Techniques for Calculations in Quantum Field Theory: A Real Example

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In this note we review the techniques used in Quantum Field Theory in a non trivial example, the Higgs boson production at a linear collider through the process $e^- e^+ \to ZZH$. The emphasis is in the use of the package `FeynCalc` for `Mathematica` and both the trace techniques and the helicity amplitudes method are discussed.

I.   INTRODUCTION

It is always a problem when the students want to attack a more complex problem that needs large trace calculations and numerical integrations. In normal textbooks this part it is normally not covered and, at most, is left as an exercise. My experience is that, for the first time, the students need some help. I will try to cover in this article all the details of a real calculation, from the first stages to the final plots for the cross section. To this end we pick a non-trivial problem, the Higgs boson production at a linear collider in association with a pair of Z bosons [1], that is,

$$e^- e^+ \to ZZH \ . \tag{1}$$

This corresponds, in the limit of neglecting the electron mass, to the following Feynman diagrams,

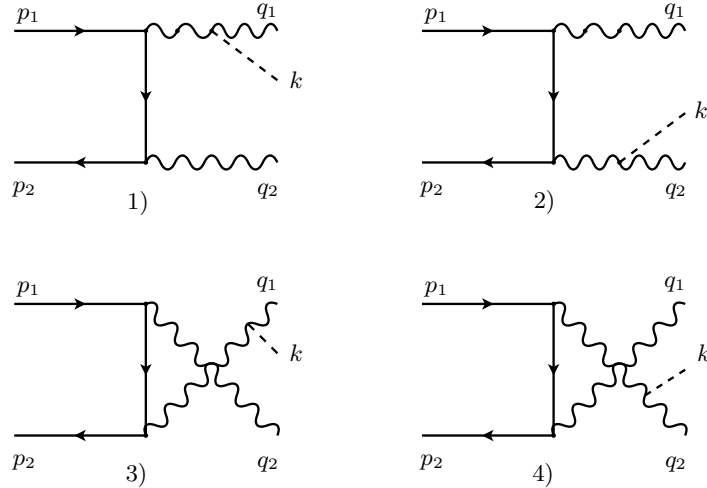Figure 1: Feynman diagrams for $e^- e^+ \to ZZH$.

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II. THE AMPLITUDES

We start by noting that the amplitudes can be written in the following form:

$$M_i = \left(\frac{g}{\cos\theta_W}\right)^3 \overline{v}(p_2)\,\Gamma_i\,u(p_1) \tag{2}$$

where

$$\Gamma_1 \;=\; \epsilon^\mu(q_1)\epsilon^\nu(q_2)\,\gamma^\nu\,(g_V - g_A\gamma_5)\,(-\not{p}_2 + \not{q}_2)\gamma^\mu\,(g_V - g_A\gamma_5)\,\frac{1}{D_e(1)D_Z(1)}$$

$$\Gamma_2 \;=\; \epsilon^\mu(q_1)\epsilon^\nu(q_2)\,\gamma^\nu\,(g_V - g_A\gamma_5)\,(\not{p}_1 - \not{q}_1)\gamma^\mu\,(g_V - g_A\gamma_5)\,\frac{1}{D_e(2)D_Z(2)}$$

$$\Gamma_3 \;=\; \epsilon^\mu(q_1)\epsilon^\nu(q_2)\,\gamma^\mu\,(g_V - g_A\gamma_5)\,(\not{p}_1 - \not{q}_2)\gamma^\nu\,(g_V - g_A\gamma_5)\,\frac{1}{D_e(3)D_Z(3)}$$

$$\Gamma_4 \;=\; \epsilon^\mu(q_1)\epsilon^\nu(q_2)\,\gamma^\mu\,(g_V - g_A\gamma_5)\,(\not{p}_2 + \not{q}_1)\gamma^\nu\,(g_V - g_A\gamma_5)\,\frac{1}{D_e(4)D_Z(4)} \tag{3}$$

The denominators of the propagators are:

$$D_Z(1) \;=\; D_Z(3) = (q_1 + k)^2 - M_Z^2 + iM_Z\Gamma_Z \tag{4}$$

$$D_Z(2) \;=\; D_Z(4) = (q_2 + k)^2 - M_Z^2 + iM_Z\Gamma_Z \tag{5}$$

$$D_e(1) \;=\; (-p_2 + q_2)^2 \quad;\quad D_e(2) = (p_1 - q_1)^2 \tag{6}$$

$$D_e(3) \;=\; (p_1 - q_2)^2 \quad;\quad D_e(4) = (-p_2 + q_1)^2 \tag{7}$$

We should note that in arriving at these expressions we took $m_e = 0$. It can be shown that the terms in the numerator of the propagator of the Z proportional to the momentum exactly cancel, and therefore we have put them to zero from the start.

III. THE SPIN AVERAGED $|M|^2$

The normal technique is to transform the spin averaged $|M|^2$ into traces. We get

$$\overline{|M|^2} = \frac{1}{4}\left(\frac{g}{\cos\theta_W}\right)^6 M_Z^2 \sum_{i=1}^4 \sum_{j=1}^4 \text{Tr}\left[\not{p}_2\,\Gamma_i^{\mu\nu}\,\not{p}_1\,\overline{\Gamma}_j^{\beta\alpha}\right]\left(-g_{\mu\alpha} - \frac{q_{1\mu}q_{1\alpha}}{M_Z^2}\right)\left(-g_{\nu\beta} - \frac{q_{2\nu}q_{2\beta}}{M_Z^2}\right) \tag{8}$$

where

$$\Gamma_i = \Gamma_i^{\alpha\beta}\,\epsilon^\alpha(q_1)\epsilon^\beta(q_2) \quad;\quad \overline{\Gamma}_i^{\beta\alpha} = \gamma^0\left(\Gamma_i^{\alpha\beta}\right)^\dagger\gamma^0 \tag{9}$$

These traces are not so easy to evaluate because they are traces of eight Dirac γ matrices. The best is to resort to automatic evaluation using some algebraic program. Nowadays, it seems to me that the more interesting approach is to use the package FeynCalc [2, 3] for Mathematica. In the appendix it is shown an example of a program that does that for the first diagram. It should be easy to generalize to all the diagrams including interference terms. For the following it is convenient to introduce the following notation

$$\overline{|M|^2} \;\equiv\; \frac{1}{4}\left(\frac{g}{\cos\theta_W}\right)^6 \frac{\sqrt{2}\pi^4}{M_Z^2}\sum_{i,j} Y_{ij}$$

$$= 64\pi^4\,G_F^3 M_Z^4 \sum_{i,j} Y_{ij} \tag{10}$$

where the Y_{ij} are dimensionless and defined by,

$$Y_{ij} = \frac{M_Z^4}{\sqrt{2}\pi^4}\text{Tr}\left[\not{p}_2\,\Gamma_i^{\mu\nu}\,\not{p}_1\,\overline{\Gamma}_j^{\beta\alpha}\right]\left(-g_{\mu\alpha} - \frac{q_{1\mu}q_{1\alpha}}{M_Z^2}\right)\left(-g_{\nu\beta} - \frac{q_{2\nu}q_{2\beta}}{M_Z^2}\right) \tag{11}$$

IV.   THE CROSS SECTION

We choose the kinematics in the CM frame shown in Fig 2. The cross section is then given by (there is an extra

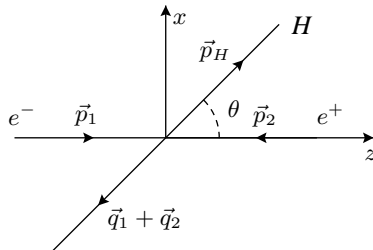
Figure 2: Kinematics

factor of 1/2 because there are two identical particles in the final state),

$$d\sigma = \frac{1}{(2\pi)^5}\frac{1}{2s}\frac{1}{2}\,\overline{|M|^2}\,\delta^4(p_1+p_2-p_H-q_1-q_2)\frac{d^3p_H}{2E_H}\frac{d^3q_1}{2q_1^0}\frac{d^3q_2}{2q_2^0} \tag{12}$$

Using Eq. (10) we can write then,

$$\frac{d\sigma}{dE_H d\cos\theta} = \frac{G_F^3 M_Z^4\, p_H}{s}\int_0^{2\pi}\frac{d\varphi}{2\pi}\int\frac{d^3q_1}{2q_1^0}\frac{d^3q_2}{2q_2^0}\,\delta^4(p_1+p_2-p_H-q_1-q_2)\sum_{ij}Y_{ij}\,. \tag{13}$$

To proceed we must do the integrations over the two Z bosons. For this we evaluate the following quantities

$$X_{ij} = \int\frac{d^3q_1}{2q_1^0}\frac{d^3q_2}{2q_2^0}\,\delta^4(\Delta-q_1-q_2)\,Y_{ij}\,. \tag{14}$$

where $\Delta = p_1+p_2-p_H$. These quantities are Lorentz invariant and can be evaluated in any reference frame. However the calculation it is particularly simple in the CM frame of the two Z bosons. In that frame

$$\Delta = (m_{12},0,0,0) \tag{15}$$

where $m_{12}^2 = (q_1+q_2)^2 = (p_1+p_2-p_H)^2$ is the invariant mass of the pair. We then get (q_1^* are the vectors in the CM frame of the pair)

$$\begin{aligned}
X_{ij} &= \frac{1}{4}\int\frac{d^3q_1^*}{q_1^{*0}q_2^{*0}}\,\delta(m_{12}-q_1^{*0}-q_2^{*0})Y_{ij}\\[2mm]
&= \frac{1}{4}\int\frac{|\vec{q}*_1|dq_1^{*0}\,d\Omega_1^*}{q_1^{*0}}\,\delta(m_{12}-2q_1^{*0})Y_{ij}\\[2mm]
&= \frac{1}{8}\sqrt{1-\frac{4M_Z^2}{m_{12}^2}}\int d\Omega_1^*\,Y_{ij}
\end{aligned} \tag{16}$$

and the final form for the cross section is

$$\sigma = \sum_{ij}\sigma_{ij} \tag{17}$$

where

$$\sigma_{ij} = \frac{G_F^3 M_Z^4}{s}\int_{M_H}^{E_H^{\max}}dE_H\sqrt{E_H^2-M_H^2}\int_0^{\pi}d\theta\sin\theta\int_0^{\pi}d\theta^*\sin\theta^*\int_0^{2\pi}d\varphi^*\frac{1}{8}\sqrt{1-\frac{4M_Z^2}{m_{12}^2}}\,Y_{ij} \tag{18}$$

and we have already used the fact that the final result does not depend on the azimuthal angle φ of the H and have done that integration. It is crucial to understand the meaning of the angles in the previous expression. The angle θ is the scattering angle of the Higgs boson in the CM of the e^-e^+ system (the lab frame), while the angles (θ^*,φ^*) are the angles of one of the Z bosons (the one with momentum q_1) in the CM frame of the pair of Z bosons. It is in these variables that the integrations are simple to set, but of course, there remains the difficulty of relating the 4-momenta in that frame to the lab frame. We will discuss in generality this question in the following section.

V. KINEMATICS: HOW TO RELATE DIFFERENT REFERENCE FRAMES

As we have seen in the previous section we have to relate the coordinates of 4-vectors in different reference frames. The best way is to do this in steps, using in sequence elementary rotations and boosts. We always perform boosts along the z axis, so the relevant Lorentz transformation is

$$x = \mathbf{Boost_z}(\beta) \cdot x' \tag{19}$$

where $x(x')$ are the coordinates in the reference frame $S(S')$, respectively, and the reference frame S' moves with relative speed β along the z axis with respect to S. The explicit form of the matrix $\mathbf{Boost_z}(\beta)$ is

$$\mathbf{Boost_z}(\beta) = \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} \tag{20}$$

For rotations, we can always use a sequence of rotations along the coordinate axis. So the relevant rotations, always written with the convention

$$x = \mathbf{Rot_x}(\theta) \cdot x' \quad ; \quad x = \mathbf{Rot_y}(\theta) \cdot x' \quad ; \quad x = \mathbf{Rot_z}(\theta) \cdot x' \tag{21}$$

are

$$\mathbf{Rot_x}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad \mathbf{Rot_y}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{22}$$

and

$$\mathbf{Rot_z}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{23}$$

Now, if we look at the kinematics shown in Fig. 2, we can easily realize that the correct transformation laws for q_1^* and q_2^* in the CM frame of the Z boson pair are,

$$q_{1,2} = \mathbf{Rot_y}(\theta + \pi) \cdot \mathbf{Boost_z}(\beta) \cdot q_{1,2}^* \tag{24}$$

where

$$\begin{aligned} \beta &= \sqrt{1 - \frac{1}{\gamma^2}} \\ \gamma &= \frac{E_{12}}{m_{12}} \\ m_{12} &= \sqrt{(q_1 + q_2)^2} = \sqrt{s + M_H^2 - 2\sqrt{s}E_H} \\ E_{12} &= \frac{s + m_{12}^2 - M_H^2}{2\sqrt{s}} \end{aligned} \tag{25}$$

Taking in account that

$$\begin{cases} q_1^{*0} = \frac{1}{2}m_{12} \\ q_1^{*1} = q_{12}^{\mathrm{CM}} \sin\theta^* \cos\varphi^* \\ q_1^{*2} = q_{12}^{\mathrm{CM}} \sin\theta^* \sin\varphi^* \\ q_1^{*3} = q_{12}^{\mathrm{CM}} \cos\theta^* \end{cases} \qquad \begin{cases} q_2^{*0} = \frac{1}{2}m_{12} \\ q_2^{*1} = -q_{12}^{\mathrm{CM}} \sin\theta^* \cos\varphi^* \\ q_2^{*2} = -q_{12}^{\mathrm{CM}} \sin\theta^* \sin\varphi^* \\ q_2^{*3} = -q_{12}^{\mathrm{CM}} \cos\theta^* \end{cases} \tag{26}$$

where

$$q_{12}^{\mathrm{CM}} = \sqrt{\frac{m_{12}^2}{4} - M_Z^2} \tag{27}$$

we can now use Eq. (24) to obtain q_1 and q_2 in the lab frame, as a function of $(E_H, \theta, \theta^*, \varphi^*)$.

VI. NUMERICAL INTEGRATION

With the kinematics completely implemented, we have to perform the integrations and obtain the cross sections. In few problems can the integrals that appear in the evaluation of cross sections be done analytically. Most of the time we have to revert to numerical methods to evaluate the integrals. There are many ways to do evaluate the integrals numerically. A good library is `CUBA package`[4]. It can be linked either with `C/C++` or with `Fortran` programs. It has several methods with the same calling structure, so that one can substitute one method for another and check for accuracy. The manual comes with the package. I have done a gaussian integration subroutine in `Fortran` called `gauss.f` that uses the same calling conventions, and therefore can also be tested in the same way. The codes for these programs can be found in a Web page that I have created [5].

Using these methods one should be able to reproduce the final result it is shown in Fig. 3 and is in agreement with Ref.[1].
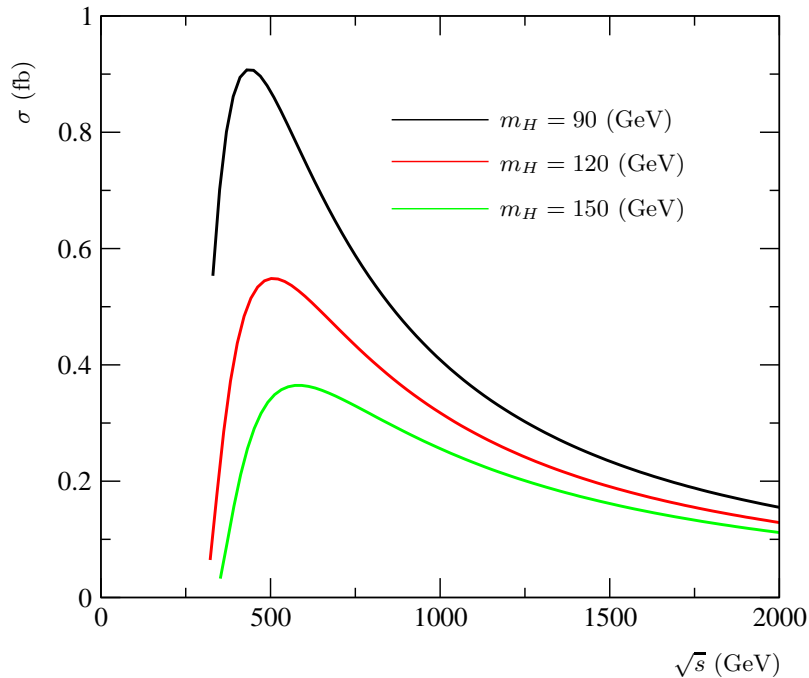

Figure 3: $\sigma(s)$ for $M_H = 90, 120, 150$ GeV.

VII. THE HELICITY AMPLITUDES TECHNIQUE

The example done in the previous sections requires already quite complex calculations. As the number of diagrams, N, grow the difficulty of the calculations grows as N^2 making a process with a large number of diagrams hard to handle. There is another way of making the calculations, where the complexity only grows linearly with N. This is the helicity amplitude method. This method is specially simple in the case of massless fermions and photons, but it can be generalized to the case of massive fermions and massive gauge bosons. As we will see the price to pay there is to have to integrate over more variables, but if one uses Monte Carlo methods, the efficiency does not change very much with the number of integrations.

A. Spinor products

In this section we will follow the work done by Kleiss and collaborators [6–8]. Let us start by considering massless fermions. Then the spinors can be taken as eigenstates of helicity, and there is no distinction between u and v spinors. We will denote these spinors by $u_\pm(p)$ with $p^2 = 0$. It is convenient to define the left and right projectors

$$\gamma_\pm = \frac{1 \pm \gamma_5}{2} \tag{28}$$

Then our spinors satisfy

$$\gamma_+ \not{p} = u_+(p)\overline{u}_+(p) \quad ; \quad \gamma_- \not{p} = u_-(p)\overline{u}_-(p) \quad ; \quad \not{p} = u_+(p)\overline{u}_+(p) + u_-(p)\overline{u}_-(p) \tag{29}$$

With these spinors we can form two independent *spinor products* [7],

$$s(p_1, p_2) = \overline{u}_+(p_1)u_-(p2) = -s(p_2, p_1)$$

$$t(p_1, p_2) = \overline{u}_-(p_1)u_+(p2) = s^*(p_2, p_1) \tag{30}$$

with the normalization

$$|s(p_1, p_2)|^2 = 2p_1 \cdot p_2 \tag{31}$$

In order to perform explicitly calculations it is necessary to have a formula for these spinor products. This is given by[6],

$$s(p_1, p_2) = \left(p_1^2 + ip_1^3\right) \sqrt{\frac{p_2^0 - p_2^1}{p_1^0 - p_1^1}} - \left(p_2^2 + ip_2^3\right) \sqrt{\frac{p_1^0 - p_1^1}{p_2^0 - p_2^1}} \tag{32}$$

and using it it is trivial to verify Eq. (31). A very important relation is ($\sigma = \pm$),

$$\overline{u}_\sigma(p_1)\gamma_\mu u_\sigma(p_2)\,\gamma^\mu = 2u_\sigma(p_2)\overline{u}_\sigma(p_1) + 2u_{-\sigma}(p_1)\overline{u}_{-\sigma}(p_2) \tag{33}$$

which shows that the spinors are normalized in such a way that

$$\overline{u}_\sigma(p)\gamma_\mu u_\sigma(p) = 2p_\mu \ . \tag{34}$$

Using Eq. (29) one can show the following useful relations:

$$\overline{u}_+(p_1)\not{p}_2\not{p}_3 \cdots \not{p}_{2n-1}u_-(p_{2n}) = s(p_1, p_2)s^*(p_3, p_2)s(p_3, p_4) \cdots s(p_{2n-1}, p_{2n})$$

$$\overline{u}_-(p_1)\not{p}_2\not{p}_3 \cdots \not{p}_{2n-1}u_+(p_{2n}) = s^*(p_2, p_1)s(p_2, p_3)s^*(p_4, p_3) \cdots s^*(p_{2n}, p_{2n-1})$$

$$\overline{u}_+(p_1)\not{p}_2\not{p}_3 \cdots \not{p}_{2n}u_+(p_{2n+1}) = s(p_1, p_2)s^*(p_3, p_2)s(p_3, p_4) \cdots s^*(p_{2n+1}, p_{2n})$$

$$\overline{u}_-(p_1)\not{p}_2\not{p}_3 \cdots \not{p}_{2n}u_-(p_{2n+1}) = s^*(p_2, p_1)s(p_2, p_3)s^*(p_4, p_3) \cdots s(p_{2n}, p_{2n+1}) \tag{35}$$

where p_i are massless momenta. When we have two fermion lines connected by the contraction of a Dirac γ matrix, then one has to use Eq. (33). For instance,

$$\overline{u}_+(p_1)\gamma_\mu u_+(p_2)\,\overline{u}_+(p_3)\gamma^\mu u_+(p_4) = 2s(p_3, p_1)s^*(p_4, p_2) \ . \tag{36}$$

Using the above relations one can transform all the amplitudes involving massless fermions in terms of spinor products.

As an example, let us evaluate the Bhabha cross section in QED in the limit of massless electrons (high energy limit). We have the two diagrams of Fig 4, corresponding to the amplitudes,
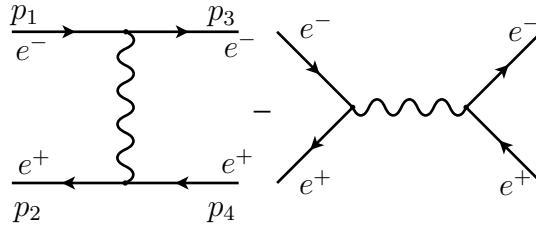

Figure 4: Diagrams for Bhabha scattering

$$M_1 = \frac{e^2}{t}\overline{u}(p_3)\gamma^\mu u(p_1)\overline{v}(p_2)\gamma_\mu v(p_4)$$

$$M_2 = -\frac{e^2}{s}\overline{v}(p_2)\gamma^\mu u(p_1)\overline{u}(p_3)\gamma_\mu v(p_4) \tag{37}$$

with $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$. One normally now goes for the spin averaged amplitude squared using the trace technique and obtaining easily,

$$\frac{1}{4} \sum_{\text{spins}} |M_1 + M_2|^2 = 2e^4 \left[\frac{t^2 + (s+t)^2}{s^2} + \frac{s^2 + (s+t)^2}{t^2} + 2\frac{(s+t)^2}{st} \right] \tag{38}$$

Let us now use the helicity amplitude technique to recover the same result. First we start by noticing that from the 16 possible helicity amplitudes $M(\sigma_1 \sigma_2; \sigma_3 \sigma_4)$ only 6 are non zero. These are $M(++;++)$, $M(--;--)$, $M(+-;+-)$, $M(-+;-+)$, $M(++;--)$ and $M(--;++)$. Using the compact notation

$$s_{ij} = s(p_i, p_j) \tag{39}$$

and using relations of the type of Eq. (36), we get

$$M(++;++) = M_1(++;++) + M_2(++;++) = 2e^2 \left[\frac{s_{23} s_{41}^*}{t} - \frac{s_{32} s_{41}^*}{s} \right] \tag{40}$$

$$M(--;--) = M_1(--;--) + M_2(--;--) = 2e^2 \left[\frac{s_{32}^* s_{14}}{t} - \frac{s_{23}^* s_{14}}{s} \right] \tag{41}$$

$$M(+-;+-) = M_1(+-;+-) = 2e^2 \frac{s_{12}^* s_{34}}{t} \tag{42}$$

$$M(-+;-+) = M_1(-+;-+) = 2e^2 \frac{s_{21} s_{43}^*}{t} \tag{43}$$

$$M(++;--) = M_2(++;-) = -2e^2 \frac{s_{13}^* s_{24}}{s} \tag{44}$$

$$M(--;++) = M_2(--;++) = -2e^2 \frac{s_{31} s_{42}^*}{s} \tag{45}$$

Now

$$\begin{aligned}
\overline{|M|^2} &= \frac{1}{4} \left[|M(++;++)|^2 + |M(--;--)|^2 + |M(+-;+-)|^2 \right. \\
&\qquad \left. + |M(-+;-+)|^2 + |M(++;--)|^2 + |M(--;++)|^2 \right] \\
&= e^4 \left[2\frac{|s_{23}|^2 |s_{41}|^2}{t^2} + 2\frac{|s_{32}|^2 |s_{41}|^2}{s^2} + 4\frac{|s_{23}|^2 |s_{41}|^2}{st} \right. \\
&\qquad \left. 2\frac{|s_{12}|^2 |s_{34}|^2}{t^2} + 2\frac{|s_{13}|^2 |s_{24}|^2}{s^2} \right]
\end{aligned} \tag{46}$$

and using

$$\begin{aligned}
|s_{23}|^2 &= -u = t + s = |s_{41}|^2 \\
|s_{12}|^2 &= s = |s_{34}|^2 \\
|s_{13}|^2 &= t = |s_{24}|^2
\end{aligned} \tag{47}$$

we finally recover Eq. (38). It should be noted that for this simple problem there is probably no gain in using this technique instead of just taking the traces. However in more complex problems, like $e^- e^+ \to e^- e^+ e^- e^+$ the gain is enormous. Also in these cases one normally calculates the amplitudes numerically as complex numbers, using Eq.(32), and then takes the absolute value, without having to transform them in scalar products.

B. Polarizations for massless gauge bosons

In the previous section we only considered gauge bosons in the internal lines. What happens if we have gauge bosons in the external lines? In this section we will show, first for massless gauge bosons (photons), how to handle

the sum over polarizations. We want to implement, using massless spinors, the result

$$\sum_\lambda \epsilon^\mu(k,\lambda)\epsilon^{*\mu}(k,\lambda) = -g^{\mu\nu} + \text{terms proportional to } k. \tag{48}$$

where the terms proportional to k can be neglected because of gauge invariance. Following Kleiss[7, 8] this can be done with the identification

$$\epsilon^\mu(k,\lambda) = \frac{1}{(4k\cdot p)^{1/2}}\,\overline{u}_\lambda(k)\gamma^\mu u_\lambda(p) \tag{49}$$

where p is any light-like 4-vector not proportional to k. As this polarization vector will always appear contracted with a γ_μ matrix we can use Eq. (33) to write,

$$\not{\epsilon}(k,\lambda) = 2u_\lambda(p)\overline{u}_\lambda(k) + 2u_{-\lambda}(k)\overline{u}_{-\lambda}(p) \tag{50}$$

which shows that we can indeed use for the polarization vectors massless spinors. Before we proceed with a simple example let us show that Eq. (49) and Eq. (48) are indeed consistent with each other. We have $(N = \sqrt{4k\cdot p})$,

$$\begin{aligned}
\sum_\lambda \epsilon^\mu(k,\lambda)\epsilon^{*\nu}(k,\lambda) &= \frac{1}{N^2}\left[\overline{u}_+(k)\gamma^\mu u_+(p)\overline{u}_+(p)\gamma^\nu u_+(k) + \overline{u}_-(k)\gamma^\mu u_-(p)\overline{u}_-(p)\gamma^\nu u_-(k)\right] \\
&= \frac{1}{N^2}\left(\text{Tr}\left[\gamma^\mu \not{p}\gamma^\nu \gamma_+ \not{k}\right] + \text{Tr}\left[\gamma^\mu \not{p}\gamma^\nu \gamma_- \not{k}\right]\right) \\
&= -g^{\mu\nu} + \frac{p^\mu k^\nu + p^\nu k^\mu}{k\cdot p} \tag{51}
\end{aligned}$$

As an example of this technique we consider the process $e^-e^+ \to \gamma\gamma$. We have two diagrams as shown in Fig. 5. The

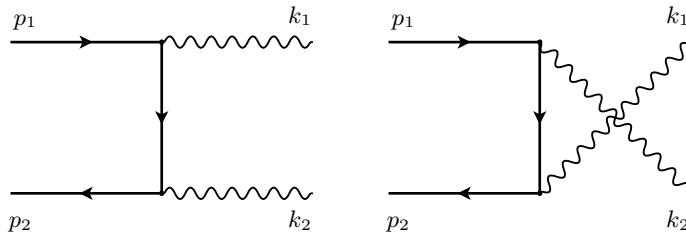Figure 5: Diagrams for $e^-e^+ \to \gamma\gamma$

corresponding amplitudes are given by

$$\begin{aligned}
M_1 &= e^2\,\overline{v}(p_2)\not{\epsilon}(k_2)\,(\not{p}_1 - \not{k}_1)\not{\epsilon}(k_1)u(p_1)\frac{1}{t} \\
M_2 &= e^2\,\overline{v}(p_2)\not{\epsilon}(k_1)\,(\not{p}_1 - \not{k}_2)\not{\epsilon}(k_2)u(p_1)\frac{1}{u}
\end{aligned}$$

$$\tag{52}$$

where $t = (p_1 - k_1)^2$ and $u = (p_1 - k_2)^2$. It is not difficult to show, using the usual trace techniques, that the interference vanishes and that we obtain,

$$\sum_{\text{spins}}\sum_{\lambda_1,\lambda_2}|M|^2 = 8e^4\frac{u}{t} + 8e^4\frac{t}{u} = 8e^4\frac{u^2 + t^2}{ut} \tag{53}$$

Now let us use the helicity amplitudes technique to reproduce the same result. By looking at Eq. (52) one immediately realizes that the electron and the positron must have the same helicity. For the photons it is more subtle.

We are free to choose the reference vector in the polarization vector, Eq. (49), in order to simplify the calculation. The only restriction is that in $\epsilon(k, \lambda)$ the reference vector p is not proportional to k. We can even make different choices for different diagrams. For instance, if we choose for M_1 the reference p_1 for $\epsilon(k_1, \lambda)$ and p_2 for $\epsilon(k_2, \lambda)$ and for M_2 the reverse choice, that is, p_2 for $\epsilon(k_1, \lambda)$ and p_1 for $\epsilon(k_2, \lambda)$, we immediately realize that the only non vanishing amplitudes are,

$$
\begin{aligned}
M(++;-+) &= 4e^2 \frac{s(p_1, k_1)s(p_2, k_2)s^*(k_1, p_2)s^*(p_1, k_1)}{t} \frac{1}{N(p_1, k_1)^2} \\
&= 2e^2 \frac{s(p_2, k_2)s^*(k_1, p_2)}{t} \\
M(++;+-) &= 4e^2 \frac{s(p_1, k_2)s(p_2, k_1)s^*(k_2, p_2)s^*(p_1, k_2)}{u} \frac{1}{N(p_1, k_2)^2} \\
&= 2e^2 \frac{s(p_2, k_1)s^*(k_2, p_2)}{u} \\
M(--;+-) &= 4e^2 \frac{s(p_1, k_1)s(p_2, k_1)s^*(k_2, p_2)s^*(p_1, k_1)}{t} \frac{1}{N(p_1, k_1)^2} \\
&= 2e^2 \frac{s(p_2, k_1)s^*(k_2, p_2)}{t} \\
M(--;-+) &= 4e^2 \frac{s(p_1, k_2)s(p_2, k_2)s^*(k_1, p_2)s^*(p_1, k_2)}{u} \frac{1}{N(p_1, k_2)^2} \\
&= 2e^2 \frac{s(p_2, k_2)s^*(k_1, p_2)}{u}
\end{aligned}
\tag{54}
$$

From here we obtain

$$
\sum_{\sigma_1, \sigma_2, \lambda_1, \lambda_2} |M(\sigma_1, \sigma_2; \lambda_1, \lambda_2)|^2 = 8e^4 \frac{u}{t} + 8e^4 \frac{u}{t} = 8e^4 \frac{u^2 + t^2}{ut}
\tag{55}
$$

in agreement with Eq. (53).

If we had chosen another reference frame, the result would be the same but it would be more difficult to realize it. For instance, suppose that we take the same convention for M_2 as we did for M_1. Then we can easily show that,

$$
\begin{aligned}
M(++;++) &= 0 \\
M(++;+-) &= 4e^2 \frac{s(p_2, k_2)s(p_2, k_1)s^*(k_2, p_1)s^*(p_1, k_2)}{u} \frac{1}{N(p_1, k_1)^2} \\
M(++;-+) &= 4e^2 \frac{s(p_1, k_2)s(p_2, p_1)s^*(p_1, k_1)s^*(p_1, p_2)}{u} \frac{1}{N(p_1, k_1)^2} \\
&\quad +4e^2 \frac{s(p_1, k_1)s(p_2, k_2)s^*(k_1, p_2)s^*(p_1, k_1)}{t} \frac{1}{N(p_1, k_1)^2} \\
M(++;--) &= 4e^2 \frac{s(p_1, p_2)s(p_2, p_1)s^*(p_1, k_1)s^*(p_1, k_2)}{u} \frac{1}{N(p_1, k_1)^2} \\
&\quad +4e^2 \frac{s(p_2, k_2)s(p_2, p_1)s^*(k_2, k_1)s^*(p_1, k_2)}{u} \frac{1}{N(p_1, k_1)^2} \\
M(--;++) &= M(++;--)^* \\
M(--;+-) &= M(++;-+)^* \\
M(--;-+) &= M(++;+-)^* \\
M(--;--) &= 0
\end{aligned}
\tag{56}
$$

These expressions look different from those in Eq. (54). However choosing a given kinematics and using Eq. (32) one can show that they give the same result. For instance in Eq. (54) the amplitude $M(++;--)$ vanishes while in

Eq. (56) it does not seem to vanish. However, choosing the kinematics

$$
\begin{aligned}
p_1 &= (\sqrt{s}/2, 0, 0, \sqrt{s}/2) \\
p_2 &= (\sqrt{s}/2, 0, 0, -\sqrt{s}/2) \\
k_1 &= (\sqrt{s}/2, 0, \sqrt{s}/2 \sin\theta, \sqrt{s}/2 \cos\theta) \\
k_2 &= (\sqrt{s}/2, 0, -\sqrt{s}/2 \sin\theta, -\sqrt{s}/2 \cos\theta)
\end{aligned}
\tag{57}
$$

from Eq. (32) we obtain

$$
\begin{aligned}
s(p_1, p_2) &= i\, 2\frac{\sqrt{s}}{2} \\
s(p_1, k_1) &= i\, \frac{\sqrt{s}}{2} - i\, \frac{\sqrt{s}}{2} e^{-i\theta} \\
s(p_1, k_2) &= i\, \frac{\sqrt{s}}{2} + i\, \frac{\sqrt{s}}{2} e^{-i\theta} \\
s(p_2, k_1) &= -i\, \frac{\sqrt{s}}{2} - i\, \frac{\sqrt{s}}{2} e^{-i\theta} \\
s(p_2, k_2) &= -i\, \frac{\sqrt{s}}{2} + i\, \frac{\sqrt{s}}{2} e^{-i\theta} \\
s(k_2, k_1) &= -i\, 2\frac{\sqrt{s}}{2} e^{-i\theta}
\end{aligned}
\tag{58}
$$

Using these explicit expressions one can show that

$$
\begin{aligned}
s(p_1, p_2)s^*(p_1, k_1) &+ s(p_2, k_2)s^*(k_2, k_1) = \\
&= \left(\frac{\sqrt{s}}{2}\right)^2 \left[2i\left(-i + ie^{i\theta}\right) + \left(-i + ie^{-i\theta}\right)\left(2\,ie^{i\theta}\right)\right] \\
&= \left(\frac{\sqrt{s}}{2}\right)^2 \left(2 - 2e^{i\theta} + 2e^{i\theta} - 2\right) = 0
\end{aligned}
\tag{59}
$$

which is needed to verify that $M(++;--)$ in Eq. (56) vanishes identically. In a similar way one can verify that the expressions in Eq. (56) are equivalent to those in Eq. (54). To summarize, one can choose the more convenient reference frame to simplify the calculations.

C. Polarizations for massive gauge bosons

In the previous section we learned how to use massless spinors to describe the polarization vectors of massless gauge bosons (photons). However, for calculations within the Standard Model we also need to known how to describe massive gauge bosons, like the W and Z. The most important property that we have to retain is the polarization sum, that is,

$$
\sum_\lambda \epsilon^\mu(q, \lambda)\epsilon^{*\nu}(q, \lambda) = -g^{\mu\nu} + \frac{q^\mu q^\nu}{M_V^2}
\tag{60}
$$

where $q^2 = M_V^2$ and $V = W, Z$. This can be achieved if we make the assignment [7],

$$
\epsilon^\mu(q) \to a^\mu = \overline{u}_-(r_1)\gamma^\mu u_-(r_2)
\tag{61}
$$

where $r_{1,2}$ are two massless 4-vectors such that

$$
q = r_1 + r_2
\tag{62}
$$

and make the correspondence,

$$
\sum_\lambda \epsilon^\mu(q, \lambda)\epsilon^{*\nu}(q, \lambda) \to \frac{3}{8\pi M_V^2} \int d\Omega\, a^\mu a^{*\nu}
\tag{63}
$$

where $d\Omega$ is the solid angle of one of the 4-vectors, say r_1, in the rest frame of the gauge boson. Let us show that this is indeed the case. We have,

$$
\begin{aligned}
\int d\Omega \; a^\mu a^{*\nu} &= \int d\Omega \; \overline{u}_-(r_1)\gamma^\mu u_-(r_2)\overline{u}_-(r_2)\gamma^\mu u_-(r_1) \\
&= \int d\Omega \; \mathrm{Tr}\left[\gamma_-\slashed{r}_1\gamma^\mu\gamma_-\slashed{r}_2\gamma^\nu\right] \\
&= \int d\Omega \; (2r_1^\mu r_2^\nu + 2r_2^\mu r_1^\nu - 2r_1\cdot r_2 g^{\mu\nu})
\end{aligned}
\tag{64}
$$

This last integral can be easily done by realizing that it can only depend on the metric tensor and on the 4-vector q. Using $q^2 = 2r_1\cdot r_2 = M_V^2$ we have

$$
\begin{aligned}
I^{\mu\nu} &= \int d\Omega \left(2r_1^\mu r_2^\nu + 2r_2^\mu r_1^\nu - M_V^2 \, g^{\mu\nu}\right) \\
&= M_V^2 g^{\mu\nu} A + q^\mu q^\nu B
\end{aligned}
\tag{65}
$$

A and B can be obtained by multiplying Eq. (65) by $g^{\mu\nu}$ and by $q^\mu q^\nu$ and using the fact that $2q\cdot r_1 = q^2 = M_V^2$. We get,

$$
\begin{aligned}
4A + B &= -8\pi \\
A + B &= 0
\end{aligned}
\tag{66}
$$

which gives $A = -B = -8\pi/3$. We therefore obtain

$$
\int d\Omega \; a^\mu a^{*\nu} = \frac{8\pi M_V^2}{3}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_V^2}\right)
\tag{67}
$$

in agreement with Eq. (63). In the next section we will give an example of the use of this technique for massive gauge bosons.

VIII.   HELICITY AMPLITUDES FOR $e^-e^+ \to ZZH$

In this section we will use the process $e^-e^+ \to ZZH$ as an example of the helicity amplitudes technique.

A.   The Helicity Amplitudes

The amplitudes corresponding to the four diagrams of Fig. 1 are,

$$
M_1 = \left(\frac{g}{\cos\theta_W}\right)^3 M_Z \, \overline{u}(p_2)\slashed{\epsilon}(q_2)\left(g_-\gamma_- + g_+\gamma_+\right)\left(-\slashed{p}_2 + \slashed{q}_2\right)\slashed{\epsilon}(q_1)\left(g_-\gamma_- + g_+\gamma_+\right)u(p_1)\frac{1}{D_e(1)D_Z(1)}
$$

$$
M_2 = \left(\frac{g}{\cos\theta_W}\right)^3 M_Z \, \overline{u}(p_2)\slashed{\epsilon}(q_2)\left(g_-\gamma_- + g_+\gamma_+\right)\left(\slashed{p}_1 - \slashed{q}_1\right)\slashed{\epsilon}(q_1)\left(g_-\gamma_- + g_+\gamma_+\right)u(p_1)\frac{1}{D_e(2)D_Z(2)}
$$

$$
M_3 = \left(\frac{g}{\cos\theta_W}\right)^3 M_Z \, \overline{u}(p_2)\slashed{\epsilon}(q_1)\left(g_-\gamma_- + g_+\gamma_+\right)\left(\slashed{p}_1 - \slashed{q}_2\right)\slashed{\epsilon}(q_2)\left(g_-\gamma_- + g_+\gamma_+\right)u(p_1)\frac{1}{D_e(3)D_Z(3)}
\tag{68}
$$

$$
M_4 = \left(\frac{g}{\cos\theta_W}\right)^3 M_Z \, \overline{u}(p_2)\slashed{\epsilon}(q_1)\left(g_-\gamma_- + g_+\gamma_+\right)\left(-\slashed{p}_2 + \slashed{q}_1\right)\slashed{\epsilon}(q_2)\left(g_-\gamma_- + g_+\gamma_+\right)u(p_1)\frac{1}{D_e(4)D_Z(4)}
$$

where

$$
g_- = g_V + g_A \quad ; \quad g_+ = g_V - g_A
\tag{69}
$$

and the denominators were given in Eq. (4). From Eq. (68) one immediately realizes that the electron and the positron must have the same helicity. We use Eq. (61) and Eq. (33) to write,

$$\begin{aligned}
\not{q}(q_2) &= 2u_-(w_2)\overline{u}_-(w_1) + 2u_+(w_1)\overline{u}_+(w_2)\\
\not{q}(q_1) &= 2u_-(r_2)\overline{u}_-(r_1) + 2u_+(r_1)\overline{u}_+(r_2)
\end{aligned} \tag{70}$$

where the massless 4-vectors $w_{1,2}$ and $r_{1,2}$ satisfy

$$q_1 = r_1 + r_2 \quad ; \quad q_2 = w_1 + w_2 \tag{71}$$

Then, the non-vanishing amplitudes $M(\sigma_1\sigma_2)$ are:

$$\begin{aligned}
M_1(++) &= \left(\frac{g}{\cos\theta_W}\right)^3 M_Z \frac{4g_+^2}{D_e(1)D_Z(1)}\Big[-s(p_2,w_2)s^*(p_2,w_1)s(p_2,r_2)s^*(p_1,r_1)\\
&\qquad\qquad\qquad + s(p_2,w_2)s^*(w_2,w_1)s(w_2,r_2)s^*(p_1,r_1)\Big]\\[2mm]
M_1(--) &= \left(\frac{g}{\cos\theta_W}\right)^3 M_Z \frac{4g_-^2}{D_e(1)D_Z(1)}\Big[-s^*(w_1,p_2)s(w_2,p_2)s^*(r_1,p_2)s(r_2,p_1)\\
&\qquad\qquad\qquad + s^*(w_1,p_2)s(w_2,w_1)s^*(r_1,w_1)s(r_2,p_1)\Big]\\[2mm]
M_2(++) &= \left(\frac{g}{\cos\theta_W}\right)^3 M_Z \frac{4g_+^2}{D_e(2)D_Z(2)}\Big[+s(p_1,r_2)s(p_2,w_2)s^*p_1,r_1)s^*(p_1,w_1)\\
&\qquad\qquad\qquad - s(p_2,w_2)s(r_1,r_2)s^*(p_1,r_1)s^*(r_1,w_1)\Big]\\[2mm]
M_2(--) &= \left(\frac{g}{\cos\theta_W}\right)^3 M_Z \frac{4g_-^2}{D_e(2)D_Z(2)}\Big[+s(r_2,p_1)s(w_2,p_1)s^*(r_1,p_1)s^*(w_1,p_2)\\
&\qquad\qquad\qquad - s(r_2,p_1)s(w_2,r_2)s^*(r_1,r_2)s^*(w_1,p_2)\Big]\\[2mm]
M_3(++) &= \left(\frac{g}{\cos\theta_W}\right)^3 M_Z \frac{4g_+^2}{D_e(3)D_Z(3)}\Big[+s(p_1,w_2)s(p_2,r_2)s^*(p_1,r_1)s^*(p_1,w_1)\\
&\qquad\qquad\qquad - s(p_2,r_2)s(w_1,w_2)s^*(p_1,w_1)s^*(w_1,r_1)\Big]\\[2mm]
M_3(--) &= \left(\frac{g}{\cos\theta_W}\right)^3 M_Z \frac{4g_-^2}{D_e(3)D_Z(3)}\Big[+s(r_2,p_1)s(w_2,p_1)s^*(r_1,p_2)s^*(w_1,p_1)\\
&\qquad\qquad\qquad - s(r_2,w_2)s(w_2,p_1)s^*(r_1,p_2)s^*(w_1,w_2)\Big]\\[2mm]
M_4(++) &= \left(\frac{g}{\cos\theta_W}\right)^3 M_Z \frac{4g_+^2}{D_e(4)D_Z(4)}\Big[+s(p_2,r_2)s(r_2,w_2)s^*(p_1,w_1)s^*(r_2,r_1)\\
&\qquad\qquad\qquad - s(p_2,r_2)s(p_2,w_2)s^*(p_1,w_1)s^*(p_2,r1)\Big]\\[2mm]
M_4(--) &= \left(\frac{g}{\cos\theta_W}\right)^3 M_Z \frac{4g_-^2}{D_e(4)D_Z(4)}\Big[+s(r_2,r_1)s(w_2,p_1)s^*(r_1,p_2)s^*(w_1,r_1)\\
&\qquad\qquad\qquad - s(r_2,p_2)s(w_2,p_1)s^*(r_1,p_2)s^*(w_1,p_2)\Big]
\end{aligned} \tag{72}$$

Although these expressions are not too difficult to obtain, one can use the FeynCalc program to evaluate them. In the Appendix A 2 we show a program that evaluates the amplitudes in terms of the spinor products. The only point worth noticing is that from the definitions, Eq. (30), we can obtain

$$\begin{aligned}
\overline{u}(p_1)u(p_2) &= s(p_1,p_2) + s^*(p_2,p_1)\\
\overline{u}(p_1)\gamma_5 u(p_2) &= -s(p_1,p_2) + s^*(p_2,p_1)
\end{aligned} \tag{73}$$

which allows to write the spinors in terms of spinor products.

B. The Cross Section

To write the expression for the cross section we start by noticing that

$$\overline{|M|^2} = \frac{1}{4} \sum_{\sigma_i,\sigma_j} \sum_{\lambda_1,\lambda_2} |M(\sigma_1,\sigma_2,\lambda_1,\lambda_2)|^2$$

$$= \frac{1}{4} \left(\frac{3}{8\pi M_Z^2}\right)^2 \int d\Omega_{\vec{r}_1} \int d\Omega_{\vec{w}_1} \sum_{\sigma_i,\sigma_j} |M(\sigma_1\sigma_2)|^2 \tag{74}$$

$$= 64\pi^4 G_F^3 M_Z^4 \left(\frac{9}{64\pi^6\sqrt{2}}\right) \int d\Omega_{\vec{r}_1} \int d\Omega_{\vec{w}_1} \sum_{\sigma_i,\sigma_j} |Y(\sigma_1\sigma_2)|^2$$

where we have defined the *reduced amplitude*, $Y(\sigma_1\sigma_2)$, by

$$M(\sigma_1\sigma_2) \equiv \left(\frac{g}{\cos\theta_W}\right)^3 M_Z \; Y(\sigma_1\sigma_2) \tag{75}$$

Now, using the results of section IV, we can write,

$$\sigma = \frac{G_F^3 M_Z^4}{s} \int_{M_H}^{E_H^{\max}} dE_H \sqrt{E_H^2 - M_H^2} \int_0^\pi d\theta \sin\theta \int_0^\pi d\theta^* \sin\theta^* \int_0^{2\pi} d\varphi^* \int d\Omega_{\vec{r}_1} \int d\Omega_{\vec{w}_1}$$

$$\sqrt{1 - \frac{4M_Z^2}{m_{12}^2}} \left(\frac{9}{512\pi^6\sqrt{2}}\right) \sum_{\sigma_i,\sigma_j} |Y(\sigma_1\sigma_2)|^2 \tag{76}$$

The price to pay in this method is the increase in the number of integrations. Now, instead of the four integrations in Eq. (18), we have to do eight integrations. This can be a problem if we use the Gauss method, but if instead we use a Monte Carlo integration method, the increase in complexity is acceptable. The only non trivial problem is to write all the 4-vectors in the center of mass frame. This has been done in section V for the 4-vectors q_1, q_2, but we also have to be able to write the massless 4-vectors r_i, w_i in that frame. This will explained in the next section.

C. The Kinematics

The basic kinematics was explained in section V. The expressions for the 4-vectors q_i in the center of mass frame as still given by Eq. (26). For the 4-vectors r_i and w_i we have,

$$r_{1,2} = \mathbf{Rot_y}(\theta + \pi) \cdot \mathbf{Boost_z}(\beta) \cdot \mathbf{Rot_z}(\varphi^*) \cdot \mathbf{Rot_y}(\theta^*) \cdot \mathbf{Boost_z}(\beta_1) \; r_{1,2}^{\mathrm{CM}}$$

$$w_{1,2} = \mathbf{Rot_y}(\theta + \pi) \cdot \mathbf{Boost_z}(\beta) \cdot \mathbf{Rot_z}(\varphi^* + \pi) \cdot \mathbf{Rot_y}(\pi - \theta^*) \cdot \mathbf{Boost_z}(\beta_1) \; w_{1,2}^{\mathrm{CM}} \tag{77}$$

where

$$\gamma_1 = \frac{m_{12}}{2M_Z} \quad , \quad \beta_1 = \sqrt{1 - \frac{1}{\gamma_1^2}} \tag{78}$$

and

$$\begin{cases} r_{1,2}^{\mathrm{CM}\,0} = \frac{1}{2}m_Z \\ r_{1,2}^{\mathrm{CM}\,1} = \pm\frac{1}{2}m_Z \sin\theta_r \cos\varphi_r \\ r_{1,2}^{\mathrm{CM}\,2} = \pm\frac{1}{2}m_Z \sin\theta_r \sin\varphi_r \\ r_{1,2}^{\mathrm{CM}\,3} = \pm\frac{1}{2}m_Z \cos\theta_r \end{cases} \qquad \begin{cases} w_{1,2}^{\mathrm{CM}\,0} = \frac{1}{2}m_Z \\ w_{1,2}^{\mathrm{CM}\,1} = \pm\frac{1}{2}m_Z \sin\theta_w \cos\varphi_w \\ w_{1,2}^{\mathrm{CM}\,2} = \pm\frac{1}{2}m_Z \sin\theta_w \sin\varphi_w \\ w_{1,2}^{\mathrm{CM}\,3} = \pm\frac{1}{2}m_Z \cos\theta_w \end{cases} \tag{79}$$

Using Eq. (79) and Eq. (77) it is now straightforward to express r_i and w_i in the center of mass frame and to perform the integrations in Eq. (76).

IX. CONCLUSIONS

In this note we review the techniques used in Quantum Field Theory in a non trivial example, the Higgs boson production at a linear collider through the process $e^-e^+ \rightarrow ZZH$. The emphasis is in the use of the package FeynCalc for Mathematica and both the trace techniques and the helicity amplitudes method are discussed.

Appendix A: INPUT FOR FEYNCALC

1. Input for the Traces Method

```
(* Begin Program *)
(* Definitions *)

dm[mu_]:=DiracMatrix[mu]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]


kin={sp[p1,p1]->0,sp[p2,p2]->0,sp[q1,q1]->MZ^2,sp[q2,q2]->MZ^2,\
sp[p1,p2]->rs^2/2}
NumProp[p_,m_]:=(ds[p] + m )
Zcoup:= (gv - ga dm[5])
SumPol[mu_,nu_,k_,M_]:=(-mt[mu,nu]+fv[k,mu] fv[k,nu]/M^2)

Gamma1= dm[nu] . Zcoup . NumProp[-p2+q2,0] . dm[mu] . Zcoup
Gamma1Bar= dm[a] . Zcoup . NumProp[-p2+q2,0] . dm[b] . Zcoup

Line11=ds[p2] . Gamma1 . ds[p1] . Gamma1Bar

ans11:= Simplify[Contract[SumPol[mu,a,q1,MZ] SumPol[nu,b,q2,MZ] \
Tr[Line11]] DE[1]^2 DZ[1] DZc[1] /. kin ]

DotPro1={sp[p1,q1]->dot[p1,q1],sp[p1,q2]->dot[p1,q2],sp[p2,q1]->dot[p2,q1]}
DotPro2={sp[p2,q2]->dot[p2,q2],sp[q1,q2]->dot[q1,q2]}
DotProducts=Flatten[{DotPro1,DotPro2}]

PropZ1={DZ[1]->(MH^2+2*dot[q1,k] -I MZ GZ)/((MH^2+2*dot[q1,k])^2+(MZ*GZ)^2)}
PropZ1c={DZc[1]->(MH^2+2*dot[q1,k] +I MZ GZ)/((MH^2+2*dot[q1,k])^2+(MZ*GZ)^2)}
PropZ2={DZ[2]->(MH^2+2*dot[q2,k] -I MZ GZ)/((MH^2+2*dot[q2,k])^2+(MZ*GZ)^2)}
PropZ2c={DZc[2]->(MH^2+2*dot[q2,k] +I MZ GZ)/((MH^2+2*dot[q2,k])^2+(MZ*GZ)^2)}
PropZ=Flatten[{PropZ1,PropZ1c,PropZ2,PropZ2c}]

PropE1={DE[1]->1/(MZ^2-2*dot[p2,q2]),DE[2]->1/(MZ^2-2*dot[p1,q1])}
PropE2={DE[3]->1/(MZ^2-2*dot[p1,q2]),DE[4]->1/(MZ^2-2*dot[p2,q1])}
PropE=Flatten[{PropE1,PropE2}]

simp=Flatten[{kin,DotProducts,PropZ,PropE]

res11= Simplify[Expand[ans11 /. simp]]

(* End Program *)
```

2.   Input for the Helicity Amplitudes Method

```
(* Begin Program *)

(* Definitions *)
dp[s_]:= (1 + s DiracMatrix[5])/2
U[p_,s_]:= dp[s] . Spinor[p,0]
UBar[p_,s_]:= SpinorUBar[p,0] . dp[-s]
Myds[p_] := U[p,1] UBar[p,1] + U[p,-1] UBar[p,-1]
PolS[k_,p_,l_]:= 2 ( U[p,l]  UBar[k,l] ) + 2 ( U[k,-l]  UBar[p,-l] )
PolSV[k_,r1_,r2_]:= PolS[r1,r2,-1]
gvga= gm dp[-1] + gp dp[1]

(* The amplitudes *)

M1[s1_,s2_]:=UBar[p2,s2] . PolSV[q2,w1,w2] . gvga . \
(-Myds[p2]+Myds[w1]+Myds[w2]) . PolSV[q1,r1,r2] . gvga . U[p1,s1]
res1[s1_,s2_]:=DiracSimplify[DotSimplify[M1[s1,s2]]]/D1

M2[s1_,s2_]:=UBar[p2,s2] . PolSV[q2,w1,w2] . gvga . \
(Myds[p1]-Myds[r1]-Myds[r2]) . PolSV[q1,r1,r2] . gvga .  U[p1,s1]
res2[s1_,s2_]:=DiracSimplify[DotSimplify[M2[s1,s2]]]/D2

M3[s1_,s2_]:=UBar[p2,s2] . PolSV[q1,r1,r2] . gvga . \
(Myds[p1]-Myds[w1]-Myds[w2]) . PolSV[q2,w1,w2] .  gvga . U[p1,s1]
res3[s1_,s2_]:=DiracSimplify[DotSimplify[M3[s1,s2]]]/D3

M4[s1_,s2_]:=UBar[p2,s2] . PolSV[q1,r1,r2] . gvga . \
(-Myds[p2]+Myds[r1]+Myds[r2]) . PolSV[q2,w1,w2] .  gvga . U[p1,s1]
res4[s1_,s2_]:=DiracSimplify[DotSimplify[M4[s1,s2]]]/D4


(* Simplifications *)

vlist={p1,p2,r1,r2,w1,w2}

simp1=Table[Spinor[vlist[[i]],0] . Spinor[vlist[[j]],0] -> \
MySP[vlist[[i]],vlist[[j]]] + MySPc[vlist[[j]],vlist[[i]]],{i,1,6},{j,1,6}] \
/. {MySP[p_, p_] -> 0, MySPc[q_, q_] -> 0}

simp2=Table[Spinor[vlist[[i]],0] . DiracMatrix[5] . Spinor[vlist[[j]],0] -> \
-MySP[vlist[[i]],vlist[[j]]] + MySPc[vlist[[j]],vlist[[i]]],{i,1,6},{j,1,6}] \
/. {MySP[p_, p_] -> 0, MySPc[q_, q_] -> 0}

simp=Flatten[{simp1,simp2}];

M[s1_,s2_]:=Expand[res1[s1,s2]+res2[s1,s2]+res3[s1,s2]+res4[s1,s2] /. simp]

(* End Program *)
```

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