{Div}\left[\hat{T}_4^{\mu\nu\alpha}\right]=0\\ \text{Div}\left[\hat{T}_4^{\mu\nu\alpha\beta}\right] &= \frac{i}{16\pi^2}\frac{1}{24}\Delta_\epsilon\left[g^{\mu\nu}g^{\alpha\beta}+g^{\mu\beta}g^{\alpha\nu}+g^{\mu\alpha}g^{\nu\beta}\right]\end{aligned} \tag{C.95}$$

C.9 Passarino-Veltman Integrals

C.9.1 The general definition

The description of the previous sections works well if one just wants to calculate the divergent part of a diagram or to show the cancellation of divergences in a set of diagrams. If one actually wants to numerically calculate the integrals the task is normally quite complicated. Except for the *self-energy* type of diagrams the integration over the Feynman parameters is normally quite difficult.

To overcome this problem a scheme was first proposed by Passarino and Veltman [16]. These scheme with the conventions of [17, 18] was latter implemented in the `Mathematica` package `FeynCalc` [18, 19] and, for numerical evaluation, in the `LoopTools` package [20]. The numerical evaluation follows the code developed earlier by van Oldenborgh [21].

We will now describe this scheme. We will write the generic one-loop tensor integral as

$$T_n^{\mu_1\cdots\mu_p}\equiv\frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\;\frac{k^{\mu_1}\cdots k^{\mu_p}}{D_0D_1D_2\cdots D_{n-1}} \tag{C.96}$$

where we follow for the momenta the conventions of section C.2 and Fig. C.1 and defined $D_0\equiv D_n$ and $m_n=m_0$ so that $D_0=k^2-m_0^2$ (remember that $r_n\equiv r_0=0$. The main difference between this definition and the previous one Eq. (C.7) is that a factor of $\frac{i}{16\pi^2}$ is taken out. This is because, as we have seen in section C.3 these integrals always give that prefactor. So with our new convention that prefactor **has** to included in the end. Factoring out the i has also the convenience of dealing with real functions in many cases.[2] From all those integrals in Eq. (C.96) the scalar integrals are, has we have seen, of particular importance and deserve a special notation. It can be shown that there are only four independent such integrals, namely ($4-d=\epsilon$)

$$A_0(m_0^2)=\frac{(2\pi\mu)^\epsilon}{i\pi^2}\int d^dk\,\frac{1}{k^2-m_0^2} \tag{C.97}$$

$$B_0(r_{10}^2,m_0^2,m_1^2)=\frac{(2\pi\mu)^\epsilon}{i\pi^2}\int d^dk\prod_{i=0}^{1}\frac{1}{\left[(k+r_i)^2-m_i^2\right]} \tag{C.98}$$

[2] The one loop functions are in general complex, but in some cases they can be real. These cases correspond to the situation where cutting the diagram does not corresponding to a kinematically allowed process.

$$C_0(r_{10}^2, r_{12}^2, r_{20}^2, m_0^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^dk \prod_{i=0}^{2} \frac{1}{[(k+r_i)^2 - m_i^2]} \quad \text{(C.99)}$$

$$D_0(r_{10}^2, r_{12}^2, r_{23}^2, r_{30}^2, r_{20}^2, r_{13}^2, m_0^2, m_1^2, \ldots, m_3^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^dk \prod_{i=0}^{3} \frac{1}{[(k+r_i)^2 - m_i^2]} \quad \text{(C.100)}$$

where

$$r_{ij}^2 = (r_i - r_j)^2 \quad ; \quad \forall\, i,j = (0, n-1) \quad \text{(C.101)}$$

Remember that with our conventions $r_0 = 0$ so $r_{i0}^2 = r_i^2$. In all these expressions the $i\epsilon$ part of the denominator factors is suppressed. The general one-loop tensor integrals are not independent. Their decomposition is not unique. We follow the conventions of [18, 20] to write

$$B^\mu \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^\mu \prod_{i=0}^{1} \frac{1}{[(k+r_i)^2 - m_i^2]} \quad \text{(C.102)}$$

$$B^{\mu\nu} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^\mu k^\nu \prod_{i=0}^{1} \frac{1}{[(k+r_i)^2 - m_i^2]} \quad \text{(C.103)}$$

$$C^\mu \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^\mu \prod_{i=0}^{2} \frac{1}{[(k+r_i)^2 - m_i^2]} \quad \text{(C.104)}$$

$$C^{\mu\nu} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^\mu k^\nu \prod_{i=0}^{2} \frac{1}{[(k+r_i)^2 - m_i^2]} \quad \text{(C.105)}$$

$$C^{\mu\nu\rho} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^\mu k^\nu k^\rho \prod_{i=0}^{2} \frac{1}{[(k+r_i)^2 - m_i^2]} \quad \text{(C.106)}$$

$$D^\mu \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^\mu \prod_{i=0}^{3} \frac{1}{[(k+r_i)^2 - m_i^2]} \quad \text{(C.107)}$$

$$D^{\mu\nu} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^\mu k^\nu \prod_{i=0}^{3} \frac{1}{[(k+r_i)^2 - m_i^2]} \quad \text{(C.108)}$$

$$D^{\mu\nu\rho} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^\mu k^\nu k^\rho \prod_{i=0}^{3} \frac{1}{[(k+r_i)^2 - m_i^2]} \quad \text{(C.109)}$$

$$D^{\mu\nu\rho\sigma} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^\mu k^\nu k^\rho k^\sigma \prod_{i=0}^{3} \frac{1}{[(k+r_i)^2 - m_i^2]} \quad \text{(C.110)}$$

These integrals can be decomposed in terms of (reducible) functions in the following way:

$$B^\mu = r_1^\mu B_1 \quad \text{(C.111)}$$

$$B^{\mu\nu} = g^{\mu\nu} B_{00} + r_1^\mu r_1^\nu B_{11} \quad \text{(C.112)}$$

$$C^\mu = r_1^\mu C_1 + r_2^\mu C_2 \quad \text{(C.113)}$$

$$C^{\mu\nu} = g^{\mu\nu} C_{00} + \sum_{i=1}^{2} r_i^\mu r_j^\nu C_{ij} \quad \text{(C.114)}$$

$$C^{\mu\nu\rho} = \sum_{i=1}^{2} \left(g^{\mu\nu} r_i^\rho + g^{\nu\rho} r_i^\mu + g^{\rho\mu} r_i^\nu\right) C_{00i} + \sum_{i,j,k=1}^{2} r_i^\mu r_j^\nu r_k^\rho \, C_{ijk} \tag{C.115}$$

$$D^\mu = \sum_{i=1}^{3} r_i^\mu \, D_i \tag{C.116}$$

$$D^{\mu\nu} = g^{\mu\nu} \, D_{00} + \sum_{i,j=1}^{3} r_i^\mu r_j^\nu \, D_{ij} \tag{C.117}$$

$$D^{\mu\nu\rho} = \sum_{i=1}^{3} \left(g^{\mu\nu} r_i^\rho + g^{\nu\rho} r_i^\mu + g^{\rho\mu} r_i^\nu\right) D_{00i} + \sum_{i,j,k=1}^{3} r_i^\mu r_j^\nu r_k^\rho \, D_{ijk} \tag{C.118}$$

$$\begin{aligned} D^{\mu\nu\rho\sigma} = {} & \left(g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}\right) D_{0000} \\ & + \sum_{i,j=1}^{3} \Big(g^{\mu\nu} r_i^\rho r_j^\sigma + g^{\nu\rho} r_i^\mu r_j^\sigma + g^{\mu\rho} r_i^\nu r_j^\sigma + g^{\mu\sigma} r_i^\nu r_j^\rho \qquad \text{(C.119)} \\ & \qquad + g^{\nu\sigma} r_i^\mu r_j^\rho + g^{\rho\sigma} r_i^\mu r_j^\nu \Big) \, D_{00ij} \\ & + \sum_{i,j,k,l=1}^{3} r_i^\mu r_j^\nu r_k^\rho r_l^\sigma \, D_{ijkl} \qquad \text{(C.120)} \end{aligned}$$

All coefficient functions have the same arguments as the corresponding scalar functions and are totally symmetric in their indices. In the `FeynCalc` [22] package one generic notation is used,

$$\mathrm{PaVe}\left[\mathrm{i}, \mathrm{j}, \ldots, \{\mathrm{r}_{10}^2, \mathrm{r}_{12}^2, \ldots\}, \{\mathrm{m}_0^2, \mathrm{m}_1^2, \ldots\}\right] \tag{C.121}$$

for instance

$$B_{11}(r_{10}^2, m_0^2, m_1^2) = \mathrm{PaVe}\left[1, 1, \{\mathrm{r}_{10}^2\}, \{\mathrm{m}_0^2, \mathrm{m}_1^2\}\right] \tag{C.122}$$

All these coefficient functions are not independent and can be reduced to the scalar functions. `FeynCalc` provides the command `PaVeREduce[...]` to accomplish that. This is very useful if one wants to check for cancellation of divergences or for gauge invariance where a number of diagrams have to cancel.

C.9.2 The divergences

The package `LoopTools` provides ways to numerically check for the cancellation of divergences. However it is useful to know the divergent part of the Passarino-Veltman integrals. Only a small number of these integrals are divergent. They are

$$\mathrm{Div}\left[\mathrm{A}_0(\mathrm{m}_0^2)\right] = \Delta_\epsilon \, m_0^2 \tag{C.123}$$

$$\mathrm{Div}\left[\mathrm{B}_0(\mathrm{r}_{10}^2, \mathrm{m}_0^2, \mathrm{m}_1^2)\right] = \Delta_\epsilon \tag{C.124}$$

$$\mathrm{Div}\left[\mathrm{B}_1(\mathrm{r}_{10}^2, \mathrm{m}_0^2, \mathrm{m}_1^2)\right] = -\frac{1}{2} \, \Delta_\epsilon \tag{C.125}$$

$$\mathrm{Div}\left[\mathrm{B}_{00}(\mathrm{r}_{10}^2, \mathrm{m}_0^2, \mathrm{m}_1^2)\right] = \frac{1}{12} \, \Delta_\epsilon \, \left(3m_0^2 + 3m_1^2 - r_{10}^2\right) \tag{C.126}$$

$$\mathrm{Div}\left[\mathrm{B}_{11}(\mathrm{r}_{10}^2, \mathrm{m}_0^2, \mathrm{m}_1^2)\right] = \frac{1}{3} \, \Delta_\epsilon \tag{C.127}$$

$$\begin{aligned}
\mathrm{Div}\left[\mathrm{C}_{00}(\mathrm{r}_{10}^2,\mathrm{r}_{12}^2,\mathrm{r}_{20}^2,\mathrm{m}_0^2,\mathrm{m}_1^2,\mathrm{m}_2^2)\right] &= \frac{1}{4}\,\Delta_\epsilon && \text{(C.128)}\\
\mathrm{Div}\left[\mathrm{C}_{001}(\mathrm{r}_{10}^2,\mathrm{r}_{12}^2,\mathrm{r}_{20}^2,\mathrm{m}_0^2,\mathrm{m}_1^2,\mathrm{m}_2^2)\right] &= -\frac{1}{12}\,\Delta_\epsilon && \text{(C.129)}\\
\mathrm{Div}\left[\mathrm{C}_{002}(\mathrm{r}_{10}^2,\mathrm{r}_{12}^2,\mathrm{r}_{20}^2,\mathrm{m}_0^2,\mathrm{m}_1^2,\mathrm{m}_2^2)\right] &= -\frac{1}{12}\,\Delta_\epsilon && \text{(C.130)}\\
\mathrm{Div}\left[\mathrm{D}_{0000}(\mathrm{r}_{10}^2,\ldots,\mathrm{m}_0^2,\ldots)\right] &= \frac{1}{24}\,\Delta_\epsilon && \text{(C.131)}\\
& && \text{(C.132)}
\end{aligned}$$

These results were obtained with the package LoopTools, after reducing to the scalar integrals with the command `PaVeReduce`, but they can be verified by comparing with our results of section C.8, after factoring out the $i/(16\pi^2)$.

C.9.3 Useful results for PV integrals

Although the PV approach is intended primarily to be used numerically there are situations where one wants to have explicit results. These can be useful to check cancellation of divergences or because in some simple cases the integrals can be done analytically. We note that as our conventions for the momenta are the same in sections C.9 and C.7 one can read immediately the integral representation of the PV in terms of the Feynman parameters just by comparing both expressions, not forgetting to take out the $i/(16\pi^2)$ factor. For instance, from Eq. (C.114) for $C^{\mu\nu}$ and Eq. (C.88) for $I_3^{\mu\nu}$ we get

$$C_{12}(r_1^2,r_{12}^2,r_2^2,m_0^2,m_1^2,m_2^2) = -\Gamma(3)\frac{2}{4}\int_0^1 dx_1\int_0^{1-x_1} dx_2\,\frac{x_1x_2}{\mathrm{C}} \tag{C.133}$$

with

$$\begin{aligned}
C &= x_1^2\,r_1^2 + x_2^2\,r_2^2 + x_1\,x_2\,(r_1^2+r_2^2-r_{12}^2) + x_1\,m_1^2 + x_2\,m_2^2\\
&\quad +(1-x_1-x_2)\,m_0^2 - x_1\,r_1^2 - x_2\,r_2^2
\end{aligned} \tag{C.134}$$

Explicit expression for A_0

This integral is trivial. There is no Feynman parameter and the integral can be read from Eq. (C.84). We get, after factoring out the $i/(16\pi^2)$,

$$A_0(m^2) = m^2\left(\Delta_\epsilon + 1 - \ln\frac{m^2}{\mu^2}\right) \tag{C.135}$$

Explicit expressions for the B functions

Function B_0

The general form of the integral $B_0(p^2,m_1^2,m_2^2)$ can be read from Eq. (C.86). We obtain

$$B_0(p^2, m_0^2, m_1^2) = \Delta_\epsilon - \int_0^1 dx \ln\left[\frac{-x(1-x)p^2 + xm_1^2 + (1-x)m_0^2}{\mu^2}\right] \tag{C.136}$$

From this expression one can easily get the following results,

$$B_0(0, m_0^2, m_1^2) = \Delta_\epsilon + 1 - \frac{m_0^2 \ln \frac{m_0^2}{\mu^2} - m_1^2 \ln \frac{m_1^2}{\mu^2}}{m_0^2 - m_1^2} \tag{C.137}$$

$$B_0(0, m_0^2, m_1^2) = \frac{A_0(m_0^2) - A_0(m_1^2)}{m_0^2 - m_1^2} \tag{C.138}$$

$$B_0(0, m^2, m^2) = \Delta_\epsilon - \ln \frac{m^2}{\mu^2} = \frac{A_0(m^2)}{m^2} - 1 \tag{C.139}$$

$$B_0(m^2, 0, m^2) = \Delta_\epsilon + 2 - \ln \frac{m^2}{\mu^2} = \frac{A_0(m^2)}{m^2} + 1 \tag{C.140}$$

$$B_0(0, 0, m^2) = \Delta_\epsilon + 1 - \ln \frac{m^2}{\mu^2} = \frac{A_0(m^2)}{m^2} \tag{C.141}$$

Function B_0'

The derivative of the B_0 function with respect to p^2 appears many times. From Eq. (C.136) one can derive an integral representation,

$$B_0'(p^2, m_0^2, m_1^2) = \int_0^1 dx \, \frac{x(1-x)}{-p^2 x(1-x) + xm_1^2 + (1-x)m_0^2} \tag{C.142}$$

An important particular case corresponds to $B_0'(m^2, m_0^2, m^2)$ that appears in the self-energy of the electron. In this case m is the electron mass and $m_0 = \lambda$ is the photon mass that one has to introduce to regularize the IR divergent integral. The integral in this case reduces to

$$\begin{aligned} B_0'(m^2, \lambda^2, m^2) &= \int_0^1 dx \, \frac{x(1-x)}{m^2 x^2 + (1-x)\lambda^2} \\ &= -\frac{1}{m^2} - \frac{1}{2m^2} \ln \frac{\lambda^2}{m^2} \end{aligned} \tag{C.143}$$

It is clear that in the limit $\lambda \to 0$ this integral diverges. Another limit that it is useful (for instance is needed in the vacuum polarization, see section C.10.1), is

$$B_0'(0, m^2, m^2) = \frac{1}{6m^2} \tag{C.144}$$

that can be easily obtained from Eq. (C.142).

Function B_1

The explicit expression can be read from Eq. (C.86). We have

$$B_1(p^2, m_0^2, m_1^2) = -\frac{1}{2}\Delta_\epsilon + \int_0^1 dx x \ln\left[\frac{-x(1-x)p^2 + xm_1^2 + (1-x)m_0^2}{\mu^2}\right] \tag{C.145}$$

For $p^2 = 0$ this integral can be easily evaluated to give

$$B_1(0, m_0^2, m_1^2) = -\frac{1}{2}\Delta_\epsilon + \frac{1}{2}\ln\left(\frac{m_0^2}{\mu^2}\right) + \frac{-3 + 4t - t^2 - 4t\ln t + 2t^2\ln t}{4(-1+t)^2} \tag{C.146}$$

where we defined

$$t = \frac{m_1^2}{m_0^2} \tag{C.147}$$

From Eq. (C.146) one can shown that even for $p^2 = 0$ B_1 is **not** a symmetric function of the masses,

$$B_1(p^2, m_0^2, m_1^2) \neq B_1(p^2, m_1^2, m_0^2) \tag{C.148}$$

As this might appear strange let us show with one example how the coefficient functions are tied to our conventions about the order of the momenta and Feynman parameters. Let us consider the contribution to the self-energy of a fermion of mass m_f of the exchange of a scalar with mass m_s. We can consider the two choices in Fig. C.5,

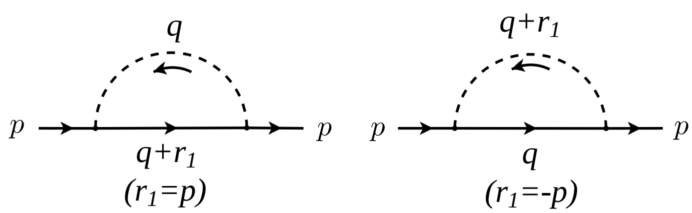


Figure C.5:

Now with the first choice (diagram on the left of Fig. C.5) we have

$$\begin{aligned} -i\Sigma_1 &= \frac{i}{16\pi^2}\left[(\not{p} + m_f)B_0(p^2, m_s^2, m_f^2) + \not{p}B_1(p^2, m_s^2, m_f^2)\right] \\ &= \frac{i}{16\pi^2}\left[\not{p}\left(B_0(p^2, m_s^2, m_f^2) + B_1(p^2, m_s^2, m_f^2)\right) + m_f B_0(p^2, m_s^2, m_f^2)\right] \end{aligned} \tag{C.149}$$

while with the second choice we have

$$-i\Sigma_2 = \frac{i}{16\pi^2}\left[-\not{p}B_1(p^2, m_f^2, m_s^2) + m_f B_0(p^2, m_f^2, m_s^2)\right] \tag{C.150}$$

How can these two expressions be equal? The reason has precisely to do with the non symmetry of B_1 with respect to the mass entries. In fact from Eq. (C.145) we have

$$
\begin{aligned}
B_1(p^2,m_0^2,m_1^2) &= -\frac{1}{2}\Delta_\epsilon + \int_0^1 dx\, x \ln\left[\frac{-x(1-x)p^2 + x m_1^2 + (1-x)m_0^2}{\mu^2}\right] \\
&= -\frac{1}{2}\Delta_\epsilon + \int_0^1 dx (1-x) \ln\left[\frac{-x(1-x)p^2 + (1-x) m_1^2 + x m_0^2}{\mu^2}\right] \\
&= -\frac{1}{2}\Delta_\epsilon + \left(\Delta_\epsilon - B_0(p^2,m_1^2,m_0^2)\right) - \left(\frac{1}{2}\Delta_\epsilon + B_1(p^2,m_1^2,m_0^2)\right) \\
&= -\left(B_0(p^2,m_1^2,m_0^2) + B_1(p^2,m_1^2,m_0^2)\right) \qquad \text{(C.151)}
\end{aligned}
$$

where we have changed variables ($x \to 1-x$) in the integral and used the definitions of B_0 and B_1. We have then, remembering that $B_0(p^2,m_s^2,m_f^2) = B_0(p^2,m_f^2,m_s^2)$,

$$
B_1(p^2,m_f^2,m_s^2) = -\left(B_0(p^2,m_s^2,m_f^2) + B_1(p^2,m_s^2,m_f^2)\right) \qquad \text{(C.152)}
$$

and therefore Eqs. (C.149) and (C.150) are equivalent.

Explicit expressions for the C functions

In Eq. (C.133) we have already given the general form of C_{12}. The other functions are very similar. In the following we just present the results for the particular case of $p^2 = 0$. This case is important in many situations where it is a good approximation to neglect the external momenta in comparison with the masses of the particles in the loop. We also warn the reader that the coefficient functions C_i, C_{ij} obtained from `LoopTools` are not well defined in this limit. Hence there is some utility in given them here.

Function C_0

$$
\begin{aligned}
C_0(0,0,0,m_0^2,m_1^2,m_2^2) &= -\Gamma(3)\frac{1}{2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{x_1 m_1^2 + x_2 m_2^2 + (1-x_1-x_2)m_0^2} \\
&= -\frac{1}{m_0^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{x_1 t_1 + x_2 t_2 + (1-x_1-x_2)} \\
&= -\frac{1}{m_0^2}\frac{-t_1\ln t_1 + t_1 t_2 \ln t_1 + t_2 \ln t_2 - t_1 t_2 \ln t_2}{(-1+t_1)(t_1-t_2)(-1+t_2)} \qquad \text{(C.153)}
\end{aligned}
$$

where

$$
t_1 = \frac{m_1^2}{m_0^2} \quad ; \quad t_2 = \frac{m_2^2}{m_0^2} \qquad \text{(C.154)}
$$

Using the properties of the logarithms one can show that in this limit C_0 is a symmetric function of the masses. This expression is further simplified when two of the masses are equal, as it happens in the $\mu \to e\gamma$ problem. Then $t = t_1 = t_2$,

$$
C_0(0,0,0,m_0^2,m_1^2,m_1^2) = -\frac{1}{m_0^2}\frac{-1+t-\ln t}{(-1+t)^2} \qquad \text{(C.155)}
$$

in agreement with Eq.(20) of [23]. In the case of equal masses for all the loop particles we have

$$C_0(0,0,0,m_0^2,m_0^2,m_0^2) = -\frac{1}{2m_0^2} \tag{C.156}$$

Before we close this section on C_0 there is another particular case when it is useful to have an explicit case for it. This in the case when it is IR divergent as in the QED vertex. The function needed is $C_0(m^2,m^2,0,m^2,\lambda^2,m^2)$. Using the definition we have

$$\begin{aligned}
C_0(m^2,m^2,0,m^2,\lambda^2,m^2) &= -\int_0^1 dx_1 \int_0^{1-x1} dx_2 \frac{1}{m^2(1-2x_1+x_1^2)+x_1\lambda^2} \\
&= -\int_0^1 dx_1 \frac{1-x_1}{m^2(1-x_1)^2+x_1\lambda^2} \\
&= -\int_0^1 dx \frac{x}{m^2x^2+(1-x)\lambda^2} \\
&= \frac{1}{2m^2}\ln\frac{\lambda^2}{m^2} = -B_0'(m^2,\lambda^2,m^2) - \frac{1}{m^2}
\end{aligned} \tag{C.157}$$

We have verified numerically, using `LoopTools`[20, 21], that Eqs. (C.157), (C.143) and (C.144) are verified.

Function C_{00}

$$\begin{aligned}
C_{00}(0,0,0,m_0^2,m_1^2,m_2^2) &= \Gamma(3)\frac{1}{4}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left[\Delta_\epsilon - \ln\left(\frac{C}{\mu^2}\right)\right] \\
&= \frac{1}{4}\Delta_\epsilon - \frac{1}{2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ln\left[\frac{x_1m_1^2+x_2m_2^2+(1-x_1-x_2)m_0^2}{\mu^2}\right] \\
&= \frac{1}{4}\left(\Delta_\epsilon - \ln\frac{m_0^2}{\mu^2}\right) + \frac{3}{8} - \frac{t_1^2}{4(t_1-1)(t_1-t_2)}\ln t_1 \\
&\quad + \frac{t_2^2}{4(t_2-1)(t_1-t_2)}\ln t_2
\end{aligned} \tag{C.158}$$

where, as before

$$t_1 = \frac{m_1^2}{m_0^2} \quad ; \quad t_2 = \frac{m_2^2}{m_0^2} \tag{C.159}$$

Using the properties of the logarithms one can show that in this limit C_{00} is a symmetric function of the masses. This expression is further simplified when two of the masses are equal. Then $t=t_1=t_2$,

$$\begin{aligned}
C_{00}(0,0,0,m_0^2,m_1^2,m_1^2) &= \frac{1}{4}\left(\Delta_\epsilon - \ln\frac{m_0^2}{\mu^2}\right) - \frac{-3+4t-t^2-4t\,\ln t+2t^2\,\ln t}{8(t-1)^2} \\
&= -\frac{1}{2}B_1(0,m_0^2,m_1^2)
\end{aligned} \tag{C.160}$$

Functions C_i and C_{ij}

We recall that the definition of the coefficient functions is not unique, it is tied to a particular convention for assigning the loop momenta and Feynman parameters, as shown in Fig. C.1. For the particular case of the C functions we show our conventions in Fig. C.6.

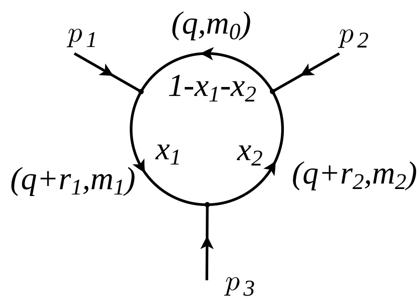


Figure C.6:

With the same techniques we obtain,

$$
\begin{aligned}
C_1(0,0,0,m_0^2,m_1^2,m_2^2) &= \frac{1}{m_0^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1}{x_1 t_1 + x_2 t_2 + (1-x_1-x_2)} \\
&= -\frac{1}{m_0^2}\left[\frac{t_1}{2(-1+t_1)(t_1-t_2)} - \frac{t_1(t_1-2t_2+t_1 t_2)}{2(-1+t_1)^2(t_1-t_2)^2}\ln t_1 \right. \\
&\quad \left. + \frac{{t_2}^2 - 2t_1{t_2}^2 + {t_1}^2{t_2}^2}{2(-1+t_1)^2(t_1-t_2)^2(-1+t_2)}\ln t_2\right] \qquad \text{(C.161)}
\end{aligned}
$$

$$
\begin{aligned}
C_2(0,0,0,m_0^2,m_1^2,m_2^2) &= \frac{1}{m_0^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_2}{x_1 t_1 + x_2 t_2 + (1-x_1-x_2)} \\
&= -\frac{1}{m_0^2}\left[-\frac{t_2}{2(t_1-t_2)(-1+t_2)} + \frac{\ln t_1}{2(-1+t_1)(-1+t_2)^2} \right. \\
&\quad \left. + \frac{2t_1 t_2 - 2{t_1}^2 t_2 - {t_2}^2 + {t_1}^2{t_2}^2}{2(-1+t_1)(t_1-t_2)^2(-1+t_2)^2}\ln\left(\frac{t_1}{t_2}\right)\right] \qquad \text{(C.162)}
\end{aligned}
$$

$$
C_{ij}(0,0,0,m_0^2,m_1^2,m_2^2) = -\frac{1}{m_0^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_i x_j}{x_1 t_1 + x_2 t_2 + (1-x_1-x_2)} \qquad \text{(C.163)}
$$

where we have not written explicitly the C_{ij} for $i,j=1,2$ because they are rather lengthy. However a simple `Fortran` program can be developed [15] to calculate all the three point functions in the zero external limit case. This is useful because in this case some of the functions from `LoopTools` will fail. Notice that the C_i and C_{ij} functions are not symmetric in their arguments. This a consequence of their non-uniqueness, they are tied to a particular convention. This is very important when ones compares with other results. However using their definition one can get some relations. For instance we can show

$$C_1(0,0,0,m_0^2,m_1^2,m_2^2) = C_1(0,0,0,m_2^2,m_1^2,m_0^2) \tag{C.164}$$

$$\begin{aligned} C_2(0,0,0,m_0^2,m_1^2,m_2^2) &= -C_0(0,0,0,m_2^2,m_1^2,m_0^2) - C_1(0,0,0,m_2^2,m_1^2,m_0^2) \\ &\quad -C_2(0,0,0,m_2^2,m_1^2,m_0^2) \end{aligned} \tag{C.165}$$

In the limit $m_1 = m_2$ we get the simple expressions,

$$\begin{aligned} C_1(0,0,0,m_0^2,m_1^2,m_1^2) &= C_2(0,0,0,m_0^2,m_1^2,m_1^2) \\ &= -\frac{1}{m_0^2}\frac{3-4t+t^2+2\ln t}{4(-1+t)^3} \end{aligned} \tag{C.166}$$

$$\begin{aligned} C_{11}(0,0,0,m_0^2,m_1^2,m_1^2) &= C_{22}(0,0,0,m_0^2,m_1^2,m_1^2) = 2\ C_{12}(0,0,0,m_0^2,m_1^2,m_1^2) \\ &= -\frac{1}{m_0^2}\frac{-11+18t-9t^2+2t^3-6\ln t}{18(-1+t)^4} \end{aligned} \tag{C.167}$$

in agreement with Eqs. (21-22) of [23]. The case of masses equal gives

$$C_1(0,0,0,m_0^2,m_0^2,m_0^2) = C_2(0,0,0,m_0^2,m_0^2,m_0^2) = \frac{1}{6m_0^2} \tag{C.168}$$

$$C_{11}(0,0,0,m_0^2,m_0^2,m_0^2) = C_{22}(0,0,0,m_0^2,m_0^2,m_0^2) = -\frac{1}{12m_0^2} \tag{C.169}$$

$$C_{12}(0,0,0,m_0^2,m_0^2,m_0^2) = -\frac{1}{24m_0^2} \tag{C.170}$$

The package PVzem

As we said before, in many situations it is a good approximation to neglect the external momenta. In this case, the loop functions are easier to evaluate and one approach is for each problem to evaluate them. However our approach here is more in the direction of automatically evaluating the one-loop amplitudes. If one does that with the use of `FeynCalc`, has we have been doing, then the result is given in terms of standard functions that can be numerically evaluated with the package `LoopTools`. However this package has problems with this limit. This is because this limit is unphysical. Let us illustrate this point calculating the functions $C_1(m^2,0,0,m_S^2,m_F^2,m_F^2)$ and $C_2(m^2,0,0,m_S^2,m_F^2,m_F^2)$ for $m_B = 100$ GeV, $m_F = 80$ GeV and m_2 ranging from 10^{-6} to 100 GeV. To better illustrate our point we show two plots with different scales on the axis.

In these plots, $C_i^{\rm Ex}$ are the exact C_i functions calculated with `LoopTools` and $C_i^{\rm Ap}$ are the C_i calculated in the zero momenta limit. We can see that only for external momenta (in this case corresponding to the mass m_2) close enough to the masses of the particles

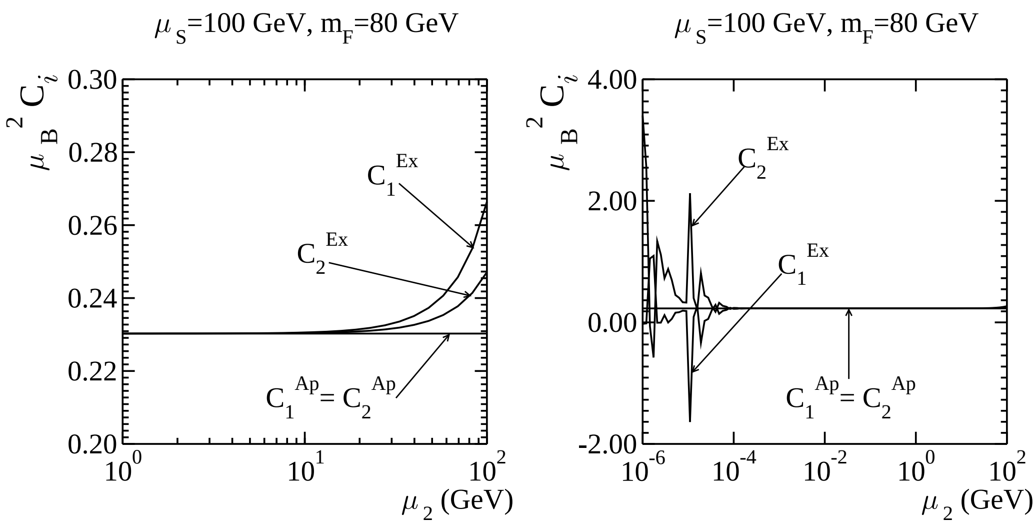


Figure C.7:

in the loop, the exact result deviates from the approximate one. However for very small values of the external momenta, `LoopTools` has numerical problems as shown in the right panel of Fig. C.7. To overcome this problem I have developed a `Fortran` package that evaluates all the C functions in the zero external momenta limit. There are no restrictions on the masses being equal or different and the conventions are the same as in `FeynCalc` and `LoopTools`, for instance,

$$c12zem(m02, m12, m22) = c0i(cc12, 0, 0, 0, m02, m12, m22) \tag{C.171}$$

where $c0i(cc12, \cdots)$ is the `LoopTools` notation and $c12zem(\cdots)$ is the notation of my package, called `PVzem`. It can be obtained from the address indicated in Ref.[15]. The approximate functions shown in Fig. C.7 were calculated using that package. We include here the `Fortran` code used to produce that figure.

```
*************************************************************
*                                                           *
*              Program LoopToolsExample                     *
*                                                           *
*   This program calculates the values used in the plots    *
*   of Figure 20. For the exact results the LoopTools       *
*   package was used. The package PVzem was used for the    *
*   approximate results.                                    *
*                                                           *
*              Version of  14/05/2012                       *
*                                                           *
*     Author: Jorge C. Romao                                *
*     e-mail: jorge.romao@ist.utl.pt                        *
*************************************************************

      program LoopToolsExample
      implicit none

*
```

```
*  LoopTools has to be used with FORTRAN programs with the
*  extension .F in order to have the header file "looptools.h"
*  preprocessed. This file includes all the definitions used
*  by LoopTools.
*
*  Functions c1zem and c2zem are provided by the package PVzem.
*

#include "looptools.h"

      integer i
      real*8 m2,mF2,mS2,m
      real*8 lgmmin,lgmmax,lgm,step
      real*8 rc1,rc2
      real*8 c1zem,c2zem

      mS2=100.d0**2
      mF2=80.d0**2

*
*  Initialize LoopTools. See the LoopTools manual for further
*  details. There you can also learn how to set the scale MU
*  and how to handle the UR and IR divergences.
*

      call ltini

      lgmmax=log10(100.d0)
      lgmmin=log10(1.d-6)
      step=(lgmmax-lgmmin)/100.d0
      lgm=lgmmin-step

      open(10,file=plot.dat,status=unknown)

      do i=1,101
         lgm=lgm+step
         m=10.d0**lgm
         m2=m**2
*
*  In LoopTools the c0i(...) are complex functions. For the
*  kinematics chosen here they are real, so we take the real
*  part for comparison.
*

      rc1=dble(c0i(cc1,m2,0.d0,0.d0,mS2,mF2,mF2))
      rc2=dble(c0i(cc2,m2,0.d0,0.d0,mS2,mF2,mF2))
      write(10,100)m,rc1*mS2,rc2*mS2,c1zem(mS2,mF2,mF2)*mS2,
     &             c2zem(mS2,mF2,mF2)*mS2
      enddo

 100  format(5(e22.14))

      call ltexi

      end
************** End of Program LoopToolsExample.F ************
```

When the above program is compiled, the location of the header file `looptools.h` must be known by the compiler. This is best achieved by using a `Makefile`. We give below, as an example, the one that was used with the above program. Depending on the installation details of `LoopTools` the paths might be different.

```
FC          =
LT          = /usr/local/lib/LoopTools
FFLAGS      =  -c -O -I$(LT)/include
LDFLAGS     =
LINKER      = $(FC)

LIB         = -L$(LT)/lib
LIBS        = -looptools

.f.o:
        $(FC) $(FFLAGS) $*.F

files   = LoopToolsExample.o PVzem.o

all:    $(files)
        $(LINKER) $(LDFLAGS) -o Example $(files) $(LIB) $(LIBS)
```

Explicit expressions for the D functions

Function D_0

The various D functions can be calculated in a similar way. However they are rather lengthy and have to handled numerically [15]. Here we just give D_0 for the equal masses case.

$$\begin{aligned} D_0(0,\cdots,0,m^2,m^2,m^2,m^2) &= \Gamma(4)\frac{1}{6}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \; \frac{1}{(m^2)^2} \\ &= \frac{1}{m^4}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \\ &= \frac{1}{6m^4} \end{aligned} \tag{C.172}$$

C.10 Examples of *1-loop* calculations with PV functions

In this section we will work out in detail a few examples of one-loop calculations using the `FeynCalc` package and the Passarino-Veltman scheme.

C.10.1 Vacuum Polarization in QED

We have done this example in section C.10.1 using the techniques described in sections C.3, C.4 and C.5. Now we will use `FeynCalc`. The first step is to write the `Matematica` program [3]. We list it below:

```
(********************** Program VacPol.m ************************)
(*
 Version compatible with FeynCalc 9.2.0

 Date: 01/06/2017
 Author: Jorge C. Romao
 email: jorge.romao@tecnico.ulisboa.pt
*)

(* First input FeynCalc *)
(* Uncomment below if you want to call from this program. If open a new
   mathematica notebook and load FeynCalc from there you should not load
   it again
*)
(*
<< FeynCalc`
*)

(* Now write the numerator of the Feynman diagram. We define the
   constant

       C=alpha/(4 pi)

   I also use the FCE notation available since FeynCalc 6. See manual
   for explanations.

*)

num:= - C Tr[GA[mu] . (GS[q] + m) . GA[nu] .  (GS[q]+GS[k]+m)]

(* Set some Options. This changed from previous versions *)

SetOptions[PaVeReduce,A0ToB0->True]
$LimitTo4 = True;

(* Define the amplitude *)

amp:=num * FeynAmpDenominator[PropagatorDenominator[q+k,m], \
                              PropagatorDenominator[q,m]]

(* Calculate the result *)

res:=(-I / Pi^2) OneLoop[q,amp]
ans=PaVeReduce[res,PaVeAutoReduce->True] //Simplify

(****************** End of Program VacPol.m ********************)
```

[3] One should check which version of `Mathematica` and `FeynCalc` is used, as conventions may change. We will indicate in which version these programs were verified. Also the output may change as `Mathematica` can order the terms differently. We will try to maintain in my web page [24] a version of the programs as updated as possible.

The output from `Mathematica` is:

```
                 2        2          2    2          2        2        2    2    2
Out[2]= (4 C (k    + 6 m   B0[0, m  , m ] - 3 (k   + 2 m ) B0[k  , m  , m ])
           2                                     2
        (k   g[mu, nu] - k[mu] k[nu])) / (9 k )
```

Now remembering that,

$$C = \frac{\alpha}{4\pi} \tag{C.173}$$

and

$$i\,\Pi_{\mu\nu}(k,\varepsilon) = -i\,k^2 P^T_{\mu\nu}\Pi(k,\varepsilon) \tag{C.174}$$

we get

$$\Pi(k,\varepsilon) = \frac{\alpha}{4\pi}\left[-\frac{4}{9} - \frac{8}{3}\frac{m^2}{k^2}\,B_0(0,m^2,m^2) + \frac{4}{3}\left(1+\frac{2m^2}{k^2}\right)\,B_0(k^2,m^2,m^2)\right] \tag{C.175}$$

To obtain the renormalized vacuum polarization one needs to know the value of $\Pi(0,\varepsilon)$. To do that one has to take the limit $k \to 0$ in Eq. (C.175). For that one uses the derivative of the B_0 function

$$B_0'(p^2,m_1^2,m_2^2) \equiv \frac{\partial}{\partial p^2}\,B_0(p^2,m_1^2,m_2^2) \tag{C.176}$$

to obtain

$$\Pi(0,\varepsilon) = \frac{\alpha}{4\pi}\left[-\frac{4}{9} + \frac{4}{3}B_0(0,m^2,m^2) + \frac{8}{3}m^2 B_0'(0,m^2,m^2)\right] \tag{C.177}$$

Using

$$B_0'(0,m^2,m^2) = \frac{1}{6m^2} \tag{C.178}$$

we finally get

$$\Pi(0,\varepsilon) = -\delta Z_3 = \frac{\alpha}{4\pi}\left[\frac{4}{3}\,B_0(0,m^2,m^2)\right] \tag{C.179}$$

and the final result for the renormalized vertex is:

$$\Pi^R(k) = \frac{\alpha}{3\pi}\left[-\frac{1}{3} + \left(1+\frac{2m^2}{k^2}\right)\left(B_0(k^2,m^2,m^2) - B_0(0,m^2,m^2)\right)\right] \tag{C.180}$$

If we want to compare with our earlier analytical results we need to know that

$$B_0(0,m^2,m^2) = \Delta_\varepsilon - \ln\frac{m^2}{\mu^2} \tag{C.181}$$

Then Eq. (C.180) reproduces the result of Eq. (4.54). The comparison between Eq. (C.180) and Eq. (4.56) can be done numerically using the package `LoopTools`[20, 21].

C.10.2 Electron Self-Energy in QED

In this section we repeat the usual calculation of using the Passarino-Veltman scheme. We start with the `Mathematica` program,

```
(******************** Program SelfEnergy.m *********************)
(*
 Version compatible with FeynCalc 9.2.0

 Date: 01/06/2017
 Author: Jorge C. Romao
 email: jorge.romao@tecnico.ulisboa.pt
*)

(* First input FeynCalc *)
(* Uncomment below if you want to call from this program. If open a new
   mathematica notebook and load FeynCalc from there you should not load
   it again
*)
(*
<< FeynCalc`
*)

(* Tell FeynCalc to reduce the result to scalar functions *)

SetOptions[PaVeReduce,A0ToB0->False,PaVeAutoReduce->True]
$LimitTo4 = True;


(* Now write the numerator of the Feynman diagram. We define the
   constant

       C= - alpha/(4 pi)

  The minus sign comes from the photon propagator. The factor
  i/(16 pi^2) is already included in this definition.

   I also use the FCE notation available since FeynCalc 6. See manual
   for explanations.

*)

num:= C GA[mu] .  (GS[p]+GS[k]+m) . GA[mu]

(* Define the amplitude *)

amp:= num \
FeynAmpDenominator[PropagatorDenominator[p+k,m], \
                   PropagatorDenominator[k]]


(* Calculate the result *)

res:=(-I / Pi^2) OneLoop[k,amp]

ans=-res;
```

```
(*
The minus sign in ans comes from the fact that -i \Sigma = diagram
*)

(* Calculate the functions A(p^2) and B(p^2) *)

A=Coefficient[ans,DiracSlash[p],0];
B=Coefficient[ans,DiracSlash[p],1];

(* Calculate deltm *)

delm=A + m B /. ScalarProduct[p,p]->m^2//Simplify

(* Calculate delZ2 *)

Ap2 = A /. ScalarProduct[p,p]->p2
Bp2 = B /. ScalarProduct[p,p]->p2

aux=2 m D[Ap2,p2] + Bp2 \
    + 2 m^2 D[Bp2,p2] /. D[B0[p2,0,m^2],p2]->DB0[p2,0,m^2]

aux2= aux /. p2->m^2

aux3= aux2 /. A0[m^2]->m^2 (B0[m^2,0,m^2] -1)

delZ2=Simplify[aux3]
(***************** End of Program SelfEnergy.m *******************)
```

The output from `Mathematica` is:

```
                                  2       2
     A =      C (2 m - 4 m B0[p , 0, m ])


             2    2           2       2    2      2      2
        C (-p  - m  B0[0, 0, m ] + (m  + p ) B0[p , 0, m ])
     B= ---------------------------------------------------
                                    2
                                   p

                                   2          2      2
  delm = -(C m (-1 + B0[0, 0, m ] + 2 B0[m , 0, m ]))

                              2       2      2      2
 delZ2 = C (-1 + B0[0, 0, m ] - 4 m  DB0[m , 0, m ])
```

We therefore get[4] (in this case $C = -\frac{\alpha}{4\pi}$)

$$A = \frac{\alpha m}{\pi} \left[-\frac{1}{2} + B_0(p^2, 0, m^2) \right] \tag{C.184}$$

$$B = \frac{\alpha}{4\pi} \left[1 + \frac{1}{p^2} A_0(m^2) - \left(1 + \frac{m^2}{p^2} \right) B_0(p^2, 0, m^2) \right] \tag{C.185}$$

$$\delta_m = \frac{3\alpha m}{4\pi} \left[-\frac{1}{3} + \frac{1}{3m^2} A_0(m^2) + \frac{2}{3} B_0(m^2, 0, m^2) \right] \tag{C.186}$$

One can check that Eq. (C.186) is in agreement with Eq. (4.80). For that one needs the following relations,

$$A_0(m^2) = m^2 \left(B_0(m^2, 0, m^2) - 1 \right) \tag{C.187}$$

$$B_0(m^2, 0, m^2) = \Delta_\varepsilon + 2 - \ln \frac{m^2}{\mu^2} \tag{C.188}$$

$$\int_0^1 dx (1+x) \ln \frac{m^2 x^2}{\mu^2} = -\frac{5}{2} + \frac{3}{2} \ln \frac{m^2}{\mu^2} \tag{C.189}$$

For δZ_2 we get

$$\delta Z_2 = \frac{\alpha}{4\pi} \left[2 - B_0(m^2, 0, m^2) + 4m^2 B_0'(m^2, \lambda^2, m^2) \right] \tag{C.190}$$

This expression can be shown to be equal to Eq. (4.83) although this is not trivial. The reason is that B_0' is IR divergent, hence the parameter λ that controls the divergence.

C.10.3 QED Vertex

In this section we repeat the usual calculation for the QED vertex using the Passarino-Veltman scheme. The `Mathematica` program should by now be easy to understand. We just list it here,

```
(******************** Program QEDVertex.m **********************)
(*
 Version compatible with FeynCalc 9.2.0

 Date: 01/06/2017
 Author: Jorge C. Romao
 email: jorge.romao@tecnico.ulisboa.pt
```

[4] One should notice that the PV functions A_0 and B_0 with one or two zero arguments are not independent. Different versions of FeynCalc, or different options, can give the output in different forms. To make the connections the following relations (see Eqs. (C.138)-(C.141)) are useful,

$$B_0(0, 0, m^2) = -1 + B_0(m^2, 0, m^2), \quad B_0(0, 0, m^2) = \frac{A_0(m^2)}{m^2}, \tag{C.182}$$

$$B_0(0, m^2, m^2) = -2 + B_0(m^2, 0, m^2), \quad B_0(0, 0, m^2) = 1 + B_0(0, m^2 m^2) \tag{C.183}$$

```
*)

(* First input FeynCalc *)
(* Uncomment below if you want to call from this program. If open a new
   mathematica notebook and load FeynCalc from there you should not load
   it again
*)
(*
<< FeynCalc`
*)

(* Tell FeynCalc to reduce the result to scalar functions *)

SetOptions[PaVeReduce,A0ToB0->True]
$LimitTo4 = True;

(* Useful Function *)

TakeDTo4 = Function[exp, expaux1 = exp /. D -> 4 - eps;
  expaux2 = Normal[Series[expaux1, {eps, 0, 1}]];
  c0 = Coefficient[expaux2, eps, 0]; c1 = Coefficient[expaux2, eps, 1];
  c1div = c1 /. PaVe[0, {z1_}, {z2_, z3_}] -> 2/eps;
  expaux3 = c0 + eps c1div // Simplify;
  Simplify[expaux3 /. eps -> 0]]


(* Now write the numerator of the Feynman diagram. We define the
   constant

       C=  alpha/(4 pi)

   The kinematics is: q = p1 -p2 and the internal momenta is k.

*)

num:=Spinor[p1,m].GA[ro].(GS[p1]-GS[k]+m).GA[mu].(GS[p2]-GS[k]+m).GA[ro].
Spinor[p2,m]


amp:=C num \
FeynAmpDenominator[PropagatorDenominator[k,lbd], \
                   PropagatorDenominator[k-p1,m], \
                   PropagatorDenominator[k-p2,m]]

(* Define the on-shell kinematics *)

onshell={ScalarProduct[p1,p1]->m^2,ScalarProduct[p2,p2]->m^2, \
         ScalarProduct[p1,p2]->m^2-q2/2}

(* Define the divergent part of the relevant PV functions*)

div={PaVe[0,{a_},{b_,c_}]->Div}

res1=(-I / Pi^2) OneLoop[k,amp]

res=res1 /. onshell
```

```
auxV1= res /.onshell
auxV2= PaVeReduce[auxV1]
auxV3= PaVeReduce[auxV2] /. div
divV=Simplify[Div*Coefficient[auxV3,Div]]

(* Check  that the divergencies do not cancel *)

testdiv:=Simplify[divV]

ans1=res;
var=Select[Variables[ans1],(Head[#]===StandardMatrixElement)&]
  Set @@ {var, {ME[1],ME[2],ME[3],ME[4]}}

(* Extract the different Matrix Elements

Mathematica writes the result in terms of 4 Standard Matrix
Elements. To have a simpler result we substitute these elements
 by simpler expressions (ME[1],ME[2],ME[3],ME[4]).

PR=GA[6]
PL=GA[7]

 {StandardMatrixElement[u[p1, m1] . PR. u[p2, m2]],

  StandardMatrixElement[u[p1, m1] . PL. u[p2, m2]],

  StandardMatrixElement[u[p1, m1] . ga[mu] . PR . u[p2, m2]],

  StandardMatrixElement[u[p1, m1] . ga[mu] . PL . u[p2, m2]]}

*)

(* We substitute PL and PR by scalar and vector Matrix Elements

ME[5] = StandardMatrixElement[u[p1, m1] .  u[p2, m2]]}

ME[6] = StandardMatrixElement[u[p1, m1] . GA[mu].  u[p2, m2]]}

*)

(* We use Gordon Identity *)

ans2=PaVeReduce[PaVeReduce[ans1]]/.
{ME[1]->ME[5]-ME[2],ME[3]->ME[6]-ME[4]}//FCE//Simplify

CE5=Coefficient[ans2, ME[5]]
CE6=Coefficient[ans2, ME[6]]
CE51=Coefficient[CE5,FV[p1,mu]]
CE52=Coefficient[CE5,FV[p2,mu]]

ans3=CE51 (FV[p1,mu]+FV[p2,mu]) ME[5] +  CE6 ME[6]

test1:=Simplify[CE51-CE52]
test2:=Simplify[ans2-ans3]
```

```
ans4= ans3 /. {(FV[p1,mu]+FV[p2,mu]) ME[5] -> 2 m ME[6] -2m ME[7]}

ans5=TakeDTo4[ans4]

CGamma:=Coefficient[ans5,ME[6]]
CSigmaAux:=Coefficient[ans5,ME[7]]

test3:=Simplify[ans5-CGamma ME[6] -CSigmaAux ME[7]]

F2:=CSigmaAux /. lbd->0//Simplify

delZ1aux= - CGamma /. q2->0 //Simplify

delZ1:= delZ1aux /. lbd->0//Simplify

F1:=CGamma + delZ1 /. lbd->0 //Simplify

(***************** End of Program QEDVertex.m ********************)
```

From this program we can obtain first the value of δZ_1. We get

```
                 2    2            2      2       2       2   2   2   2
delZ1= C (B0[0, m , m ] - 2 (B0[m , 0, m ] + 2 m  C0[0,m ,m ,m ,m , 0]))
```

which can be written as

$$
\begin{aligned}
\delta Z_1 &= \frac{\alpha}{4\pi}\big[1 - B_0(0,0,m^2) + 2B_0(0,m^2,m^2) - 2B_0(m^2,0,m^2) \\
&\quad -4m^2 C_0(m^2,m^2,0,m^2,\lambda^2,m^2)\big] \qquad \text{(C.191)}
\end{aligned}
$$

where we have introduced a small mass for the photon in the function $C_0(m^2,m^2,0,m^2,\lambda^2,m^2)$ because it is IR divergent when $\lambda \to 0$ (see Eq. (C.157)). Using the results of Eqs. (C.139), (C.140), (C.141) and Eq. (C.157) we can show the important result

$$
\delta Z_1 = \delta Z_2 \qquad \text{(C.192)}
$$

where δZ_2 was defined in Eq. (C.190). After performing the renormalization the coefficient $F_1(k^2)$ is finite and given by

```
                                 2              2   2                 2      2
        2 C q2     C (8 m  - q2) B0[0, m , m ]   2 C q2 B0[m , 0, m ]
F1   = --------- + -------------------------- - -------------------- -
          2                    2                          2
       4 m  - q2            4 m  - q2                  4 m  - q2

          2                 2   2           2        2   2    2   2
     C (8 m  - 3 q2) B0[q2, m , m ]          2        2   2    2   2
     ----------------------------- - 4 C m  C0[0, m , m , m , m , 0] +
                     2
               4 m  - q2

            2            2   2        2      2
     2 C (2 m  - q2) C0[m , m , q2, m , 0, m ]

In[5]:= F1 /. q2->0
```

```
Out[5]= 0
```

or, expanding

```
                                          2                  2   2
               q2        q2 B0[0, 0, m ]   2 q2 B0[0, m , m ]
F1 =   C (-(---------) - --------------- + ------------------ -
                     2                 2                   2
            q2 - 4 m         q2 - 4 m            q2 - 4 m

    2        2   2                   2   2         2         2   2
  8 m  B0[0, m , m ]   3 q2 B0[q2, m , m ]   8 m  B0[q2, m , m ]
  ------------------ - ------------------- + -------------------
             2                   2                     2
    q2 - 4 m             q2 - 4 m              q2 - 4 m
```

```
            2      2          2     2   2      2      2
  2 q2 B0[m , 0, m ]   4 q2 m  C0[m , m , 0, m , 0, m ]
+ ------------------ - -------------------------------- +
               2                          2
      q2 - 4 m                   q2 - 4 m

     4     2   2      2      2          2     2   2       2      2
  16 m  C0[m , m , 0, m , 0, m ]   2 q2  C0[m , m , q2, m , 0, m ]
  ------------------------------ - ------------------------------- +
                2                                 2
         q2 - 4 m                         q2 - 4 m

        2     2   2       2      2         4     2   2       2      2
  12 q2 m  C0[m , m , q2, m , 0, m ]   16 m  C0[m , m , q2, m , 0, m ]
  ---------------------------------- - ------------------------------- )
                  2                                 2
          q2 - 4 m                          q2 - 4 m
```

while the coefficient $F_2(q^2)$ does not need renormalization and it is given by,

```
              2             2   2          2      2             2   2
      -4 C m  (2 + B0[0, m , m ] - 2 B0[m , 0, m ] + B0[q2, m , m ])
 F2 = ---------------------------------------------------------------
                                     2
                                  4 m  - q2
```

and for $F_2(0)$ we get

```
                             2   2         2      2
F2[0] = -2 C (1 + B0[0, m , m ] - B0[m , 0, m ])
```

Using the results of the Appendix (see Eqs. (C.138)-(C.141)) we can show that,

$$F_2(0) = \frac{\alpha}{2\pi} \tag{C.193}$$

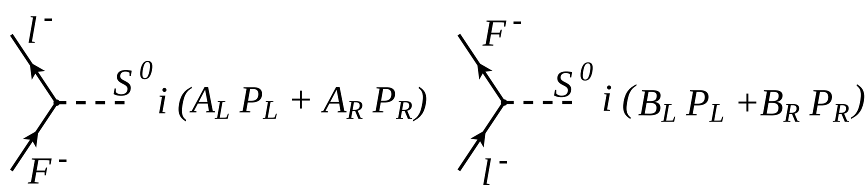
a well known result, first obtained by Schwinger even before the renormalization program was fully understood ($F_2(q^2)$ is finite).

C.11 Modern techniques in a real problem: $\mu \to e\gamma$

In the previous sections we have redone most of the QED standard textbook examples using the PV decomposition and automatic tools. Here we want to present a more complex example, the calculation of the partial width $\mu \to e\gamma$ in an arbitrary theory where the charged leptons couple to scalars and fermions, charged or neutral. This has been done in Ref.[23] for fermions and bosons of arbitrary charge Q_F and Q_B, but for simplicity I will consider here separately the cases of neutral and charged scalars.

C.11.1 Neutral scalar charged fermion loop

We will consider a theory with the following interactions,



where F^- is a fermion with mass m_F and S^0 a neutral scalar with mass m_S. In fact $B_{L,R}$ are not independent of $A_{L,R}$ but it is easier for our programming to consider them completely general. The Feynman rule for the coupling of the photon with the lepton is $-i\, e\, Q_\ell\, \gamma^\mu$ where e is the positron charge (for an electron $Q_\ell = -1$). ℓ_i^- can be any of the leptons but we will omit all indices in the program, the lepton being identified by its mass and from the assumed kinematics

$$\ell_2(p_2) \to \ell_1(p_1) + \gamma(k) \tag{C.194}$$

The diagrams contributing to the process are given in Fig. C.8,

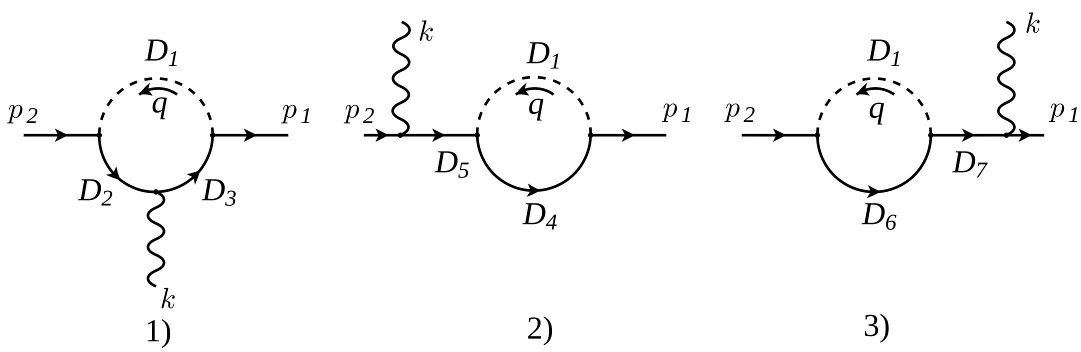


Figure C.8:

where

$$D_1 \;=\; q^2 - m_S^2 \quad ; \quad D_2 = (p_2 + q)^2 - m_F^2 \quad ; \quad D_3 = (q + p2 - k)^2 - m_F^2 \tag{C.195}$$

$$D_4 = D_3 \quad ; \quad D_6 = D_2 \quad ; \quad D_5 = (p_2 - k)^2 - m_2^2 = -2p_2 \cdot k \tag{C.196}$$

$$D_7 = (p_1 + k)^2 - m_1^2 = 2p_1 \cdot k = -D_5 \tag{C.197}$$

The amplitudes are

$$\begin{aligned} iM_1 &= \frac{e\,Q_\ell}{D_1 D_2 D_3} \overline{u}(p_1)\,(A_L P_L + A_R P_R)\,(\not{q} + \not{p}_2 - \not{k} + m_F)\,\gamma^\mu\ (\not{q} + \not{p}_2 + m_F) \\ &\qquad (B_L P_L + B_R P_R)\,u(p_2)\ \varepsilon_\mu(k) \end{aligned} \tag{C.198}$$

$$\begin{aligned} iM_2 &= \frac{e\,Q_\ell}{D_1 D_4 D_5} \overline{u}(p_1)\,(A_L P_L + A_R P_R)\,(\not{q} + \not{p}_2 - \not{k} + m_F)\,(B_L P_L + B_R P_R) \\ &\qquad (\not{p}_2 - \not{k}_2 + m_2)\,\gamma^\mu u(p_2)\ \varepsilon_\mu(k) \end{aligned} \tag{C.199}$$

$$\begin{aligned} iM_3 &= \frac{e\,Q_\ell}{D_1 D_6 D_7} \overline{u}(p_1)\gamma^\mu\ (\not{p}_1 + \not{k} + m_F)\,(A_L P_L + A_R P_R)\,(\not{q} + \not{p}_2 + m_1) \\ &\qquad (B_L P_L + B_R P_R)\,u(p_2)\ \varepsilon_\mu(k) \end{aligned} \tag{C.200}$$

On-shell the amplitude will take the form (we have $p_1 \cdot k = p_2 \cdot k$)

$$\begin{aligned} iM &= 2p_2 \cdot \varepsilon(k) \Big[\ C_L \overline{u}(p_1) P_L u(p_2) + C_R \overline{u}(p_1) P_R u(p_2) \Big] \\ &\qquad + D_L u(p_1) \not{\varepsilon} P_L u(p_2) + D_R u(p_1) \not{\varepsilon} P_R u(p_2) \end{aligned} \tag{C.201}$$

If we write the amplitude as

$$M = M_\mu\ \varepsilon^\mu(k) \tag{C.202}$$

then gauge invariance implies

$$M_\mu k^\mu = 0 \tag{C.203}$$

Imposing this condition on Eq. (C.201) we get the relations

$$D_L = -m_2 C_R - m_1 C_L \tag{C.204}$$

$$D_R = -m_1 C_R - m_2 C_L \tag{C.205}$$

Assuming these relations the amplitude can be written as

$$\begin{aligned} iM = {} & C_L \left[2p_2 \cdot \varepsilon(k) \overline{u}(p_1) P_L u(p_2) - m_1 \overline{u}(p_1) \not{\varepsilon}(k) P_L u(p_2) - m_2 \overline{u}(p_1) \not{\varepsilon}(k) P_R u(p_2)\right] \\ & + C_R \left[2p_2 \cdot \varepsilon(k) \overline{u}(p_1) P_R u(p_2) - m_2 \overline{u}(p_1) \not{\varepsilon}(k) P_L u(p_2) - m_1 \overline{u}(p_1) \not{\varepsilon}(k) P_R u(p_2)\right] \end{aligned} \tag{C.206}$$

and the decay width will be

$$\Gamma = \frac{1}{16\pi m_2^3} \left(m_2^2 - m_1^2\right)^3 \left(|C_L|^2 + |C_R|^2\right) \tag{C.207}$$

As the coefficient of $p_2 \cdot \varepsilon(k)$ only comes from the 3-point function (amplitude M_1) this justifies the usual procedure of just calculating that coefficient and forgetting about the self-energies (amplitudes M_2 and M_3). However these amplitudes are crucial for the cancellation of divergences and for gauge invariance. Now we will show the power of the automatic `FeynCalc` [22] program and calculate both the coefficients $C_{L,R}$ and $D_{L,R}$, showing the cancellation of the divergences and that the relations, Eqs. (C.204) and (C.205) needed for gauge invariance are satisfied. We start by writing the `mathematica` program:

```
(************************ Program mueg-ns.m **************************)
(*
This program calculates the COMPLETE (both the 3 point amplitude and
the two self energy type on each external line) amplitudes for
\mu -> e \gamma when the fermion line in the loop is charged and the
neutral line is a scalar. The \mu has momentum p2 and mass m2, the
electron (p1,m1) and the photon momentum k. The momentum in the loop
is q.

The assumed vertices are,

1) Electron-Scalar-Fermion:

  Spinor[p1,m1] (AL P_L + AR P_R) Spinor [pf,mf]

2) Fermion-Scalar-Muon:

   Spinor[pf,mf] (BL P_L + BR P_R) Spinor [p2,m2]
*)

dm[mu_]:=DiracMatrix[mu,Dimension->D]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]

(*
SetOptions[{B0,B1,B00,B11},BReduce->True]
 *)


gA:= AL DiracMatrix[7] + AR DiracMatrix[6]
gB:= BL DiracMatrix[7] + BR DiracMatrix[6]


num1:=Spinor[p1,m1] . gA . (ds[q]+ds[p2]-ds[k]+mf) . ds[Polarization[k]]\
      . (ds[q]+ds[p2]+mf) . gB . Spinor[p2,m2]
```

```
num2:=Spinor[p1,m1] . gA . (ds[q]+ds[p1]+mf) . gB . (ds[p1]+m2) . \
       ds[Polarization[k]] . Spinor[p2,m2]


num3:=Spinor[p1,m1] . ds[Polarization[k]] . (ds[p2]+m1) . gA . \
      (ds[q]+ds[p2]+mf) . gB . Spinor[p2,m2]



SetOptions[OneLoop,Dimension->D]

amp1:=num1 \
FeynAmpDenominator[PropagatorDenominator[q+p2-k,mf], \
                   PropagatorDenominator[q+p2,mf], \
                   PropagatorDenominator[q,ms]]
amp2:=num2 \
FeynAmpDenominator[PropagatorDenominator[q+p1,mf], \
                   PropagatorDenominator[p2-k,m2],  \
                   PropagatorDenominator[q,ms]]
amp3:=num3 \
FeynAmpDenominator[PropagatorDenominator[p1+k,m1], \
                   PropagatorDenominator[q+p2,mf],  \
                   PropagatorDenominator[q,ms]]

(* Define the on-shell kinematics *)

onshell={ScalarProduct[p1,p1]->m1^2,ScalarProduct[p2,p2]->m2^2, \
        ScalarProduct[k,k]->0,ScalarProduct[p1,k]->(m2^2-m1^2)/2,\
        ScalarProduct[p2,k]->(m2^2-m1^2)/2, \
        ScalarProduct[p2,Polarization[k]]->p2epk, \
        ScalarProduct[p1,Polarization[k]]->p2epk}


(* Define the divergent part of the relevant PV functions*)
div={B0[m1^2,mf^2,ms^2]->Div,B0[m2^2,mf^2,ms^2]->Div, \
  B0[0,mf^2,ms^2]->Div,B0[0,mf^2,mf^2]->Div,B0[0,ms^2,ms^2]->Div}

res1:=(-I / Pi^2) OneLoop[q,amp1]
res2:=(-I / Pi^2) OneLoop[q,amp2]
res3:=(-I / Pi^2) OneLoop[q,amp3]
res:=res1+res2+res3 /. onshell

auxT1:= res1 /.onshell
auxT2:= PaVeReduce[auxT1]
auxT3:= auxT2 /. div
divT:=Simplify[Div*Coefficient[auxT3,Div]]

auxS1:= res2 + res3 /.onshell
auxS2:= PaVeReduce[auxS1]
auxS3:= auxS2 /. div
divS:=Simplify[Div*Coefficient[auxS3,Div]]

(* Check cancellation of divergences
   testdiv should be zero because divT=-divS  *)
```

```
testdiv:=Simplify[divT + divS]

(* Extract the different Matrix Elements

Mathematica writes the result in terms of 8 Standard Matrix Elements.
To have a simpler result we substitute these elements by simpler
expressions (ME[1],...ME[8]). But they are not all independent. The
final result can just be written in terms of 4 Matrix Elements.

{StandardMatrixElement[p2epk u[p1,m1] . ga[6] . u[p2,m2]],

StandardMatrixElement[p2epk u[p1,m1] . ga[7] . u[p2,m2]],

StandardMatrixElement[p2epk u[p1,m1] . gs[k] . ga[6] . u[p2,m2]],

StandardMatrixElement[p2epk u[p1,m1] . gs[k] . ga[7] . u[p2,m2]],

StandardMatrixElement[u[p1,m1] . gs[ep[k]] . ga[6] . u[p2,m2]],

StandardMatrixElement[u[p1,m1] . gs[ep[k]] . ga[7] . u[p2,m2]],

StandardMatrixElement[u[p1,m1] . gs[k] . gs[ep[k]] . ga[6] . u[p2,m2]],

StandardMatrixElement[u[p1,m1] . gs[k] . gs[ep[k]] . ga[7]. u[p2,m2]]} *)


ans1=res;
var=Select[Variables[ans1],(Head[#]===StandardMatrixElement)&]
Set @@ {var, {ME[1],ME[2],ME[3],ME[4],ME[5],ME[6],ME[7],ME[8]}}
identities={ME[3]->-m1 ME[1] + m2 ME[2], ME[4]->-m1 ME[2] + m2 ME[1],
            ME[7]->-m1 ME[5] - m2 ME[6] + 2  ME[1],
            ME[8]->-m1 ME[6] - m2 ME[5] + 2  ME[2]}

ans2 =ans1 /. identities ;
ans=Simplify[ans2];

CR=Coefficient[ans,ME[1]]/2;
CL=Coefficient[ans,ME[2]]/2;
DR=Coefficient[ans,ME[5]];
DL=Coefficient[ans,ME[6]];

(* Test to see if we did not forget any term *)

test1:=Simplify[ans-2 CR*ME[1]-2 CL*ME[2]-DR*ME[5]-DL*ME[6]]


(* Test that the divergences cancel term by term *)

auxCL=PaVeReduce[CL] /. div ;
testdivCL:=Simplify[Coefficient[auxCL,Div]]

auxCR=PaVeReduce[CR] /. div ;
testdivCR:=Simplify[Coefficient[auxCR,Div]]

auxDL=PaVeReduce[DL] /. div ;
testdivDL:=Simplify[Coefficient[auxDL,Div]]
```

```
auxDR=PaVeReduce[DR] /. div ;
testdivDR:=Simplify[Coefficient[auxDR,Div]]

(* Test the gauge invariance relations *)
testGI1:=Simplify[PaVeReduce[(m2^2-m1^2)*CR - DR*m1 + DL*m2]]

testGI2:=Simplify[PaVeReduce[(m2^2-m1^2)*CL + DR*m2 - DL*m1]]

(********************** End Program mueg-ns.m  *************************)
```

We first do the tests. The output of `mathematica` is

```
(**********************  Mathematica output   ************************)
In[3]:= << FeynCalc.m

FeynCalc4.1.0.3b Type ?FeynCalc for help or visit
http://www.feyncalc.org

In[4]:= << mueg-ns.m

In[5]:= test1

Out[5]= 0

In[6]:= testdiv

Out[6]= 0

In[7]:= testdivCL

Out[7]= 0

In[8]:= testdivCR

Out[8]= 0

In[9]:= testdivDL

Out[9]= 0

In[10]:= testdivDR

Out[10]= 0

In[11]:= testGI1

Out[11]= 0

In[12]:= testGI2

Out[12]= 0
(******************  End of Mathematica output   ********************)
```

Now we obtain the results for C_L

```
(*********************  Mathematica output   ***********************)
In[13]:= CL
                                2    2    2    2    2
Out[13]= (-4 AL BL mf C0[0, m2 , m1 , mf , mf , ms ] +

                              2    2       2    2    2
    4 AL BR m2 PaVe[2, {0, m1 , m2 }, {mf , mf , ms }] -

                              2    2       2    2    2
    4 AL BL mf PaVe[2, {0, m1 , m2 }, {mf , mf , ms }] -

                                 2    2       2    2    2
    4 AR BL m1 PaVe[1, 2, {0, m1 , m2 }, {mf , mf , ms }] +

                                 2    2       2    2    2
    4 AL BR m2 PaVe[1, 2, {0, m1 , m2 }, {mf , mf , ms }] +

                                 2    2       2    2    2
    4 AL BR m2 PaVe[2, 2, {0, m1 , m2 }, {mf , mf , ms }]) / 4
```

and for C_R

```
In[15]:= CR
                                2    2    2    2    2
Out[15]= (-4 AR BR mf C0[0, m2 , m1 , mf , mf , ms ] +

                              2    2       2    2    2
    4 AR BL m2 PaVe[2, {0, m1 , m2 }, {mf , mf , ms }] -

                              2    2       2    2    2
    4 AR BR mf PaVe[2, {0, m1 , m2 }, {mf , mf , ms }] -

                                 2    2       2    2    2
    4 AL BR m1 PaVe[1, 2, {0, m1 , m2 }, {mf , mf , ms }] +

                                 2    2       2    2    2
    4 AR BL m2 PaVe[1, 2, {0, m1 , m2 }, {mf , mf , ms }] +

                                 2    2       2    2    2
    4 AR BL m2 PaVe[2, 2, {0, m1 , m2 }, {mf , mf , ms }]) / 4
(****************** End of  Mathematica output  *********************)
```

The expressions for $D_{L,R}$ are quite complicated. They are not normally calculated because they can be related to $C_{L,R}$ by gauge invariance. However the power of this automatic program can be illustrated by asking for these functions. As they are very long we calculate them by pieces. We just calculate D_L because one can easily check that $D_R = D_L(L \leftrightarrow R)$.

```
(*********************  Mathematica output   ***********************)

In[12]:= Coefficient[PaVeReduce[DL],AL BL]
```

```
                   2    2    2                 2    2    2
         m1 mf B0[m1 , mf , ms ]   m1 mf B0[m2 , mf , ms ]
Out[12]= ----------------------- - ----------------------- +
                  2     2                     2     2
               m1  - m2                    m1  - m2

           2    2        2    2    2
  m1 mf C0[m1 , m2 , 0, mf , ms , mf ]

In[13]:= Coefficient[PaVeReduce[DL],AL BR]

            2      2           2    2
         (mf  - ms ) B0[0, mf , ms ]
Out[13]= -------------------------- -
                   2 m1 m2

     2           2         2        2    2    2
  (m1  m2 - m2 mf  + m2 ms ) B0[m1 , mf , ms ]
  ------------------------------------------- +
                          2     2
               2 m1 (m1  - m2 )

        2         2         2        2    2    2
  (m1 m2  - m1 mf  + m1 ms ) B0[m2 , mf , ms ]
  -------------------------------------------
                          2     2
               2 m2 (m1  - m2 )

In[14]:= Coefficient[PaVeReduce[DL],AR BL]

                     2          2        2    2    2
         1   (-2 m1 mf  + 2 m1 ms ) B0[m1 , mf , ms ]
Out[14]= - - -------------------------------------- +
         2                        2     2
                        2 m1 (m1  - m2 )

          2          2        2    2    2
  (-2 m2 mf  + 2 m2 ms ) B0[m2 , mf , ms ]
  ---------------------------------------
                       2     2
             2 m2 (m1  - m2 )


      2      2    2        2    2    2
  + mf  C0[m1 , m2 , 0, mf , ms , mf ]

In[15]:= Coefficient[PaVeReduce[DL],AR BR]

                   2    2    2                 2    2    2
         m2 mf B0[m1 , mf , ms ]   m2 mf B0[m2 , mf , ms ]
Out[15]= ----------------------- - -----------------------
                  2     2                     2     2
               m1  - m2                    m1  - m2

               2    2        2    2    2
  + m2 mf C0[m1 , m2 , 0, mf , ms , mf ]
```

```
(******************* End of  Mathematica output  *********************)
```

From these expressions one can immediately verify that the divergences cancel in $D_{L,R}$ and that they are not present in $C_{L,R}$. To finish this section we just rewrite the $C_{L,R}$ in our usual notation. We get

$$
\begin{aligned}
C_L &= \frac{e\,Q_\ell}{16\pi^2}\Big[A_L B_L m_F \left(-C_0(0,m_2^2,m_1^2,m_F^2,m_F^2,m_S^2) - C_2(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2)\right) \\
&\qquad + A_L B_R m_2 \Big(C_2(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2) + C_{12}(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2) \\
&\qquad\qquad + C_{22}(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2)\Big) \\
&\qquad + A_R B_L m_1\, C_{12}(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2)\Big] && \text{(C.208)} \\
C_R &= C_L(L \leftrightarrow R) && \text{(C.209)}
\end{aligned}
$$

These equations are in agreement with Eqs. (32-34) and Eqs. (38-39) of Ref. [23], although some work has to be done in order to verify that[5]. This has to do with the fact that the PV decomposition functions are not independent (see the Appendix for further details on this point). We can however use the power of **FeynCalc** to verify this. We list below a simple program to accomplish that.

```
(*******************  Program lavoura-ns.m  ************************)
(*
This program tests the results of my program mueg-ns.m against the
results obtained by L. Lavoura (hepph/0302221).
*)

(* First load FeynCalc.m and mueg-ns.m *)


<< FeynCalc.m
<< mueg-ns.m

(*
Now write Lavoura integrals in the notation of FeynCalc. Be careful
with the order of the entries.
*)

c1:=PaVe[1,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]
c2:=PaVe[2,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]

d1:=PaVe[1,1,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]
d2:=PaVe[2,2,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]

f:=PaVe[1,2,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]
```

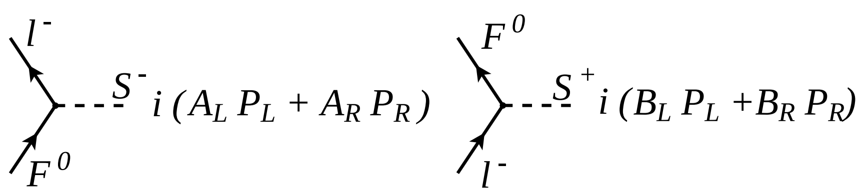
[5] An important difference between our conventions and those of Ref.[23] is that p_1 and p_2 (and obviously m_1 and m_2) are interchanged.

```
(* Write Eqs. (32)-(34) of hepph/0302221 in our notation *)

k1:=PaVeReduce[m2*(c1+d1+f)]
k2:=PaVeReduce[m1*(c2+d2+f)]
k3:=PaVeReduce[mf*(c1+c2)]

(*
Now test the results. For this we should use the equivalences:

\rho     -> AL BR
\lambda -> AR BL
\xi      -> AR BR
\nu      -> AL BL

*)

testCLALBR:=Simplify[PaVeReduce[Coefficient[CL, AL BR]-k1]]
testCLARBL:=Simplify[PaVeReduce[Coefficient[CL, AR BL]-k2]]
testCLALBL:=Simplify[PaVeReduce[Coefficient[CL, AL BL]-k3]]

testCRALBR:=Simplify[PaVeReduce[Coefficient[CR, AL BR]-k2]]
testCRARBL:=Simplify[PaVeReduce[Coefficient[CR, AR BL]-k1]]
testCRARBR:=Simplify[PaVeReduce[Coefficient[CR, AR BR]-k3]]

(***************** End of Program lavoura-ns.m  *********************)
```

One can easily check that the output of the six tests is zero, showing the equivalence between our results. And all this is done in a few seconds.

C.11.2 Charged scalar neutral fermion loop

We consider now the case of the scalar being charged and the scalar neutral. The general case of both charged [23] can also be easily implemented, but for simplicity we do not consider it here. The couplings are now



and the diagrams contributing to the process are given in Fig. C.9, where all the denominators are as in Eqs. (C.195)- (C.197) except that

$$D'_1 = q^2 - m_F^2 \quad ; \quad D'_2 = (q-p_1)^2 - m_S^2 \quad ; \quad D'_3 = (q-p_1-k)^2 - m_S^2 \tag{C.210}$$

Also the coupling of the photon to the charged scalar is, in our notation,

$$-\, ie\, Q_\ell\, (-2q + p_1 + p_2)^\mu \tag{C.211}$$

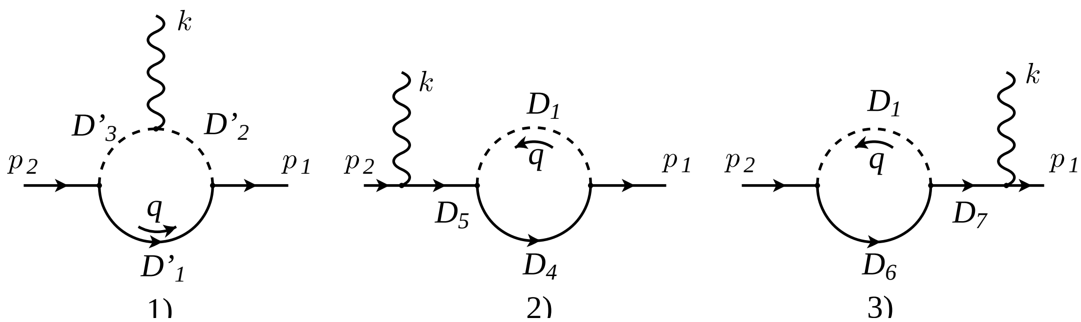


Figure C.9:

The procedure is very similar to the neutral scalar case and we just present here the `mathematica` program and the final result. All the checks of finiteness and gauge invariance can be done as before.

```
(************************ Program mueg-cs.m **************************)
(*
This program calculates the COMPLETE (both the 3 point amplitude and
the two self energy type on each external line) amplitudes for
\mu -> e \gamma when the fermion line in the loop is neutral and the
charged line is a scalar. The \mu has momentum p2 and mass m2, the
electron (p1,m1) and the photon momentum k. The momentum in the loop
is q.


The assumed vertices are,

1) Electron-Scalar-Fermion:

  Spinor[p1,m1] (AL P_L + AR P_R) Spinor [pf,mf]

2) Fermion-Scalar-Muon:

   Spinor[pf,mf] (BL P_L + BR P_R) Spinor [p2,m2]
*)

dm[mu_]:=DiracMatrix[mu,Dimension->4]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]

(*
SetOptions[{B0,B1,B00,B11},BReduce->True]
 *)
```

```
gA:= AL DiracMatrix[7] + AR DiracMatrix[6]
gB:= BL DiracMatrix[7] + BR DiracMatrix[6]

num1:= Spinor[p1,m1] . gA . (ds[q]+mf) . gB . Spinor[p2,m2]  \
PolarizationVector[k,mu] ( - 2 fv[q,mu] + fv[p1,mu] + fv[p2,mu] )

num11:=DiracSimplify[num1];

num2:=Spinor[p1,m1] . gA . (ds[q]+ds[p1]+mf) . gB . (ds[p1]+m2) . \
       ds[Polarization[k]] . Spinor[p2,m2]


num3:=Spinor[p1,m1] . ds[Polarization[k]] . (ds[p2]+m1) . gA . \
      (ds[q]+ds[p2]+mf) . gB . Spinor[p2,m2]



SetOptions[OneLoop,Dimension->D]


amp1:=num1 \
FeynAmpDenominator[PropagatorDenominator[q,mf],\
                   PropagatorDenominator[q-p1,ms],\
                   PropagatorDenominator[q-p1-k,ms]]

amp2:=num2 \
FeynAmpDenominator[PropagatorDenominator[q+p1,mf], \
                   PropagatorDenominator[p2-k,m2],  \
                   PropagatorDenominator[q,ms]]
amp3:=num3 \
FeynAmpDenominator[PropagatorDenominator[p1+k,m1], \
                   PropagatorDenominator[q+p2,mf],  \
                   PropagatorDenominator[q,ms]]

(* Define the on-shell kinematics *)

onshell={ScalarProduct[p1,p1]->m1^2,ScalarProduct[p2,p2]->m2^2, \
         ScalarProduct[k,k]->0,ScalarProduct[p1,k]->(m2^2-m1^2)/2, \
         ScalarProduct[p2,k]->(m2^2-m1^2)/2, \
         ScalarProduct[p2,Polarization[k]]->p2epk, \
         ScalarProduct[p1,Polarization[k]]->p2epk}

(* Define the divergent part of the relevant PV functions*)

div={B0[m1^2,mf^2,ms^2]->Div,B0[m2^2,mf^2,ms^2]->Div, \
     B0[0,mf^2,ms^2]->Div,B0[0,mf^2,mf^2]->Div,B0[0,ms^2,ms^2]->Div}

res1:=(-I / Pi^2) OneLoop[q,amp1]
res2:=(-I / Pi^2) OneLoop[q,amp2]
res3:=(-I / Pi^2) OneLoop[q,amp3]

res:=res1+res2+res3 /. onshell


auxT1:= res1 /.onshell
auxT2:= PaVeReduce[auxT1]
```

```
auxT3:= auxT2 /. div
divT:=Simplify[Div*Coefficient[auxT3,Div]]

auxS1:= res2 + res3 /.onshell
auxS2:= PaVeReduce[auxS1]
auxS3:= auxS2 /. div
divS:=Simplify[Div*Coefficient[auxS3,Div]]


(* Check cancellation of divergences

   testdiv should be zero because divT=-divS

*)

testdiv:=Simplify[divT + divS]

(* Extract the different Matrix Elements

Mathematica writes the result in terms of 6 Standard Matrix Elements.
To have a simpler result we substitute these elements by simpler
expressions (ME[1],...ME[6]). Not all are independent.

{StandardMatrixElement[p2epk u[p1, m1] . ga[6] . u[p2, m2]],

StandardMatrixElement[p2epk u[p1, m1] . ga[7] . u[p2, m2]],

StandardMatrixElement[p2epk u[p1, m1] . gs[k] . ga[6] . u[p2, m2]],

StandardMatrixElement[p2epk u[p1, m1] . gs[k] . ga[7] . u[p2, m2]],

StandardMatrixElement[u[p1, m1] . gs[ep[k]] . ga[6] . u[p2, m2]],

StandardMatrixElement[u[p1, m1] . gs[ep[k]] . ga[7] . u[p2, m2]]}
*)

ans1=res;
var=Select[Variables[ans1],(Head[#]===StandardMatrixElement)&]

Set @@ {var, {ME[1],ME[2],ME[3],ME[4],ME[5],ME[6]}}
identities={ME[3]->-m1 ME[1] + m2 ME[2],ME[4]->-m1 ME[2] + m2 ME[1]}

ans2 =ans1 /. identities ;
ans=Simplify[ans2];


CR=Coefficient[ans,ME[1]]/2;
CL=Coefficient[ans,ME[2]]/2;
DR=Coefficient[ans,ME[5]];
DL=Coefficient[ans,ME[6]];

(* Test to see if we did not forget any term *)

test1:=Simplify[ans-2*CR*ME[1]-2*CL*ME[2]-DR*ME[5]-DL*ME[6]]
```

```
(* Test that the divergences cancel term by term *)

auxCL:=PaVeReduce[CL] /. div ;
testdivCL:=Simplify[Coefficient[auxCL,Div]]

auxCR:=PaVeReduce[CR] /. div ;
testdivCR:=Simplify[Coefficient[auxCR,Div]]

auxDL:=PaVeReduce[DL] /. div ;
testdivDL:=Simplify[Coefficient[auxDL,Div]]

auxDR:=PaVeReduce[DR] /. div ;
testdivDR:=Simplify[Coefficient[auxDR,Div]]

(* Test the gauge invariance relations *)
testGI1:=PaVeReduce[(m2^2-m1^2)*CR - DR*m1 + DL*m2]

testGI2:=PaVeReduce[(m2^2-m1^2)*CL + DR*m2 - DL*m1]

(********************** End Program mueg-cs.m  **********************)
```

Note that although these programs look large, in fact they are very simple. Most of it are comments and tests. The output of this program gives,

```
(*********************  Mathematica output   ***********************)
In[3]:= CL

                                 2     2     2     2     2
Out[3]= (-2 AR BL m1 C0[0, m1 , m2 , ms , ms , mf ] -

                              2        2       2     2     2
    2 AR BL m1 PaVe[1, {m1 , 0, m2 }, {mf , ms , ms }] -

                              2     2          2     2     2
    4 AR BL m1 PaVe[1, {m1 , m2 , 0}, {ms , mf , ms }] -

                              2     2          2     2     2
    2 AL BL mf PaVe[1, {m1 , m2 , 0}, {ms , mf , ms }] -

                              2        2       2     2     2
    2 AL BR m2 PaVe[2, {m1 , 0, m2 }, {mf , ms , ms }] -

                              2     2          2     2     2
    2 AR BL m1 PaVe[2, {m1 , m2 , 0}, {ms , mf , ms }] +

                              2     2          2     2     2
    2 AL BR m2 PaVe[2, {m1 , m2 , 0}, {ms , mf , ms }] -

                                 2     2          2     2     2
    2 AR BL m1 PaVe[1, 1, {m1 , m2 , 0}, {ms , mf , ms }] -

                                 2     2          2     2     2
    2 AR BL m1 PaVe[1, 2, {m1 , m2 , 0}, {ms , mf , ms }] +

                                 2     2          2     2     2
    2 AL BR m2 PaVe[1, 2, {m1 , m2 , 0}, {ms , mf , ms }]) / 2
```

```
(****************** End of  Mathematica output  *******************)
```

To finish this section we just rewrite the $C_{L,R}$ in our usual notation. We get

$$
\begin{aligned}
C_L \;=\; & \frac{e\,Q_\ell}{16\pi^2}\Big[A_L B_L m_F \left(-C_1(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)\right) \\
& \qquad +A_L B_R m_2 \Big(-C_2(m_1^2,0,m_2^2,m_F^2,m_S^2,m_S^2)+C_2(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2) \\
& \qquad\qquad +C_{12}(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)\Big) \\
& \qquad + A_R B_L m_1 \left(-C_0(0,m_1^2,m_2^2,m_S^2,m_S^2,m_F^2)-C_1(m_1^2,0,m_2^2,m_F^2,m_S^2,m_S^2)\right. \\
& \qquad\qquad -2C_1(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)-C_2(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2) \\
& \qquad\qquad \left.-C_{11}(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)-C_{12}(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)\right)\Big] \\
C_R \;=\; & C_L(L\leftrightarrow R)
\end{aligned}
\tag{C.212}
$$

It is left as an exercise to write a `mathematica` program that proves that these equations are in agreement with Eqs. (35-37) and Eqs. (38-39) of Ref. [23].

Appendix D

Feynman Rules for the Standard Model

D.1 Introduction

To do actual calculations it is very important to have all the Feynman rules with consistent conventions. In this Appendix we will give the complete Feynman rules for the Standard Model in the general R_ξ gauge.

D.2 The Standard Model

One of the most difficult problems in having a consistent set of of Feynman rules are the conventions. We give here those that are important for building the SM. We will separate them by gauge group.

D.2.1 Gauge Group $SU(3)_c$

Here the important conventions are for the field strengths and the covariant derivatives. We have

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^{abc} G^b_\mu G^c_\nu, \quad a = 1, \ldots, 8 \tag{D.1}$$

where f^{abc} are the group structure constants, satisfying

$$\left[T^a, T^b\right] = i f^{abc} T^c \tag{D.2}$$

and T^a are the generators of the group. The covariant derivative of a (quark) field q in some representation T^a of the gauge group is given by

$$D_\mu q = \left(\partial_\mu - i\, g\, G^a_\mu T^a\right) q \tag{D.3}$$

In QCD the quarks are in the fundamental representation and $T^a = \lambda^a/2$ where λ^a are the Gell-Mann matrices. A gauge transformation is given by a matrix

$$U = e^{-iT^a\alpha^a} \tag{D.4}$$

and the fields transform as

$$\begin{aligned} &q \to e^{-iT^a\alpha^a} q && \delta q = -iT^a\alpha^a q \\ &G^a_\mu T^a \to U G^a_\mu T^a U^{-1} - \frac{i}{g}\partial_\mu U U^{-1} && \delta G^a_\mu = -\frac{1}{g}\partial_\mu \alpha^a + f^{abc}\alpha^b G^c_\mu \end{aligned} \tag{D.5}$$

where the second column is for infinitesimal transformations. With these definitions one can verify that the covariant derivative transforms like the field itself,

$$\delta(D_\mu q) = -i\, T^a \alpha^a (D_\mu q) \tag{D.6}$$

ensuring the gauge invariance of the Lagrangian.

D.2.2 Gauge Group $SU(2)_L$

This is similar to the previous case. We have

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\nu, \quad a = 1, \ldots, 3 \tag{D.7}$$

where, for the fundamental representation of $SU(2)_L$ we have $T^a = \sigma^a/2$ and ϵ^{abc} is the completely anti-symmetric tensor in 3 dimensions. The covariant derivative for any field ψ_L transforming non-trivially under this group is,

$$D_\mu \psi_L = \left(\partial_\mu - i\, g\, W^a_\mu T^a\right) \psi_L \tag{D.8}$$

D.2.3 Gauge Group $U(1)_Y$

In this case the group is abelian and we have

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{D.9}$$

with the covariant derivative given by

$$D_\mu \psi_R = \left(\partial_\mu + i\, g'\, Y\, B_\mu\right) \psi_R \tag{D.10}$$

where Y is the hypercharge of the field. Notice the different sign convention between Eq. (D.8) and Eq. (D.9). This is to have the usual definition[1]

$$Q = T_3 + Y\ . \tag{D.12}$$

It is useful to write the covariant derivative in terms of the mass eigenstates A_μ and Z_μ. These are defined by the relations,

$$\left\{ \begin{array}{l} W^3_\mu = Z_\mu \cos\theta_W - A_\mu \sin\theta_W \\ B_\mu = Z_\mu \sin\theta_W + A_\mu \cos\theta_W \end{array} \right. , \quad \left\{ \begin{array}{l} Z_\mu = W^3_\mu \cos\theta_W + B_\mu \sin\theta_W \\ A_\mu = -W^3_\mu \sin\theta_W + B_\mu \cos\theta_W \end{array} \right. . \tag{D.13}$$

---

[1] For this to be consistent one must also have, under hypercharge transformations, for a field of hypercharge Y,

$$\psi' = e^{+iY\alpha_Y}\psi, \quad B'_\mu = B_\mu - \frac{1}{g'}\partial_\mu \alpha_Y\ . \tag{D.11}$$

This is important when finding the ghost interactions. It would have been possible to have a minus sign in Eq. (D.10), with a definition $\theta_W \to \theta_W + \pi$. This would also mean reversing the sign in the exponent of the hypercharge transformation in Eq. (D.11) maintaining the similarity with Eq. (D.5).

Field	ℓ_L	ℓ_R	ν_L	u_L	d_L	u_R	d_R	ϕ^+	ϕ^0
T_3	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
Y	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
Q	-1	-1	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	1	0

Table D.1: Values of T_3^f, Q and Y for the SM particles.

For a field ψ_L, with hypercharge Y, we get,

$$\begin{aligned} D_\mu \psi_L &= \left[\partial_\mu - i\frac{g}{\sqrt{2}}\left(\tau^+ W^+_\mu + \tau^- W^-_\mu\right) - i\frac{g}{2}\tau_3 W^3_\mu + ig' Y B_\mu\right]\psi_L \qquad \text{(D.14)} \\ &= \left[\partial_\mu - i\frac{g}{\sqrt{2}}\left(\tau^+ W^+_\mu + \tau^- W^-_\mu\right) + ie\,Q\,A_\mu - i\frac{g}{\cos\theta_W}\left(\frac{\tau_3}{2} - Q\,\sin^2\theta_W\right)Z_\mu\right]\psi_L \end{aligned}$$

where, as usual, $\tau^\pm = (\tau_1 \pm i\tau_2)/2$ and the charge operator is defined by

$$Q = \begin{bmatrix} \frac{1}{2}+Y & 0 \\ 0 & -\frac{1}{2}+Y \end{bmatrix}, \qquad \text{(D.15)}$$

and we have used the relations,

$$e = g\sin\theta_W = g'\cos\theta_W\,, \qquad \text{(D.16)}$$

and the usual definition,

$$W^\pm_\mu = \frac{W^1_\mu \mp i\,W^2_\mu}{\sqrt{2}}\,. \qquad \text{(D.17)}$$

For a singlet of $SU(2)_L$, ψ_R we have,

$$\begin{aligned} D_\mu \psi_R &= \left[\partial_\mu + ig' Y B_\mu\right]\psi_R \\ &= \left[\partial_\mu + ie\,Q\,A_\mu + i\frac{g}{\cos\theta_W}Q\,\sin^2\theta_W Z_\mu\right]\psi_R\,. \qquad \text{(D.18)} \end{aligned}$$

We collect in Table D.1 the quantum number of the SM particles.

D.2.4 The Gauge Field Lagrangian

For completeness we write the gauge field Lagrangian. We have

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \qquad \text{(D.19)}$$

where the field strengths are given in Eqs. (D.1), and (D.9).

D.2.5 The Fermion Fields Lagrangian

Here we give the kinetic part and gauge interaction, leaving the Yukawa interaction for a next section. We have

$$\mathcal{L}_{\text{Fermion}} = \sum_{\text{quarks}} i\overline{q}\gamma^\mu D_\mu q + \sum_{\psi_L} i\overline{\psi_L}\gamma^\mu D_\mu \psi_L + \sum_{\psi_R} i\overline{\psi_R}\gamma^\mu D_\mu \psi_R \tag{D.20}$$

where the covariant derivatives are obtained with the rules in Eqs. (D.3), (D.14) and (D.18).

D.2.6 The Higgs Lagrangian

In the SM we use an Higgs doublet with the following assignments,

$$\Phi = \begin{bmatrix} \phi^+ \\ \dfrac{v + H + i\varphi_Z}{\sqrt{2}} \end{bmatrix} \tag{D.21}$$

The hypercharge of this doublet is 1/2 and therefore the covariant derivative reads

$$\begin{aligned} D_\mu \Phi &= \left[\partial_\mu - i\frac{g}{\sqrt{2}}\left(\tau^+ W^+_\mu + \tau^- W^-_\mu\right) - i\frac{g}{2}\tau_3 W^3_\mu + i\frac{g'}{2}B_\mu\right]\Phi \\ &= \left[\partial_\mu - i\frac{g}{\sqrt{2}}\left(\tau^+ W^+_\mu \tau^- W^-_\mu\right) + ie\,Q\,A_\mu - i\frac{g}{\cos\theta_W}\left(\frac{\tau_3}{2} - Q\,\sin^2\theta_W\right) Z_\mu\right]\Phi \end{aligned} \tag{D.22}$$

The Higgs Lagrangian is then

$$\mathcal{L}_{\text{Higgs}} = (D_\mu\Phi)^\dagger D_\mu\Phi + \mu^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2 \tag{D.23}$$

If we expand this Lagrangian we find the following terms

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} = \cdots &+ \frac{1}{8}g^2v^2W^3_\mu W^{\mu 3} + \frac{1}{8}g'^2v^2B_\mu B^\mu + \frac{1}{4}gg'v^2W^3_\mu B^\mu + \frac{1}{4}g^2v^2W^+_\mu W^{-\mu} \\ &+ \frac{1}{2}v\,\partial^\mu\varphi_Z\left(g'B_\mu + gW^3_\mu\right) + \frac{i}{2}gvW^-_\mu\partial^\mu\varphi^+ - \frac{i}{2}gvW^+_\mu\partial^\mu\varphi^- \end{aligned} \tag{D.24}$$

The first three terms give, after diagonalization, a massless field, the photon, and a massive one, the Z, with the relations given in Eq. (D.13), while the fourth gives the mass to the charged $W^\pm_\mu$ boson. Using Eq. (D.13) we get,

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} = \cdots &+ \frac{1}{2}M_Z^2 Z_\mu Z^\mu + M_W^2 W^+_\mu W^{-\mu} \\ &+ M_Z Z_\mu\partial^\mu\varphi_Z + iM_W\left(W^-_\mu\partial^\mu\varphi^+ - W^+_\mu\partial^\mu\varphi^-\right) \end{aligned} \tag{D.25}$$

where

$$M_W = \frac{1}{2}gv, \quad M_Z = \frac{1}{\cos\theta_W}\frac{1}{2}gv = \frac{1}{\cos\theta_W}M_W \tag{D.26}$$

By looking at Eq. (D.25) we realize that besides finding a realistic spectra for the gauge bosons, we also got a problem. In fact the terms in the last line are quadratic in the fields and complicate the definition of the propagators. We now see how one can use the needed gauge fixing to solve also this problem.

D.2.7 The Yukawa Lagrangian

Now we have to spell out the interaction between the fermions and the Higgs doublet that after spontaneous symmetry breaking gives masses to the elementary fermions. We have,

$$\mathcal{L}_{\text{Yukawa}} = -\, Y_l\, \overline{L}\, \Phi\ \ell_R - Y_d\, \overline{Q}\, \Phi\ d_R - Y_u\, \overline{Q}\, \widetilde{\Phi}\ u_R + \text{h.c.} \tag{D.27}$$

where sum is implied over generations, L (Q) are the lepton (quark) doublets and,

$$\widetilde{\Phi} = i\,\sigma_2 \Phi^* = \begin{bmatrix} \dfrac{v+H-i\varphi_Z}{\sqrt{2}} \\ -\varphi^- \end{bmatrix} \tag{D.28}$$

D.2.8 The Gauge Fixing

As it is well known, we have to gauge fix the gauge part of the Lagrangian to be able to define the propagators. We will use a generalization of the class of Lorenz gauges, the so-called R_ξ gauges. With this choice the gauge fixing Lagrangian reads

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2\xi} F_G^2 - \frac{1}{2\xi} F_A^2 - \frac{1}{2\xi} F_Z^2 - \frac{1}{\xi} F_- F_+ \tag{D.29}$$

where

$$\begin{aligned} F_G^a =& \partial^\mu G_\mu^a, \quad F_A = \partial^\mu A_\mu, \quad F_Z = \partial^\mu Z_\mu - \xi M_Z \varphi_Z \\ F_+ =& \partial^\mu W_\mu^+ - i\xi M_W \varphi^+, \quad F_- = \partial^\mu W_\mu^- + i\xi M_W \varphi^- \end{aligned} \tag{D.30}$$

One can easily verify that with these definitions we cancel the quadratic terms in Eq. (D.25).

D.2.9 The Ghost Lagrangian

The last piece in writing the SM Lagrangian is the ghost Lagrangian. As it is well known, this is given by the Fadeev-Popov prescription,

$$\begin{aligned} \mathcal{L}_{\text{Ghost}} =& \sum_{i=1}^{4} \left[\overline{c}_+ \frac{\partial(\delta F_+)}{\partial \alpha^i} + \overline{c}_- \frac{\partial(\delta F_+)}{\partial \alpha^i} + \overline{c}_Z \frac{\partial(\delta F_Z)}{\partial \alpha^i} + \overline{c}_A \frac{\partial(\delta F_A)}{\partial \alpha^i} \right] c_i \\ &+ \sum_{a,b=1}^{8} \overline{\omega}^a \frac{\partial(\delta F_G^a)}{\partial \beta^b} \omega^b \end{aligned} \tag{D.31}$$

where we have denoted by ω^a the ghosts associated with the $SU(3)_c$ transformations defined by,

$$U = e^{-iT^a \beta^a}, \quad a = 1, \ldots, 8 \tag{D.32}$$

and by $c_\pm, c_A, c_Z$ the electroweak ghosts associated with the gauge transformations,

$$U = e^{-iT^a \alpha^a}, \quad a = 1, \ldots, 3, \quad U = e^{iY\alpha^4} \tag{D.33}$$

For completeness we write here the gauge transformations of the gauge fixing terms needed to find the Lagrangian in Eq. (D.31). It is convenient to redefine the parameters as

$$\begin{aligned}
\alpha^{\pm} =& \frac{\alpha^1 \mp \alpha^2}{\sqrt{2}} \\
\alpha_Z =& \alpha^3 \cos\theta_W + \alpha^4 \sin\theta_W \\
\alpha_A =& -\alpha^3 \sin\theta_W + \alpha^4 \cos\theta_W
\end{aligned} \tag{D.34}$$

We then get

$$\begin{aligned}
\delta F_G^a =& -\partial_\mu \beta^a + g_s f^{abc} \beta^b G_\mu^c \\
\delta F_A =& -\partial_\mu \alpha_A \\
\delta F_Z =& \partial_\mu (\delta Z^\mu) - M_Z \delta\varphi_Z \\
\delta F_+ =& \partial_\mu (\delta W_\mu^+) - i M_W \delta\varphi^+ \\
\delta F_- =& \partial_\mu (\delta W_\mu^-) + i M_W \delta\varphi^-
\end{aligned} \tag{D.35}$$

Using the explicit form of the gauge transformations we can finally find the missing pieces,

$$\delta Z_\mu = -\partial_\mu \alpha_Z + ig\cos\theta_W \left(W_\mu^+ \alpha^- - W_\mu^- \alpha^+ \right) \tag{D.36}$$

$$\delta W_\mu^+ = -\partial_\mu \alpha^+ + ig\left[\alpha^+ \left(Z_\mu \cos\theta_W - A_\mu \sin\theta_w \right) - \left(\alpha_Z \cos\theta_w - \alpha_A \sin\theta_W \right) W_\mu^+ \right]$$

$$\delta W_\mu^- = -\partial_\mu \alpha^- - ig\left[\alpha^- \left(Z_\mu \cos\theta_W - A_\mu \sin\theta_w \right) - \left(\alpha_Z \cos\theta_w - \alpha_A \sin\theta_W \right) W_\mu^- \right]$$

and

$$\begin{aligned}
\delta\varphi_Z =& -\frac{1}{2} g \left(\alpha^- \varphi^+ + \alpha^+ \varphi^- \right) + \frac{g}{2\cos\theta_W} \alpha_Z (v+H) \\
\delta\varphi^+ =& -i\,\frac{g}{2}(v+H+i\varphi_Z)\alpha^+ - i\,\frac{g}{2}\frac{\cos 2\theta_W}{\cos\theta_W}\varphi^+ \alpha_Z + ie\,\varphi^+ \alpha_A \\
\delta\varphi^- =& i\,\frac{g}{2}(v+H-i\varphi_Z)\alpha^- + i\,\frac{g}{2}\frac{\cos 2\theta_W}{\cos\theta_W}\varphi^- \alpha_Z - ie\,\varphi^- \alpha_A
\end{aligned} \tag{D.37}$$

D.2.10 The Complete SM Lagrangian

Finally the complete Lagrangian for the Standard Model is obtained putting together all the pieces. We have,

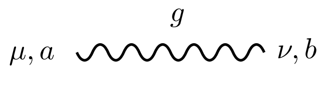
$$\mathcal{L}_{\rm SM} = \mathcal{L}_{\rm gauge} + \mathcal{L}_{\rm Fermion} + \mathcal{L}_{\rm Higgs} + \mathcal{L}_{\rm Yukawa} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm Ghost} \tag{D.38}$$

where the different terms were given in Eqs. (D.19), (D.20), (D.23), (D.27), (D.29), (D.31).

D.3 The Feynman Rules for QCD

We give separately the Feynman Rules for QCD and the electroweak part of the Standard Model.

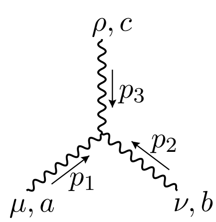
D.3.1 Propagators



$$-i\delta_{ab}\left[\frac{g_{\mu\nu}}{k^2+i\epsilon}-(1-\xi)\frac{k_\mu k_\nu}{(k^2)^2}\right] \tag{D.39}$$

$$\delta_{ab}\ \frac{i}{k^2+i\epsilon} \tag{D.40}$$

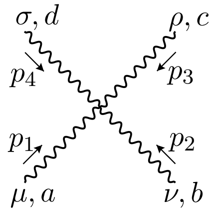
D.3.2 Triple Gauge Interactions



$$\begin{aligned} gf^{abc}[\quad & g^{\mu\nu}(p_1-p_2)^\rho + g^{\nu\rho}(p_2-p_3)^\mu \\ & +g^{\rho\mu}(p_3-p_1)^\nu] \\ & p_1+p_2+p_3=0 \end{aligned} \tag{D.41}$$

D.3.3 Quartic Gauge Interactions

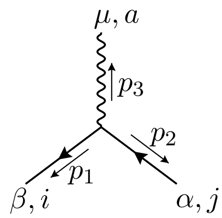
ii) Vértice quártico dos bosões de gauge



$$\begin{aligned} -ig^2\Big[\quad & f_{eab}f_{ecd}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}) \\ & +f_{eac}f_{edb}(g_{\mu\sigma}g_{\rho\nu}-g_{\mu\nu}g_{\rho\sigma}) \\ & +f_{ead}f_{ebc}(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\rho}g_{\nu\sigma})\Big] \end{aligned} \tag{D.42}$$

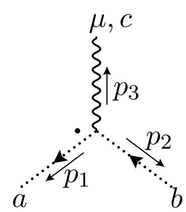
$$p_1+p_2+p_3+p_4=0$$

D.3.4 Fermion Gauge Interactions



$$ig(\gamma^{\mu})_{\beta\alpha}T^{a}_{ij} \tag{D.43}$$

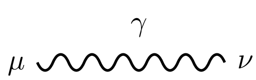
D.3.5 Ghost Interactions



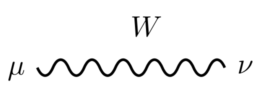
$$\begin{gathered} g\ C^{abc}p_1^{\mu} \\ p_1 \quad +p_2+p_3=0 \end{gathered} \tag{D.44}$$

D.4 The Feynman Rules for the Electroweak Theory

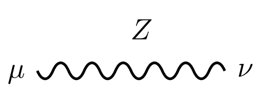
D.4.1 Propagators



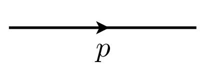
$$-\,i\left[\frac{g_{\mu\nu}}{k^2+i\epsilon}-(1-\xi)\frac{k_{\mu}k_{\nu}}{(k^2)^2}\right] \tag{D.45}$$



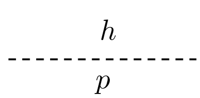
$$\frac{-ig_{\mu\nu}}{k^2-M_W^2+i\epsilon} \tag{D.46}$$



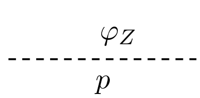
$$\frac{-ig_{\mu\nu}}{k^2-M_Z^2+i\epsilon} \tag{D.47}$$



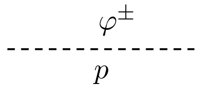
$$\frac{i(\not{p}+m_f)}{p^2-m_f^2+i\epsilon} \tag{D.48}$$



$$\frac{i}{p^2-M_h^2+i\epsilon} \tag{D.49}$$

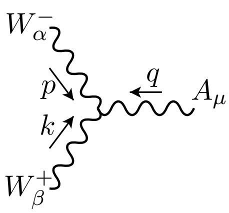


$$\frac{i}{p^2-\xi m_Z^2+i\epsilon} \tag{D.50}$$

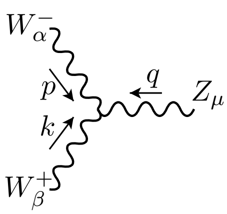


$$\frac{i}{p^2 - \xi m_W^2 + i\epsilon} \tag{D.51}$$

D.4.2 Triple Gauge Interactions

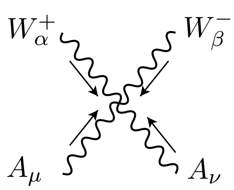


$$-\, ie \left[g_{\alpha\beta}(p-k)_\mu + g_{\beta\mu}(k-q)_\alpha + g_{\mu\alpha}(q-p)_\beta\right] \tag{D.52}$$

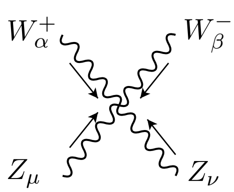


$$ig\cos\theta_W \left[g_{\alpha\beta}(p-k)_\mu + g_{\beta\mu}(k-q)_\alpha + g_{\mu\alpha}(q-p)_\beta\right] \tag{D.53}$$

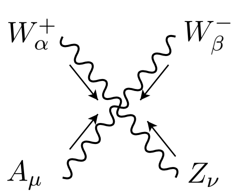
D.4.3 Quartic Gauge Interactions



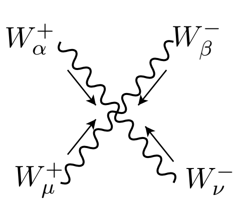
$$-\, ie^2 \left[2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right] \tag{D.54}$$



$$-\, ig^2\, \cos^2\theta_W \left[2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right] \tag{D.55}$$

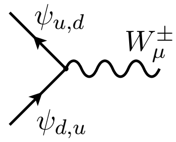


$$ieg\cos\theta_W \left[2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right] \tag{D.56}$$



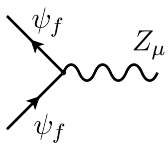
$$ig^2 \left[2g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right] \tag{D.57}$$

D.4.4 Charged Current Interaction

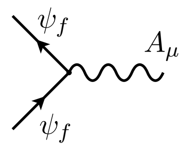


$$i\frac{g}{\sqrt{2}}\gamma_\mu\frac{1-\gamma_5}{2} \tag{D.58}$$

D.4.5 Neutral Current Interaction



$$i\frac{g}{\cos\theta_W}\gamma_\mu\left(g_V^f - g_A^f\gamma_5\right)$$

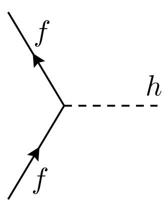


$$-ieQ_f\gamma_\mu \tag{D.59}$$

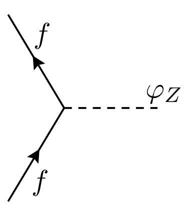
where

$$g_V^f = \frac{1}{2}T_f^3 - Q_f\sin^2\theta_W, \quad g_A^f = \frac{1}{2}T_f^3\ . \tag{D.60}$$

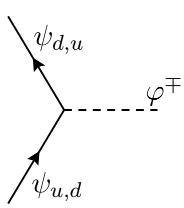
D.4.6 Fermion-Higgs and Fermion-Goldstone Interactions



$$-\,i\,\frac{g}{2}\,\frac{m_f}{m_W} \tag{D.61}$$

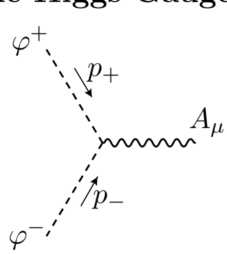


$$-\,g\,T_f^3\,\frac{m_f}{m_W}\,\gamma_5 \tag{D.62}$$

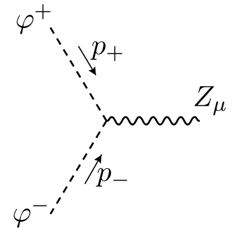


$$i\frac{g}{\sqrt{2}}\left(\frac{m_u}{m_W}P_{R,L} - \frac{m_d}{m_W}P_{L,R}\right) \tag{D.63}$$

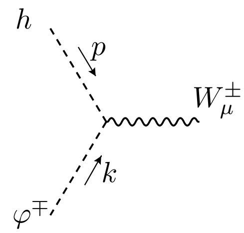
D.4.7 Triple Higgs-Gauge and Goldstone-Gauge Interactions



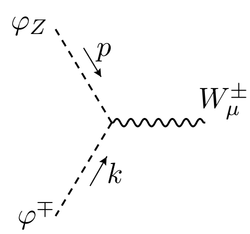
$$-\,i\,e\,\left(p_+ - p_-\right)_\mu \tag{D.64}$$



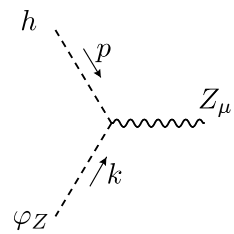
$$i\, g\, \frac{\cos 2\theta_W}{2 \cos \theta_W}\, (p_+ - p_-)_\mu \tag{D.65}$$



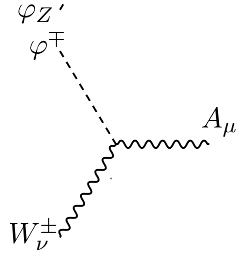
$$\mp\, \frac{i}{2}\, g\, (k-p)_\mu \tag{D.66}$$



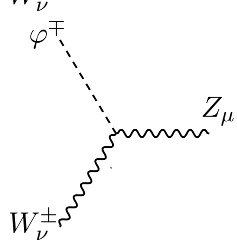
$$\frac{g}{2}\, (k-p)_\mu \tag{D.67}$$



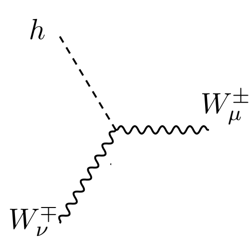
$$\frac{g}{2 \cos \theta}\, (k-p)_\mu \tag{D.68}$$



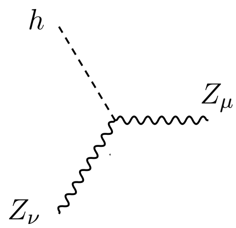
$$-\, ie\, m_W\, g_{\mu\nu} \tag{D.69}$$



$$-\, ig\, m_Z\, \sin^2 \theta_W\, g_{\mu\nu} \tag{D.70}$$

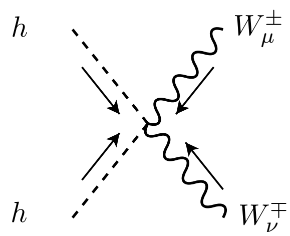


$$ig\, m_W\, g_{\mu\nu} \tag{D.71}$$

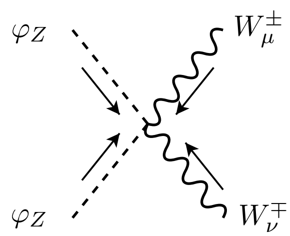


$$i \frac{g}{\cos\theta_W} m_Z \, g_{\mu\nu} \tag{D.72}$$

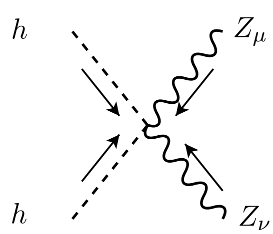
D.4.8 Quartic Higgs-Gauge and Goldstone-Gauge Interactions



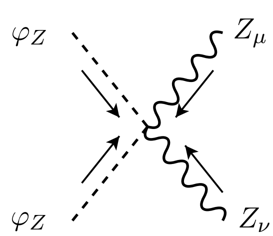
$$\frac{i}{2} g^2 \, g_{\mu\nu} \tag{D.73}$$



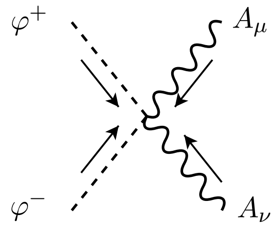
$$\frac{i}{2} g^2 \, g_{\mu\nu} \tag{D.74}$$



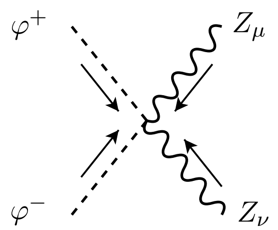
$$\frac{i}{2} \frac{g^2}{\cos^2\theta_W} g_{\mu\nu} \tag{D.75}$$



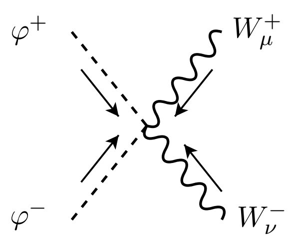
$$\frac{i}{2} \frac{g^2}{\cos^2\theta_W} g_{\mu\nu} \tag{D.76}$$



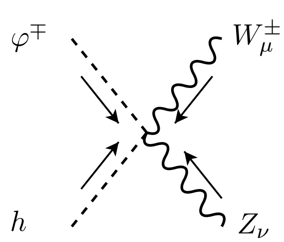
$$2i\, e^2\, g_{\mu\nu} \tag{D.77}$$



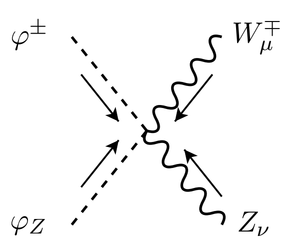
$$\frac{i}{2}\left(\frac{g\cos 2\theta_W}{\cos\theta_W}\right)^2 g_{\mu\nu} \tag{D.78}$$



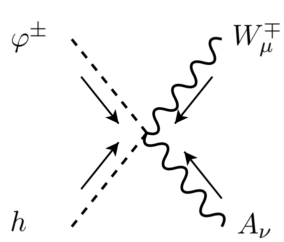
$$\frac{i}{2}\, g^2\, g_{\mu\nu} \tag{D.79}$$



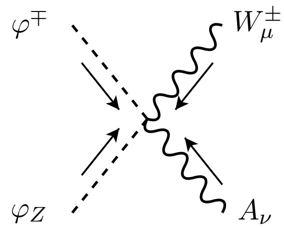
$$-\, i\, g^2\, \frac{\sin^2\theta_W}{2\cos\theta_W}\, g_{\mu\nu} \tag{D.80}$$



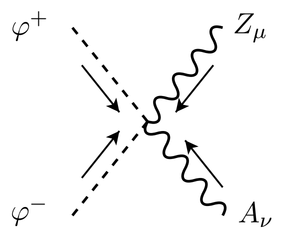
$$\mp\; g^2\, \frac{\sin^2\theta_W}{2\cos\theta_W}\, g_{\mu\nu} \tag{D.81}$$



$$-\,\frac{i}{2}\, eg\, g_{\mu\nu} \tag{D.82}$$

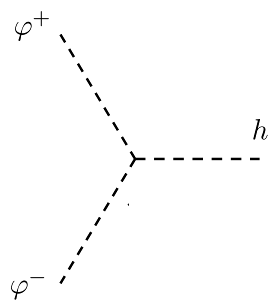


$$\pm\,\frac{1}{2}\,eg\,g_{\mu\nu} \tag{D.83}$$

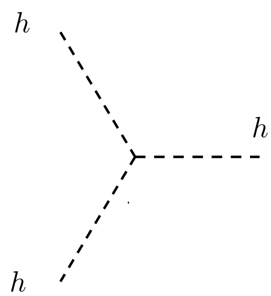


$$-\,i\,eg\,\frac{\cos 2\theta_W}{\cos\theta_W}g_{\mu\nu} \tag{D.84}$$

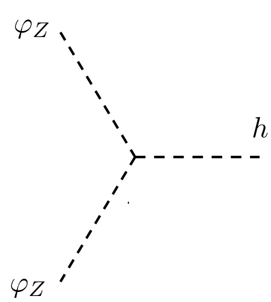
D.4.9 Triple Higgs and Goldstone Interactions



$$-\,\frac{i}{2}\,g\,\frac{m_h^2}{m_W} \tag{D.85}$$

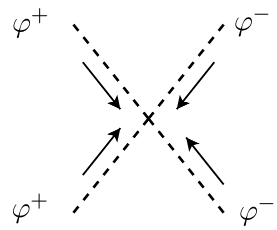


$$-\,\frac{3}{2}\,i\,g\frac{m_h^2}{m_W} \tag{D.86}$$

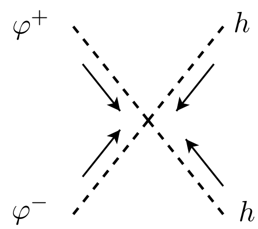


$$-\,\frac{i}{2}\,g\frac{m_h^2}{m_W} \tag{D.87}$$

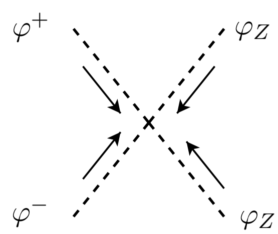
D.4.10 Quartic Higgs and Goldstone Interactions



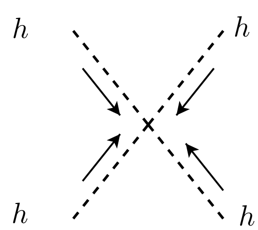
$$-\frac{i}{2}\,g^2\,\frac{m_h^2}{m_W^2} \tag{D.88}$$



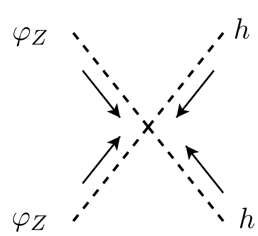
$$-\frac{i}{4}\,g^2\,\frac{m_h^2}{m_W^2} \tag{D.89}$$



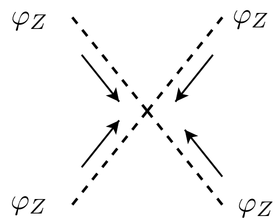
$$-\frac{i}{4}\,g^2\,\frac{m_h^2}{m_W^2} \tag{D.90}$$



$$-\frac{3}{4}\,i\,g^2\,\frac{m_h^2}{m_W^2} \tag{D.91}$$



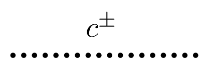
$$-\frac{i}{4}\,g^2\,\frac{m_h^2}{m_W^2} \tag{D.92}$$



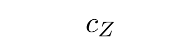
$$-\frac{3}{4}\, i\, g^2 \frac{m_h^2}{m_W^2} \tag{D.93}$$

D.4.11 Ghost Propagators

$$\frac{i}{k^2 + i\epsilon} \tag{D.94}$$

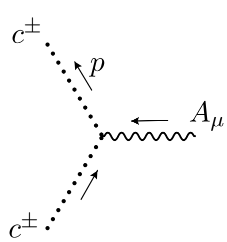


$$\frac{i}{k^2 - \xi m_W^2 + i\epsilon} \tag{D.95}$$

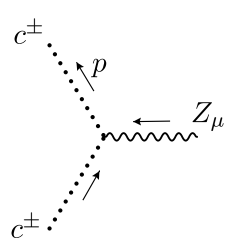


$$\frac{i}{k^2 - \xi m_Z^2 + i\epsilon} \tag{D.96}$$

D.4.12 Ghost Gauge Interactions

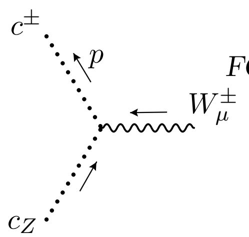


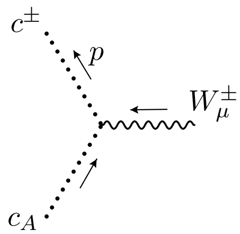
$$\mp\, ie\, p_\mu \tag{D.97}$$



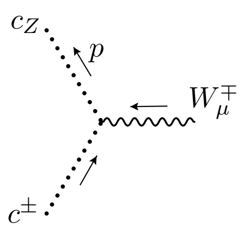
$$\pm\, ig\, \cos\theta_W\, p_\mu \tag{D.98}$$

$$\mp\, ig\, \cos\theta_W\, p_\mu \tag{D.99}$$

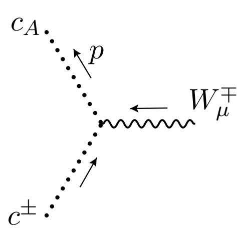




$$\pm\, ie\, p_\mu \qquad \text{(D.100)}$$

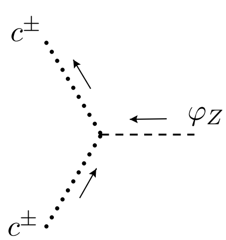


$$\mp\, ig\, \cos\theta_W\, p_\mu \qquad \text{(D.101)}$$

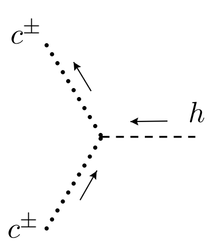


$$\pm\, ie\, p_\mu \qquad \text{(D.102)}$$

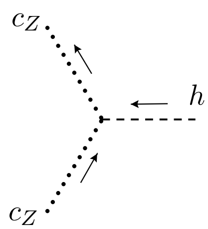
D.4.13 Ghost Higgs and Ghost Goldstone Interactions



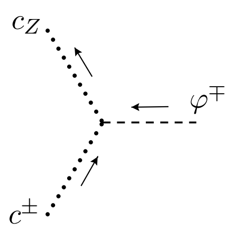
$$\pm\, \frac{g}{2}\, \xi\, m_W \qquad \text{(D.103)}$$



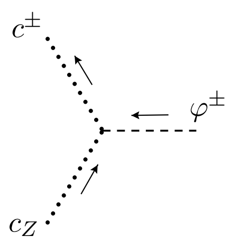
$$-\, \frac{i}{2}\, g\, \xi\, m_W \qquad \text{(D.104)}$$



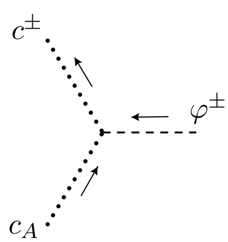
$$-\frac{ig}{2\cos\theta_W}\,\xi\,m_Z \tag{D.105}$$



$$\frac{i}{2}\,g\,\xi\,m_Z \tag{D.106}$$



$$-\,ig\,\frac{\cos 2\theta_W}{2\cos\theta_W}\,\xi\,m_W \tag{D.107}$$



$$ie\,\xi\,m_W \tag{D.108}$$

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