TÉCNICO LISBOA

# Física de Partículas
# Aula 2
# Relativistic Quantum Mechanics: Scattering and Decays

Jorge C. Romão
Instituto Superior Técnico, Departamento de Física & CFTP
A. Rovisco Pais 1, 1049-001 Lisboa, Portugal

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# Topics for this Class



- Natural units
- Golden rule for decays
- Decays into two particles
- Golden rule for cross sections
- Scattering $1 + 2 \to 3 + 4$ in the CM
- Feynman Rules for a spinless model
- Lifetime for $A$
- Scattering $A + A \to B + B$
- Higher order corrections

# Natural Units



- In particle Physics we use the natural units system where $\hbar = c = 1$ complemented with $\epsilon_0 = \mu_0 = 1$. Notice these are consistent $c^2 = 1/(\epsilon_0 \mu_0)$.
- Useful expressions

$$
\begin{aligned}
1 = c = 2.999792 \times 10^8 \text{ ms}^{-1} \quad &\to \quad 1 \text{ s} = 2.999792 \times 10^8 \text{ m} \\
1 = \hbar c = 197.327 \text{ MeV.fermi} \quad &\to \quad 1 \text{ MeV}^{-1} = 197.327 \times 10^{-15} \text{ m} \\
1 = \hbar = 1.054571 \times 10^{-34} \text{ Js} \quad &\to \quad 1 \text{ J.s} = 9.482529 \times 10^{33}
\end{aligned}
$$

- We can write everything in terms of energy

$$
\begin{aligned}
1 \text{ m} &= 5.067730 \times 10^{12} \text{ MeV}^{-1} \\
1 \text{ s} &= 1.520214 \times 10^{21} \text{ MeV}^{-1} \\
1 \text{ Kg} &= \frac{1 \text{ J.s}}{1 \text{ m}^2 \times 1 \text{ s}^{-1}} = \frac{1 \text{ J.s} \times 1 \text{ s}}{1 \text{ m}^2} = 5.613088 \times 10^{29} \text{ MeV} .
\end{aligned}
$$

- Useful relations

$$
\begin{aligned}
1 \text{ s}^{-1} &= 6.578023 \times 10^{-22} \text{ MeV} \\
1 \text{ pb} &= 2.568189 \times 10^{-15} \text{ MeV}^{-2} \\
1 \text{ GeV}^{-2} &= 3.893794 \times 10^8 \text{ pb}
\end{aligned}
$$

# Natural Units



- We could think that information is lost. However it is always possible to put back the $\hbar$ and $c$. Consider the cross section for $e^- + e^+ \to \mu^- + \mu^+$ in QED. If we neglect the masses we have

$$\sigma = \frac{4\pi\alpha^2}{s} \ \ \mathrm{GeV}^{-2}$$

where $s$ is the square of the CM energy and $\alpha = 1/137.032\cdots$, the fine structure constant.

- To go back to the SI system we use the fact that the cross section has dimensions of mass.

$$\begin{aligned} L^2 &= \left(ML^2T^{-2}\right)^{-2} \hbar^\beta c^\gamma \\ &= M^{-2}L^{-4}T^4 \left(ML^2T^{-1}\right)^\beta \left(LT^{-1}\right)^\gamma \\ &= M^{-2+\beta}\, L^{-4+2\beta+\gamma}\, T^{4-2\beta-\gamma}\,, \end{aligned}$$

with solution $\beta = 2, \gamma = 2$. Therefore

$$\sigma = \frac{4\pi\,\hbar^2 c^2 \alpha^2}{s}$$

# Golden Rule for Decays



$$\Gamma = \underbrace{\frac{1}{2m_1}}_{A} \underbrace{S}_{B} \int \underbrace{|\mathcal{M}|^2}_{C} \underbrace{(2\pi)^4 \delta^4(p_1 - \sum_{i=2}^{n} p_i) \prod_{j=2}^{n} \frac{d^3 p_j}{(2\pi)^3 2p_j^0}}_{D}$$

- A: Initial state
- B: Symmetry factor final state
- C: Invariant amplitude (Dynamics)
- D: Final State
- Dimensions of $\mathcal{M}$:

$$[\mathcal{M}] = (\text{mass})^{4-n} \equiv (\text{M})^{4-n}$$

- For $\Gamma$

$$[\Gamma] = \text{M}^{-1}\text{M}^{8-2n}\text{M}^{-4}\text{M}^{2n-2} = \text{M}$$

# Decays into two particles



- For two body decays the calculations can be done easily if particles are not polarized.
- We get

$$\begin{aligned}\Gamma =& \frac{1}{2m_1} S \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3) \frac{d^3p_2}{(2\pi)^3 2p_2^0} \frac{d^3p_3}{(2\pi)^3 2p_3^0} \\ =& \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2 \delta^4(p_1 - p_2 - p_3) \frac{d^3p_2}{p_2^0} \frac{d^3p_3}{p_3^0} \\ =& \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2 \delta\left(m_1 - \sqrt{|\vec{p}_2|^2 + m_2^2} - \sqrt{|\vec{p}_2|^2 + m_3^2}\right) \frac{d^3p_2}{p_2^0 p_3^0}\end{aligned}$$

where we have done the integration in $\vec{p}_3$, from which $\vec{p}_2 + \vec{p}_3 = 0$ with $p_i^0 = \sqrt{|\vec{p}_i|^2 + m_i^2}$.

- To continue we use spherical coordinates in momentum space

$$d^3p_2 = d\Omega_2 |\vec{p}_2|^2 d|\vec{p}_2|$$

With our assumptions $\mathcal{M}$ does not depend on the angles and the angular integration gives $4\pi$.

# Decays into two particles



- We get,

$$\Gamma = \frac{S}{8\pi m_1} \int d|\vec{p}_2||\vec{p}_2|^2 \, |\mathcal{M}|^2 \, \frac{\delta(m_1 - \sqrt{|\vec{p}_2|^2 + m_2^2} - \sqrt{|\vec{p}_2|^2 + m_3^2})}{p_2^0 p_3^0}$$

- Using

$$\delta(f(x)) = \sum_i^n \frac{\delta(x - x_i)}{|f'(x)|_{x=x_i}}$$

into

$$\delta(m_1 - \sqrt{|\vec{p}_2|^2 + m_2^2} - \sqrt{|\vec{p}_2|^2 + m_3^2}) = \frac{\delta(|\vec{p}_2| - \cdots)}{\frac{|\vec{p}_2|}{p_2^0} + \frac{|\vec{p}_2|}{p_3^0}}$$

- We finally get

$$\Gamma = \frac{S}{8\pi m_1^2} \, |\vec{p}_2| \, |\mathcal{M}|^2$$

# Golden Rule for Cross Sections



$$\sigma = \underbrace{\frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}}_{A} \underbrace{S}_{B} \int \underbrace{|\mathcal{M}|^2}_{C} \underbrace{(2\pi)^4 \delta^4(p_1 + p_2 - \sum_{i=3}^{n} p_i) \prod_{j=3}^{n} \frac{d^3 p_j}{(2\pi)^3 2p_j^0}}_{D}$$

- A: Initial state
- B: Symmetry factor for final state
- C: Invariant amplitude (Dynamics)
- D: Final state
- Dimensions of $\mathcal{M}$:

$$[\mathcal{M}] = (\text{mass})^{4-n} \equiv (\text{M})^{4-n}$$

- Dimensions of $\sigma$

$$[\sigma] = \text{M}^{-2}\text{M}^{8-2n}\text{M}^{-4}\text{M}^{2n-4} = \text{M}^{-2}$$

# Process $1+2 \to 3+4$ in the CM



- The simplest case is the process $1+2 \to 3+4$ in the CM frame.
- Even in this case is not possible to do all the integrations without knowing $\mathcal{M}$
- It is convenient to use the Mandelstam variable $s$, defined by

$$s = (p_1 + p_2)^2$$

- Expanding

$$s = m_1^2 + m_2^2 + 2 p_1 \cdot p_2$$

we get

$$p_1 \cdot p_2 = \frac{1}{2} \left( s - m_1^2 - m_2^2 \right)$$

- Therefore the initial state factor reads

$$4 \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = 4 \sqrt{s}\, |\vec{p}_1|$$

# Process $1+2 \to 3+4$ in the CM



- Show last step: Start with

$$|\vec{p}_1|^2 = E_1^2 - m_1^2 = \left(\frac{s+m_1^2-m_2^2}{2\sqrt{s}}\right)^2 - m_1^2$$

- we get

$$\begin{aligned} s|\vec{p}_1|^2 =& \frac{1}{4}\left[\left(s+m_1^2-m_2^2\right)^2 - 4sm_1^2\right] \\ =& \left[\frac{1}{4}\left(s-m_1^2-m_2^2\right)^2 - m_1^2 m_2^2\right] \\ =& (p_1 \cdot p_2)^2 - m_1^2 m_2^2 \end{aligned}$$

- and therefore

$$4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2} = 4\sqrt{s}\,|\vec{p}_1|$$

# Process $1+2 \to 3+4$ in the CM



- We have then

$$\sigma = \frac{S}{64\pi^2\sqrt{s}|\vec{p}_1|}\int |\mathcal{M}|^2\, \delta^4(p_1+p_2-p_3-p_4)\frac{d^3p_3}{p_3^0}\frac{d^3p_4}{p_4^0}$$

- Start with integration in $\vec{p}_4$,

$$\sigma = \frac{S}{64\pi^2\sqrt{s}|\vec{p}_1|}\int |\mathcal{M}|^2\, \delta(\sqrt{s}-\sqrt{|\vec{p}_3|^2+m_3^2}-\sqrt{|\vec{p}_3|^2+m_4^2})\frac{d^3p_3}{p_3^0p_4^0}$$

- Introduce spherical coordinates in the momentum $\vec{p}_3$.

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{S}{64\pi^2\sqrt{s}\,|\vec{p}_1|}\int \frac{d|\vec{p}_3||\vec{p}_3|^2}{p_3^0p_4^0}\,|\mathcal{M}|^2\, \delta(\sqrt{s}-\sqrt{|\vec{p}_3|^2+m_3^2}-\sqrt{|\vec{p}_3|^2+m_4^2})\\ &= \frac{S}{64\pi^2\sqrt{s}\,|\vec{p}_1|}\int |\mathcal{M}|^2\,\frac{d|\vec{p}_3||\vec{p}_3|^2}{p_3^0p_4^0}\,\frac{\delta(|\vec{p}_3|-\cdots)}{\frac{|\vec{p}_3|}{p_3^0}+\frac{|\vec{p}_3|}{p_3^0}}\\ &= \frac{S}{64\pi^2\sqrt{s}\,|\vec{p}_1|}\,\frac{|\vec{p}_3|}{p_3^0+p_4^0}\,|\mathcal{M}|^2 = \frac{S}{64\pi^2 s}\,\frac{|\vec{p}_3|}{|\vec{p}_1|}\,|\mathcal{M}|^2\end{aligned}$$

- To finish we have to know $\mathcal{M}$

# A spinless model

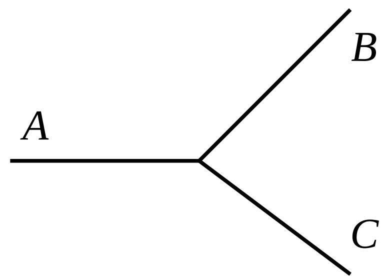


- To proceed we have to specify the rules to compute $\mathcal{M}$. These will be different for each different theory.
- Let us start with the simplest case of neutral scalars (spin 0). Consider a model with three of such particles: $A, B$ e $C$. We assume that the masses are

$$m_A > m_B + m_C$$

in such a way that the decay $A$ into $B + C$ is allowed.

- The model has only one interaction represented by the Feynman diagram




and the rule of multiplying by $-i\,g$

- The constant $g$ has dimensions of mass in this model.

# A spinless model



- The scattering $A + A \to B + B$ in lowest order is given by diagrams of Fig. 1. There are two diagrams because they are not distinguishable and must be added.

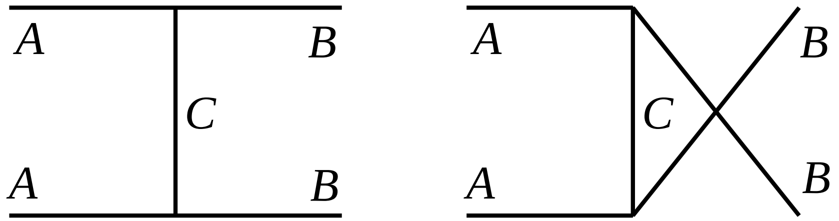



Figure 1: Process $A + A \to B + B$ in lowest order

- The scattering $A + B \to A + B$ is shown in Fig. 2.

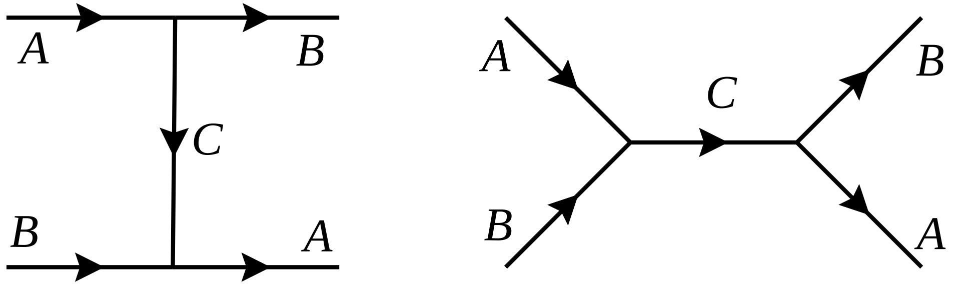



Figure 2: Process $A + B \to A + B$ in lowest order

# A spinless model



- This processes in lowest order are known as *tree level* processes.
- Higher order processes involve closed *loops*, like the corrections to the vertex shown in Fig. 3.

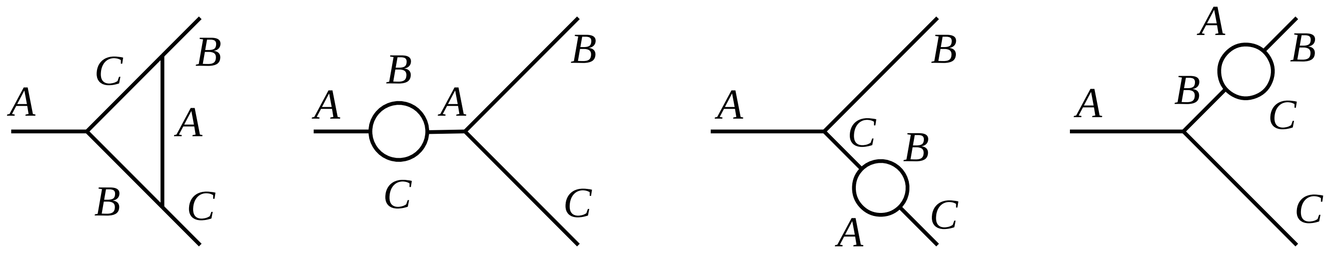



Figure 3: One loop corrections to the vertex

- In the spirit of perturbation theory these corrections, being of order $g^3$, must be smaller then the lowest order, of order $g$, and then can be neglected in first approximation.

# Feynman rules for the $ABC$ model



- Draw all distinct ways of connecting the initial state to the final state in a given order in the interaction
- For each vertex multiply by the factor

$$-i\; g$$

- For each internal line with momentum $q$ multiply by the propagator

$$\frac{i}{q^2-m^2}$$

- Apply energy-momentum conservation at each vertex
- For each loop choose one momentum $k$ for one of the internal lines and multiply by the factor

$$\int \frac{d^4 q}{(2\pi)^4}$$

- The result of the previous rules gives $-i\,\mathcal{M}$, therefore to obtain $\mathcal{M}$ multiply the final result by $i$

# Lifetime of particle $A$



- As particle $A$ can decay we will calculate its lifetime
- The Feynman diagram coincides with the vertex

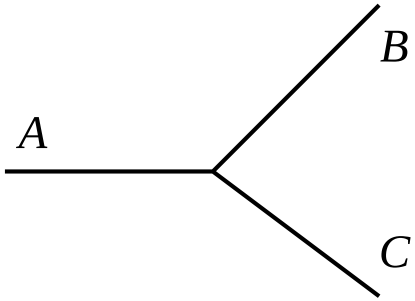



Figure 4: Decay $A \to B + C$ in lowest order

- Feynman rules give in this case

$$\mathcal{M} = g$$

- We can now use the formula for two body decays to get

$$\Gamma = \frac{g^2 |\vec{p}|}{8\pi m_A^2}$$

# Lifetime of particle $A$



- For the lifetime we get

$$\tau = \frac{1}{\Gamma} = \frac{8\pi m_A^2}{g^2 |\vec{p}|}$$

- Where the momentum in the rest frame of $A$ is given by

$$\begin{aligned} |\vec{p}| &= \sqrt{E_B^2 - m_B^2} \\ &= \sqrt{\left(\frac{m_A^2 + m_B^2 - m_C^2}{2m_A}\right)^2 - m_B^2} \\ &= \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2} \end{aligned}$$

# Scattering $A + A \to B + B$



- Consider the kinematics of Fig. 5

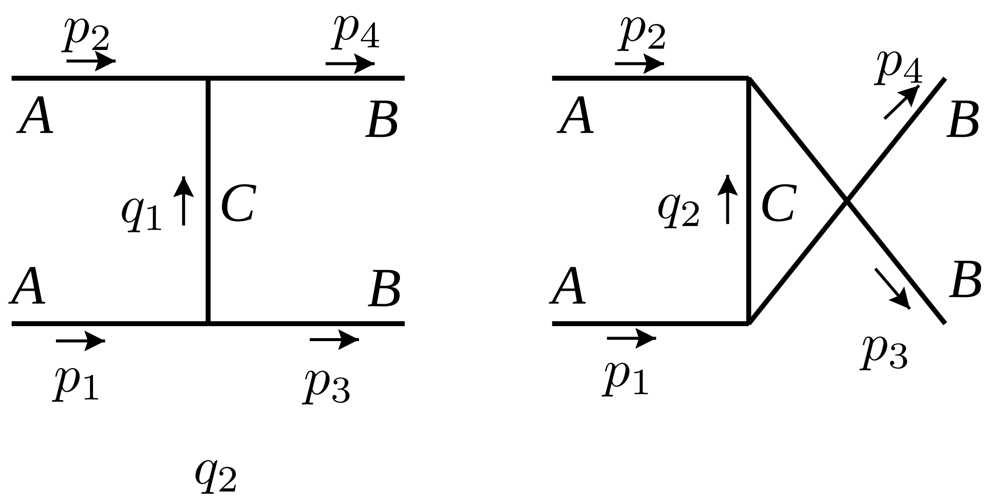



Figure 5: Kinematics for process $A + A \to B + B$

- Energy momentum conservation gives

$$q_1 = p_1 - p_3, \quad q_2 = p_1 - p_4$$

# Scattering $A + A \to B + B$



- Feynman rules give

$$\mathcal{M} = \frac{g^2}{(p_1 - p_3)^2 - m_C^2} + \frac{g^2}{(p_1 - p_4)^2 - m_C^2} = \frac{g^2}{t - m_C^2} + \frac{g^2}{u - m_C^2}$$

where we used the Mandelstam variables

- For these reason these diagrams are called $t$ and $u$ channel, respectively
- Introducing in the formula for the differential cross section in the CM we get

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} \frac{g^4}{64\pi^2 s} \frac{|\vec{p}_3|}{|\vec{p}_1|} \left[ \frac{1}{t - m_C^2} + \frac{1}{u - m_C^2} \right]^2$$

- To proceed we write $t$ and $u$ as

$$\begin{aligned} t &= (p_1 - p_3)^2 = m_A^2 + m_B^2 - 2E_1 E_3 (1 - \beta_3 \beta_1 \cos\theta) \\ u &= (p_1 - p_4)^2 = m_A^2 + m_B^2 - 2E_1 E_4 (1 + \beta_4 \beta_1 \cos\theta) \end{aligned}$$

where $\beta_i$ as the velocities in the CM and $\theta$ is the scattering angle between particles 1 and 3. Note the factor $S = 1/2$ for identical particles.

# Higher order processes



- The examples we saw are in lowest order. When we go to next order we begin to run into problems
- We are not going to show how these can be solved but let us show in a simple case what type of problems we have.
- Consider the corrections to the propagator of particle $A$, also called the *self-energy*. The Feynman diagram is shown in Fig. 6.

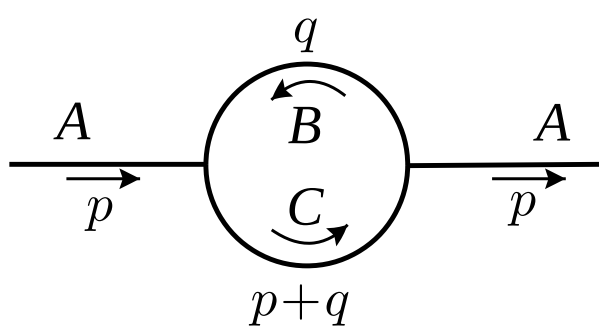



Figure 6: Self-energy of particle A

- Applying the Feynman rules we get

$$\mathcal{M} = i\, g^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{[q^2 - m_B^2][(p+q)^2 - m_C^2]}$$

# Higher order processes



- The integrations are done from $-\infty$ to $+\infty$. We immediately see that we run into problems as the integral diverges logarithmically

$$\int q^3 dq \, \frac{1}{q^4} = \int \frac{dq}{q} = \infty$$

- This problem took more then 30 years to be fully understood through a procedure known as renormalization.
- The study of this procedure is out of the scope of this course, but it can be said that it is now fully understood and we can make sense of those divergent integrals and compare the results with the experiment with great success.