TÉCNICO LISBOA

Física de Partículas
Aula 4
Quantum Field Theory and Feynman Diagrams

Jorge C. Romão
Instituto Superior Técnico, Departamento de Física & CFTP
A. Rovisco Pais 1, 1049-001 Lisboa, Portugal

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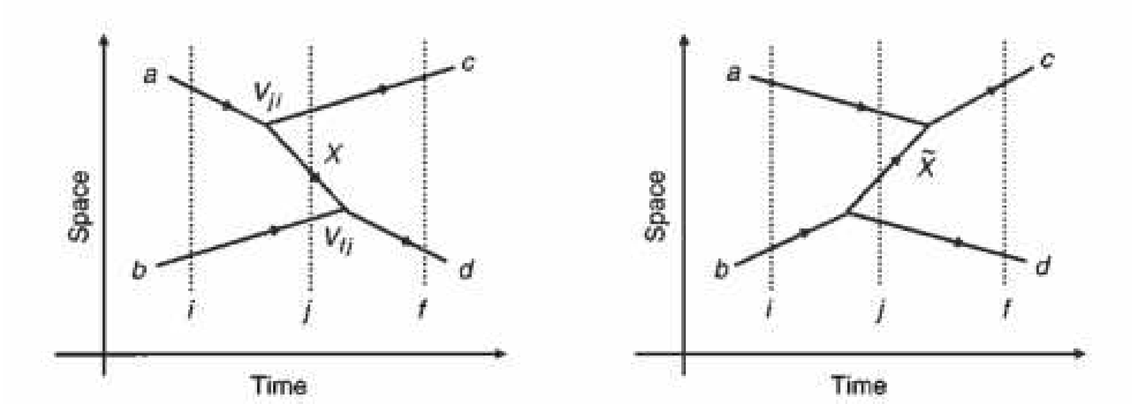
Topics for this Class

- Feynman diagrams and time ordering
- The photon
- Quantum Electrodynamics (QED)
- Feynman Rules for QED
- Examples
  - Electron-muon elastic scattering
  - Electron-positron elastic scattering
  - Compton effect
- Calculation of $e^- + e^+ \to \mu^- + \mu^+$
- Hadron production in $e^- + e^+$ collisions
  - Hadronization
  - The elementary process $e^- + e^+ \to q + \overline{q}$
  - The ratio R

We follow Chapters 5 and 6 of Mark Thomson book.

Feynman diagrams and time ordering

- Consider the process $a + b \to c + d$



- Using second order perturbation theory the first diagram is

$$T_{fi}^{ab} = \frac{\langle f|V|j\rangle \langle j|V|i\rangle}{E_i - E_j} = \frac{\langle d|V|X + b\rangle \langle c + X|V|a\rangle}{E_a + E_b - (E_c + E_X + E_b)}$$

- In non-relativistic QM one uses the transition amplitude T_{fi} while in relativistic QM one uses the Lorentz Invariant (LI) amplitude $\mathcal{M}_{fi}$. The relation is

$$T_{fi} = \mathcal{M}_{fi} \prod_k (2E_k)^{-1/2}$$

- Therefore

$$V_{ji} = \langle c + X|V|a\rangle = \frac{\mathcal{M}_{a\to c+X}}{(2E_a 2E_c 2E_X)^{1/2}}$$

Feynman diagrams and time ordering

- Take the LI amplitudes in the simplest case, a constant

$$\mathcal{M}_{a\to c+X} = g_a, \quad \mathcal{M}_{X+b\to d} = g_b$$

- Then

$$\mathcal{M}^{ab}_{fi} = (2E_a 2E_b 2E_c 2E_d)^{1/2}\, T^{ab}_{fi} = \frac{(2E_a 2E_b 2E_c 2E_d)^{1/2}}{2E_X\,(2E_a 2E_b 2E_c 2E_d)^{1/2}} \frac{g_a g_b}{E_a - E_c - E_X}$$

$$= \frac{1}{2E_X} \frac{g_a g_b}{E_a - E_c - E_X}$$

- For the second diagram we get

$$\mathcal{M}^{ba}_{fi} = \frac{1}{2E_X} \frac{g_a g_b}{E_b - E_d - E_X}$$

- The total LI amplitude is

$$\mathcal{M}_{fi} = \frac{g_a g_b}{2E_X}\left[\frac{1}{E_a - E_c - E_X} + \frac{1}{E_b - E_d - E_X}\right]$$

$$= \frac{g_a g_b}{2E_X}\left[\frac{1}{E_a - E_c - E_X} - \frac{1}{E_a - E_c + E_X}\right]$$

$$= \frac{g_a g_b}{(E_a - E_c)^2 - E_X^2}$$

$$E_a + E_b = E_c + E_d$$

$$\Downarrow$$

$$E_b - E_d = -(E_a - E_c)$$

Feynman diagrams and time ordering

- But we can relate E_X with the momenta. We have

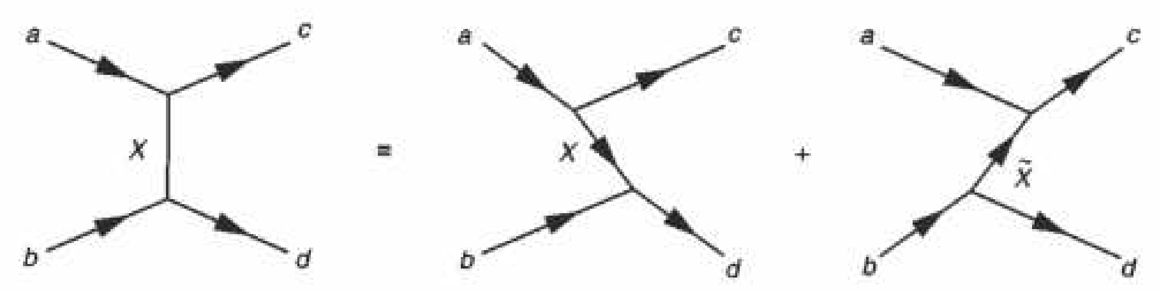
$$E_X^2 = |\vec{p}_X|^2 + m_X^2 = |\vec{p}_a - \vec{p}_c|^2 + m_X^2$$

- And therefore we get

$$\mathcal{M}_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - |\vec{p}_a - \vec{p}_c|^2 - m_X^2} = \frac{g_a g_b}{q^2 - m_X^2}$$

where $q = p_a - p_c$ is the momentum in the Feynman propagator

- Feynman diagrams represents the two time orderings, **therefore the relative position of the vertices with respect to time does not matter**



The photon: The formalism of the electromagnetism

- The (non-homogeneous) Maxwell equations are

$$\partial_\mu F^{\mu\nu} = J^\nu , \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- In free space, and using the Lorentz condition ($\partial_\mu A^\mu = 0$) we have the wave equation

$$\Box A^\mu = 0$$

that has plane waves as solutions

$$A^\mu(x) = N e^{-i\,p\cdot x}\, \epsilon^\mu(p)$$

where N is a normalization and $\epsilon^\mu(p)$ is the polarization vector that characterizes the spin of the photon

- Lorentz condition implies

$$\epsilon_\mu p^\mu = 0 \; .$$

The photon: The formalism of the electromagnetism

- We know from Classical Electrodynamics that the photon has two polarizations, but we are describing it with a 4-vector A^μ with 4 degrees of freedom.
- The Lorentz condition fixes already one of the degrees of freedom and to fully fix them we can choose the Coulomb gauge, where

$$A^0 = 0\,, \quad \vec{\nabla}\cdot\vec{A} = 0 \quad \Rightarrow \quad \epsilon^0 = 0\,, \quad \vec{\epsilon}\cdot\vec{p} = 0$$

meaning that the two polarizations are perpendicular to the direction of motion. Taking this as the z axis we have

$$\epsilon(p,1) = (0,1,0,0), \quad \epsilon(p,2) = (0,0,1,0)$$

- These vectors obey

$$\epsilon_\mu p^\mu = 0\,, \quad \epsilon_\mu(p,1)\epsilon^\mu(p,2) = 0\,, \quad \epsilon_\mu(p,\lambda)\epsilon^\mu(p,\lambda) = -1$$

$$\sum_{\text{Pol}} \epsilon^*_\mu \epsilon_\mu = -g_{\mu\nu}$$

Quantum Electrodynamics (QED)

- Quantum Electrodynamics (QED) is the theory of the interaction of electrons (and positrons) with photons. Later we will discuss the Lagrangian of QED as the basic example of the construction of a gauge theory. Here we just discuss the form of the interaction
- We saw that from Dirac equation we can derive a conserved probability current given by

$$j^\mu = \overline{\psi}\gamma^\mu\psi, \quad \partial_\mu j^\mu = 0$$

If we multiply by the electron charge, $q_e = -e$, where e is the proton charge, we get the electromagnetic current

$$J^\mu = -ej^\mu = -e\overline{\psi}\gamma^\mu\psi$$

- This is the current that appears in Maxwell equations. How does this current interact with the photon? From classical Electrodynamics we know that the Lagrangian for a non-relativistic particle with charge q in interaction with the electromagnetic field is

$$L = \frac{1}{2}mv^2 - q\phi + q\vec{A}\cdot\vec{v}$$

Quantum Electrodynamics (QED)

- In Quantum Field Theory we define a Lagrangian density

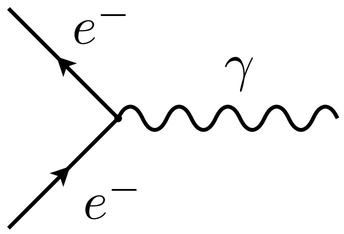
$$L \equiv \int d^3x \mathcal{L}$$

- This gives for the interaction

$$\mathcal{L}_{\rm int} = -J^\mu A_\mu = e\overline{\psi}\gamma^\mu\psi A_\mu = -eQ_e\overline{\psi}\gamma^\mu\psi A_\mu$$

where $Q_e = -1$.

- In the language of Feynman diagrams we describe the interaction by the vertex



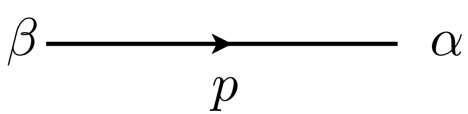
$$-ie\, Q_e \gamma^\mu$$

- We see that the Feynman rule for the vertex corresponds to *take out* the fields from the interaction Lagrangian and in the end multiply the result by i

Feynman Rules for QED

We are going now to list the complete set of Feynman rules for QED. They follow very closely what we have seen in the ABC model with the necessary modifications for spinors and antiparticles

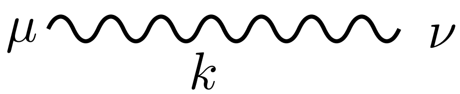
1. For a given process draw all the distinct diagrams that connect the initial state with the final state
2. For each electron entering the diagram a factor $u(p,s)$. If the electron is leaving the diagram a factor $\overline{u}(p,s)$
3. For each positron leaving the diagram a factor $v(p,s)$. If it enters the diagram a factor of $\overline{v}(p,s)$.
4. For each photon in the initial sate a polarization vector $\varepsilon_\mu(k)$ and for the final state $\varepsilon^*_\mu(k)$.
5. For each internal electron line, the propagator



$$S_{F\alpha\beta}(p) = i\frac{(\not p + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon}$$

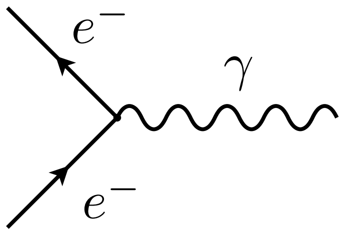
Feynman Rules for QED

6. For each internal photon line the propagator (in Feynman's gauge)



$$D_{F\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2 + i\varepsilon}$$

7. For each vertex



$$(-ieQ_e\gamma^\mu)_{\alpha\beta}$$

where $e = |e|$, is the positron charge. For the electron then $Q_e = -1$

8. For each internal momenta not fixed by energy-momentum conservation (*loops*) a factor

$$\int \frac{d^4q}{(2\pi)^4}$$

Feynman Rules for QED

9. For each fermion loop a sign (-1).
10. A factor -1 between diagrams that differ by an odd number of permutations (Fermi statistics)
11. The result of the application of the previous rules gives $-i\,\mathcal{M}$, therefore to get $\mathcal{M}$ we have to multiply the final result by i.

Comments

1. Rules 9) and 10) are difficult to explain with second quantization and Wick's theorem
2. To correctly write the fermionic lines. notice that in the end they have to give a complex number, that is a 1×1 matrix in Dirac space. To get this we should follow the rule that we start always at the head of the arrow.
3. The denominators of the propagators have the same form as in the ABC model. As we have seen they can be understood from summing over the different time orderings for the vertices connected by the propagator. The numerators are different for electrons (spin 1/2) and photons (spin 1). They correspond to the sum over spins or polarizations.

Simple processes in QED

- If we only consider two particles in the final state the number of processes in lowest order is very small

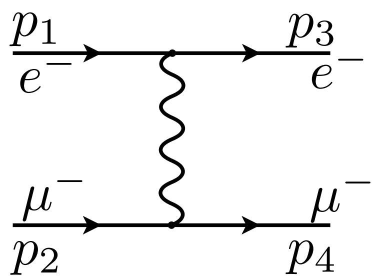
Process	Comment
$\mu^- + e^- \to \mu^- + e^-$	in QED
$e^- + e^+ \to e^- + e^+$	Bhabha scattering
$\gamma + e^- \to \gamma + e^-$	Compton effect
e^-+ Nucleus(Z) $\to e^-+$ Nucleus(Z) $+\gamma$	Bremsstrahlung
$e^- + e^+ \to \gamma + \gamma$	Pair annihilation
$e^- + e^- \to e^- + e^-$	Möller scattering
$\gamma + \gamma \to e^- + e^+$	Pair creation
$\gamma+$ Nucleus(Z) $\to$ Nucleus(Z) $+e^- + e^+$	Pair creation

Table 1: Simple processes in QED

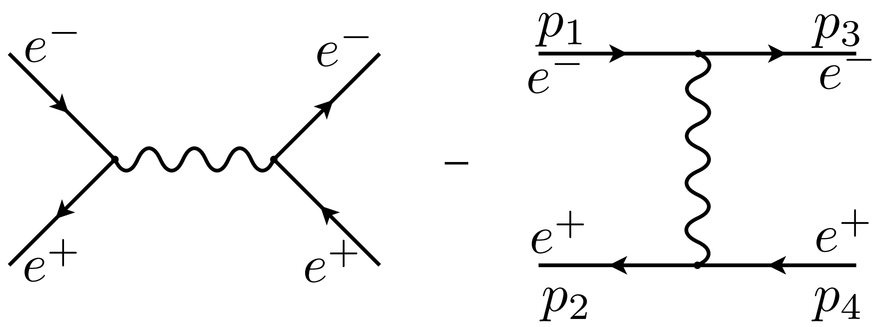
- We will show the diagrams for the first three and will learn how to calculate using a simpler, but important, case: $e^- + e^+ \to \mu^-\mu^+$

Diagrams for three processes

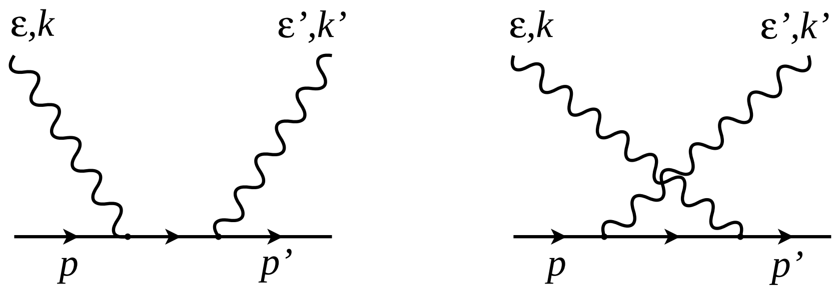
- $\mu^- + e^- \to \mu^- + e^-$



- $e^- + e^+ \to e^- + e^+$ (Bhabha scattering)

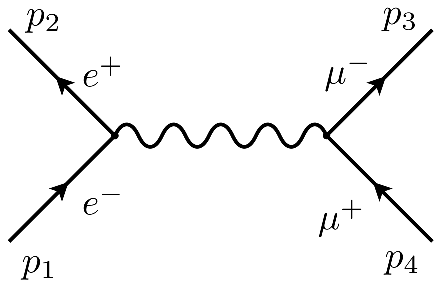


- $\gamma + e^- \to \gamma + e^-$ (Compton effect)

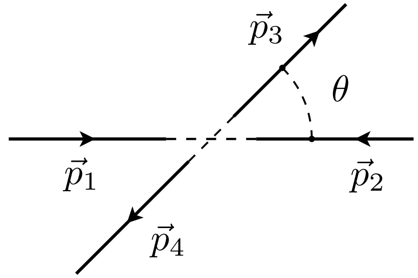


Calculating the process $e^- + e^+ \to \mu^- + \mu^+$

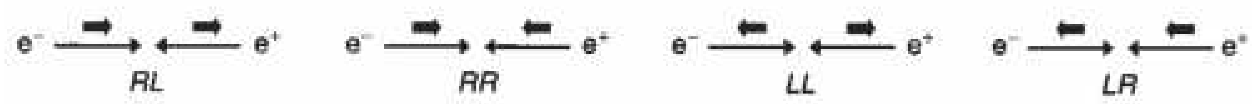
- The amplitude for this process is



$$\mathcal{M} = i\,\overline{v}(p_2)(ie\gamma^\mu)u(p_1)\,\frac{-i\,g_{\mu\nu}}{(p_1+p_2)^2}\,\overline{u}(p_3)(ie\gamma^\nu)v(p_4)$$
$$= -\frac{e^2}{s}\overline{v}(p_2)\gamma^\mu u(p_1)\ \overline{u}(p_3)\gamma_\mu v(p_4)$$



- We sum over the the final state spins and take the average over the initial state (non-polarized beams)
- For the initial state (in the CM)



- For the final state (in the CM)

- Here

$R \equiv\uparrow$ positive helicity, $\quad L \equiv\downarrow$ negative helicity

Calculating the process $e^- + e^+ \to \mu^- + \mu^+$

- Therefore we want to calculate

$$\begin{aligned}\langle|\mathcal{M}_{fi}|^2\rangle =&\frac{1}{4}\left[|\mathcal{M}(\uparrow\uparrow;\uparrow\uparrow)|^2+|\mathcal{M}(\uparrow\uparrow;\uparrow\downarrow)|^2+\cdots\right.\\ &+|\mathcal{M}(\uparrow\downarrow;\uparrow\uparrow)|^2+|\mathcal{M}(\uparrow\downarrow;\uparrow\downarrow)|^2+\cdots\\ &+|\mathcal{M}(\downarrow\uparrow;\uparrow\uparrow)|^2+|\mathcal{M}(\downarrow\uparrow;\uparrow\downarrow)|^2+\cdots\\ &\left.+|\mathcal{M}(\downarrow\downarrow;\uparrow\uparrow)|^2+|\mathcal{M}(\downarrow\downarrow;\uparrow\downarrow)|^2+\cdots\right]\end{aligned}$$

- We take all masses to zero. Then the four momenta are:

$$\begin{aligned} p_1 &= E(1,0,0,1), & p_2 &= E(1,0,0,-1)\\ p_3 &= E(1,\sin\theta,0,\cos\theta), & p_4 &= E(1,-\sin\theta,0,-\cos\theta)\end{aligned}$$

- The helicity spinors are ($|\vec{p}| = E$, with $E = \sqrt{s}/2$):

$$u_\uparrow = \sqrt{E}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{i\phi}\\ \cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{bmatrix}\ u_\downarrow = \sqrt{E}\begin{bmatrix}-\sin\left(\frac{\theta}{2}\right)\\ \cos\left(\frac{\theta}{2}\right)e^{i\phi}\\ \sin\left(\frac{\theta}{2}\right)\\ -\cos\left(\frac{\theta}{2}\right)e^{i\phi}\end{bmatrix}\ v_\uparrow = \sqrt{E}\begin{bmatrix}\sin\left(\frac{\theta}{2}\right)\\ -\cos\left(\frac{\theta}{2}\right)e^{i\phi}\\ -\sin\left(\frac{\theta}{2}\right)\\ \cos\left(\frac{\theta}{2}\right)e^{i\phi}\end{bmatrix}\ v_\downarrow = \sqrt{E}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{i\phi}\\ \cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{bmatrix}$$

Calculating the process $e^- + e^+ \to \mu^- + \mu^+$

- Making the substitutions

$$p_1 : \theta \to 0, \phi \to 0, \qquad p_2 : \theta \to \pi, \phi \to \pi$$
$$p_3 : \theta \to \theta, \phi \to 0, \qquad p_4 : \theta \to \pi - \theta, \phi \to \pi$$

- We get for the initial sate

$$u_\uparrow(p_1) = \sqrt{E}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \; u_\downarrow(p_1) = \sqrt{E}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}, \; v_\uparrow(p_2) = \sqrt{E}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \; v_\downarrow(p_2) = \sqrt{E}\begin{bmatrix}0\\-1\\0\\-1\end{bmatrix}$$

- and for the final state

$$u_\uparrow(p_3) = \sqrt{E}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)\\ \cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)\end{bmatrix} u_\downarrow(p_3) = \sqrt{E}\begin{bmatrix}-\sin\left(\frac{\theta}{2}\right)\\ \cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)\\ -\cos\left(\frac{\theta}{2}\right)\end{bmatrix} v_\uparrow(p_4) = \sqrt{E}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)\\ -\cos\left(\frac{\theta}{2}\right)\\ -\sin\left(\frac{\theta}{2}\right)\end{bmatrix} v_\downarrow(p_4) = \sqrt{E}\begin{bmatrix}\sin\left(\frac{\theta}{2}\right)\\ -\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)\\ -\cos\left(\frac{\theta}{2}\right)\end{bmatrix}$$

Calculating the process $e^- + e^+ \to \mu^- + \mu^+$

- We write the LI amplitude as

$$\mathcal{M}(h_1, h_2, h_3, h_4) = -\frac{e^2}{s}\, J_{u_1 v_2}(h_1, h_2)^\mu J_{u_3 v_4}(h_3, h_4)_\mu$$

where $h_i = \uparrow, \downarrow$, and the currents,

$$J_{u_1 v_2}(h_1, h_2)^\mu = \overline{v}(p_2, h_2)\gamma^\mu u(p_1, h_1),\; J_{u_3 v_4}(h_3, h_4)^\mu = \overline{u}(p_3, h_3)\gamma^\mu v(p_4, h_4)$$

- We want to calculate the components of these 4-vectors. It has to be done component by component. For instance

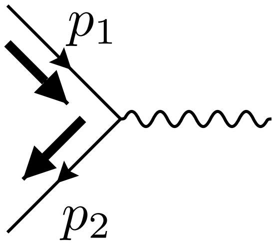
$$J_{u_1 v_2}(\uparrow, \uparrow)^0 = (\sqrt{E})^2 v^\dagger(p_2, \uparrow)\gamma^0\gamma^0 u(p_1, \uparrow) = E\,[1, 0, -1, 0]\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = 0$$

$$J_{u_1 v_2}(\uparrow, \downarrow)^2 = (\sqrt{E})^2 v^\dagger(p_2, \downarrow)\gamma^0\gamma^2 u(p_1, \uparrow)$$

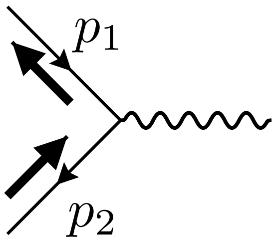
$$= E\,[0, -1, 0, -1]\begin{bmatrix}0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ 0 & -i & 0 & 0\\ i & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = E\,[0, -1, 0, -1]\begin{bmatrix}0\\i\\0\\i\end{bmatrix} = -2i\,E$$

Calculating the process $e^- + e^+ \to \mu^- + \mu^+$

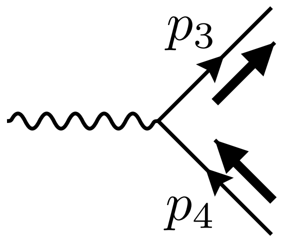
- Although this is straightforward, it is a bit boring. But we can program the procedure in `Mathematica` and get the final results that are quite simple.
- We get that the only non-zero currents are



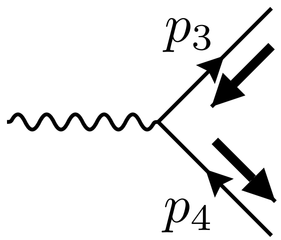
$$J_{u_1 v_2}(\uparrow,\downarrow) = \sqrt{s}\ (0,-1,-i,0)$$



$$J_{u_1 v_2}(\downarrow,\uparrow) = \sqrt{s}\ (0,-1,i,0)$$



$$J_{u_3 v_4}(\uparrow,\downarrow) = \sqrt{s}\ (0,-\cos\theta,i,\sin\theta)$$



$$J_{u_3 v_4}(\downarrow,\uparrow) = \sqrt{s}\ (0,-\cos\theta,-i,\sin\theta)$$

Calculating the process $e^- + e^+ \to \mu^- + \mu^+$

- Therefore we get

$$\mathcal{M}(\uparrow\downarrow;\uparrow\downarrow) = -\frac{e^2}{s}\left[\sqrt{s}(0,-1,-i,0)\right]\cdot\left[\sqrt{s}(0,-\cos\theta,i,\sin\theta)\right]$$
$$= \frac{e^2}{s} s\left(1+\cos\theta\right) \equiv 4\pi\alpha\left(1+\cos\theta\right)$$

- Similarly

$$|\mathcal{M}(\uparrow\downarrow;\uparrow\downarrow)|^2 = |\mathcal{M}(\downarrow\uparrow;\downarrow\uparrow)|^2 = (4\pi\alpha)^2\left(1+\cos\theta\right)^2$$
$$|\mathcal{M}(\uparrow\downarrow;\downarrow\uparrow)|^2 = |\mathcal{M}(\downarrow\uparrow;\uparrow\downarrow)|^2 = (4\pi\alpha)^2\left(1-\cos\theta\right)^2$$

- And

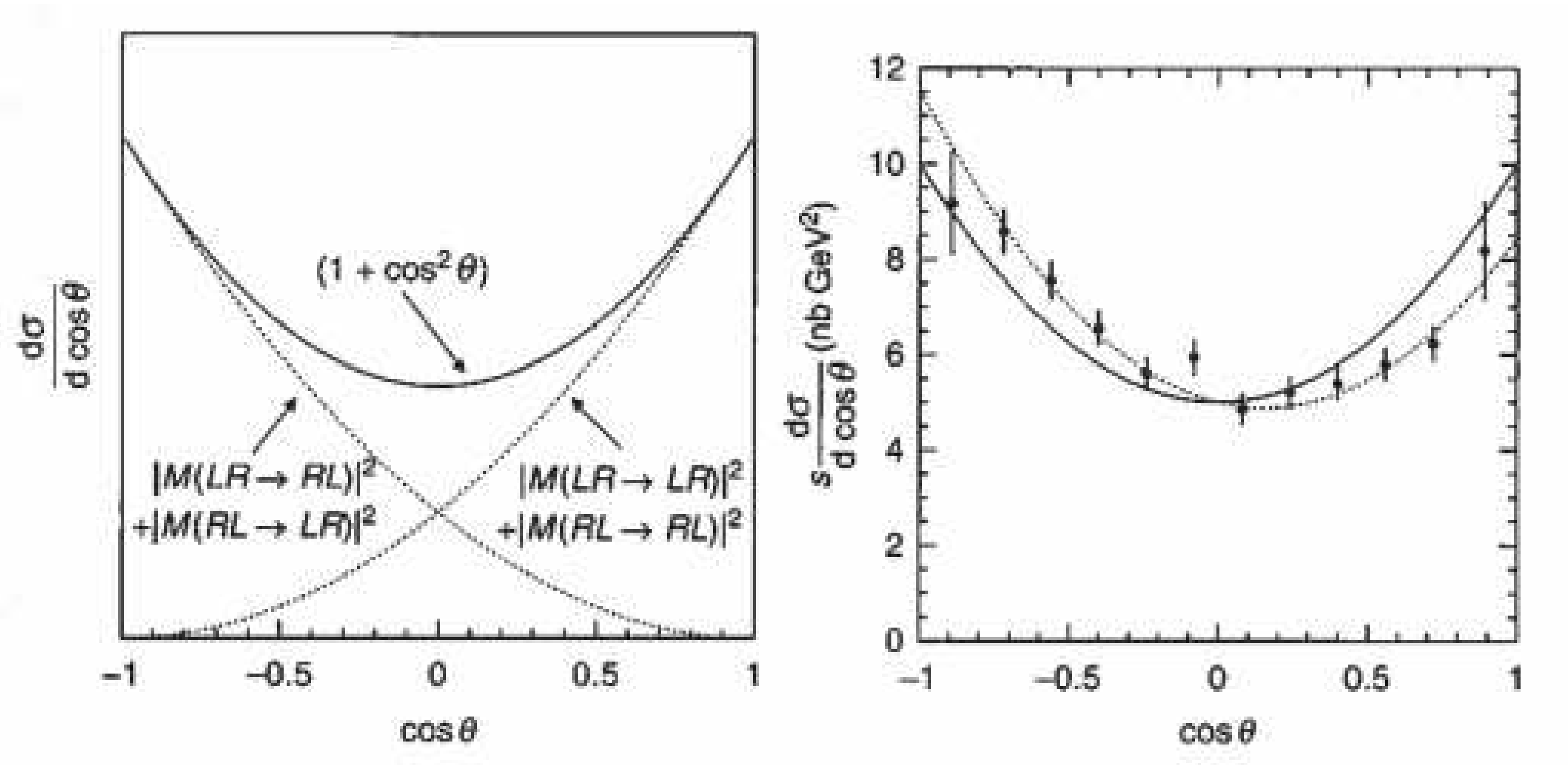
$$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{1}{4}(4\pi\alpha)^2\left[2(1+\cos\theta)^2+2(1-\cos\theta)^2\right]$$
$$= (4\pi\alpha)^2\left(1+\cos^2\theta\right)$$

- For the cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\langle|\mathcal{M}|^2\rangle = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

$$\sigma = \frac{4\pi\alpha^2}{3s}$$

Calculating the process $e^- + e^+ \to \mu^- + \mu^+$

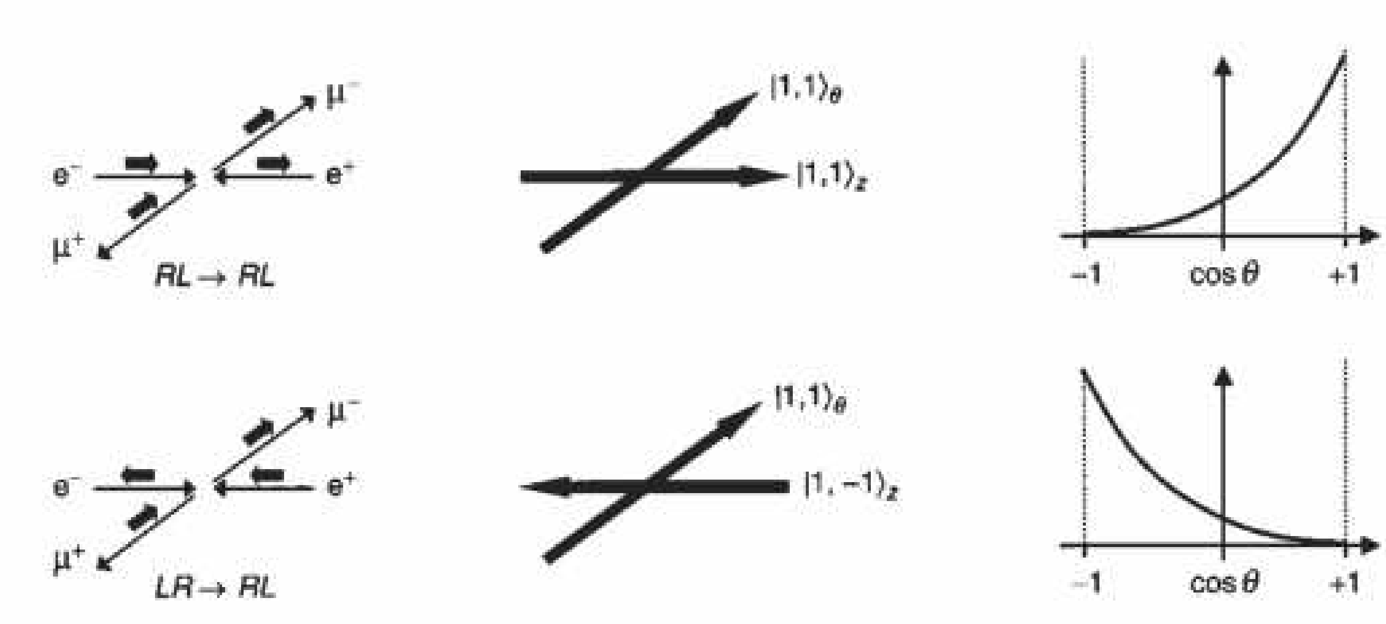


- From JADE experiment (Bartel et al. (1985)
- Solid curve is QED prediction. Dotted curve includes electroweak corrections

$e^- + e^+ \to \mu^- + \mu^+$: Understanding the result (1)

- The only non-zero amplitudes are those where the spins add up to ± 1

Photon ⬇ **Spin 1**



$1 + \cos\theta$

$1 - \cos\theta$

- For spin 1 along the θ direction we have

$$|1,+1\rangle_\theta = \frac{1}{2}(1-\cos\theta)\,|1,-1\rangle_z\ \frac{1}{\sqrt{2}}\sin\theta)\,|1,0\rangle_z\ \frac{1}{2}(1+\cos\theta)\,|1,1\rangle_z$$

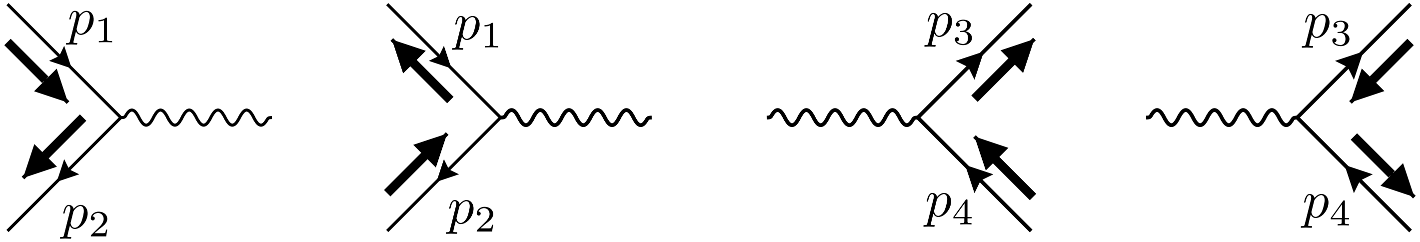
- And

$$\mathcal{M}(\uparrow\downarrow;\uparrow\downarrow) \propto {}_\theta\langle 1,+1|1,+1\rangle_z = \frac{1}{2}(1+\cos\theta)$$

$$\mathcal{M}(\downarrow\uparrow;\uparrow\downarrow) \propto {}_\theta\langle 1,+1|1,-1\rangle_z = \frac{1}{2}(1-\cos\theta)$$

$e^- + e^+ \to \mu^- + \mu^+$: Understanding the result (2)

- We see that (for zero masses) the non-zero interactions **do not flip the spin**



- This is due to a property known as **chirality**. Define the projectors

$$P_R = \frac{1}{2}(1+\gamma_5) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad P_L = \frac{1}{2}(1-\gamma_5) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$P_L + P_R = 1,\ P_L^2 = P_L,\ P_R^2 = P_R,\ P_L P_R = P_R P_L = 0$$

- Then, for the massless case

$$P_R u_\uparrow = u_\uparrow,\ P_L u_\uparrow = 0,\ P_R u_\downarrow = 0,\ P_L u_\downarrow = 0$$

$$P_R v_\uparrow = 0,\ P_L v_\uparrow = v_\uparrow,\ P_R v_\downarrow = v_\downarrow,\ P_L v_\downarrow = 0$$

I use L,R for chirality and $\uparrow,\downarrow$ for helicity

- For zero masses **Helicity $\equiv$ Chirality**

$e^- + e^+ \to \mu^- + \mu^+$: Understanding the result (3)

- Define

$$\psi = (P_L + P_R)\psi = \psi_L + \psi_R, \quad \psi_L \equiv P_L \psi, \quad \psi_R \equiv P_R \psi$$

- Then

$$\begin{aligned}\overline{\psi} =& \overline{\psi}_L + \overline{\psi}_R = (P_L\psi)^\dagger \gamma^0 + (P_R\psi)^\dagger \gamma^0 = \left(P_L^2\psi\right)^\dagger \gamma^0 + \left(P_R^2\psi\right)^\dagger \gamma^0 \\ &= (P_L\psi)^\dagger P_L^\dagger \gamma^0 + (P_R\psi)^\dagger P_R^\dagger \gamma^0 = (P_L\psi)^\dagger P_L \gamma^0 + (P_R\psi)^\dagger P_R \gamma^0 \\ &= (\psi_L)^\dagger \gamma^0 P_R + (\psi_R)^\dagger \gamma^0 P_L^\dagger = \overline{\psi}_L P_R + \overline{\psi}_R P_L\end{aligned}$$

- Therefore

$$\begin{aligned}\overline{\psi}\gamma^\mu\psi &= \left(\overline{\psi}_L P_R + \overline{\psi}_R P_L\right) \gamma^\mu \left(\psi_L + \psi_R\right) \\ &= \overline{\psi}_L \gamma^\mu \psi_L + \overline{\psi}_R \gamma^\mu \psi_R\end{aligned}$$

and

$$m\overline{\psi}\psi = m\left(\overline{\psi}_L \psi_R + \overline{\psi}_R \psi_L\right)$$

The QED interaction preserves Chirality

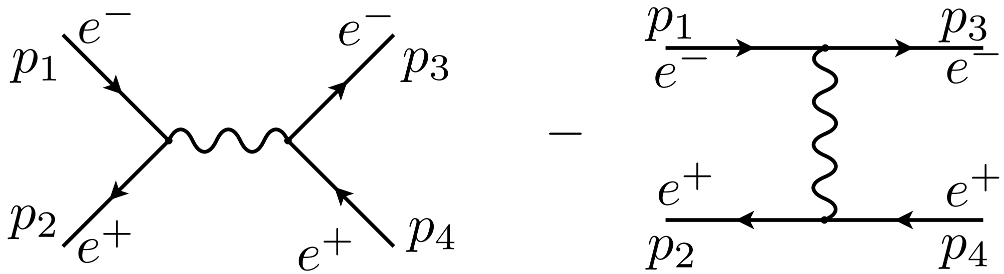
In the massless limit

$\Downarrow$

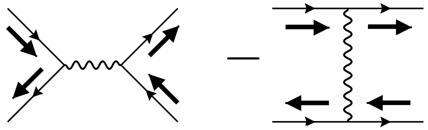
Preserves Helicity

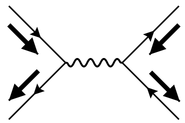
How to calculate other processes: Bhabha scattering (1)

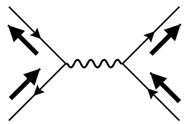
- For $e^- + e^+ \to e^- + e^+$ (Bhabha scattering) we have two diagrams

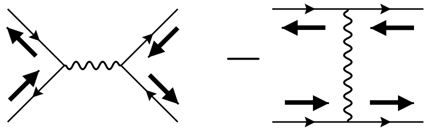


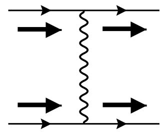
- Using what we have learned, for $m_e = 0$, we have

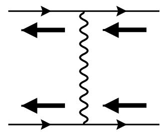
$\mathcal{M}(\uparrow,\downarrow;\uparrow,\downarrow) =$ 

$\mathcal{M}(\uparrow,\downarrow;\downarrow,\uparrow) =$ 

$\mathcal{M}(\downarrow,\uparrow;\uparrow,\downarrow) =$ 

$\mathcal{M}(\downarrow,\uparrow;\downarrow,\uparrow) =$ 

$\mathcal{M}(\uparrow,\uparrow;\uparrow,\uparrow) =$ 

$\mathcal{M}(\downarrow,\downarrow;\downarrow,\downarrow) =$ 

t-channel currents

- We have in a obvious notation for $p_1 + p_2 \to p_3 + p_4$ processes

$$J_{u_1 u_3}(\uparrow,\uparrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, i\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right)$$

$$J_{u_1 u_3}(\downarrow,\downarrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, -i\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right)$$

$$J_{v_1 v_3}(\uparrow,\uparrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, i\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right)$$

$$J_{v_1 v_3}(\downarrow,\downarrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, -i\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right)$$

$$J_{u_2 u_4}(\uparrow,\uparrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}, i\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}\right)$$

$$J_{u_2 u_4}(\downarrow,\downarrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}, -i\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}\right)$$

$$J_{v_2 v_4}(\uparrow,\uparrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}, i\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}\right)$$

$$J_{v_2 v_4}(\downarrow,\downarrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}, -i\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}\right)$$

Mathematica Notebook

How to calculate other processes: Bhabha scattering (2)

- The general amplitude is

$$\mathcal{M}(h_1, h_2; h_3, h_4) = -\frac{e^2}{s} J_{u_1 v_2}(h_1, h_2) \cdot J_{u_3 v_4}(h_3, h_4) + \frac{e^2}{t} J_{u_1 u3}(h_1, h_3) \cdot J_{v_2 v_4}(h_2, h_4)$$

- Summing the six non-zero helicity amplitudes we get finally

$$\begin{aligned}\left\langle |\mathcal{M}|^2 \right\rangle =& 2e^4 \left[\frac{t^2 + (s+t)^2}{s^2} + \frac{s^2 + (s+t)^2}{t^2} + 2\frac{(s+t)^2}{st} \right] \\ =& 2e^4 \left[\frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} - \frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)} + \frac{1+\cos^2\theta}{2} \right]\end{aligned}$$

where

$$t = -\frac{s}{2}(1 + \cos\theta) = -s\cos^2\frac{\theta}{2}, \quad u = -\frac{s}{2}(1 - \cos\theta) = -s\sin^2\frac{\theta}{2}$$

u-channel Amplitudes

- To be able to calculate all the processes with electrons and positrons we also need the u-channel amplitudes.

$$J_{u_1u_4}(\uparrow,\uparrow) = \sqrt{s}\ \left(\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}, -i\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)$$

$$J_{u_1u_4}(\downarrow,\downarrow) = \sqrt{s}\ \left(-\sin\frac{\theta}{2}, \cos\frac{\theta}{2}, -i\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right)$$

$$J_{u_2u_3}(\uparrow,\uparrow) = \sqrt{s}\ \left(-\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}, i\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)$$

$$J_{u_2u_3}(\downarrow,\downarrow) = \sqrt{s}\ \left(\sin\frac{\theta}{2}, \cos\frac{\theta}{2}, i\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right)$$

$$J_{v_1v_4}(\uparrow,\uparrow) = \sqrt{s}\ \left(-\sin\frac{\theta}{2}, \cos\frac{\theta}{2}, i\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right)$$

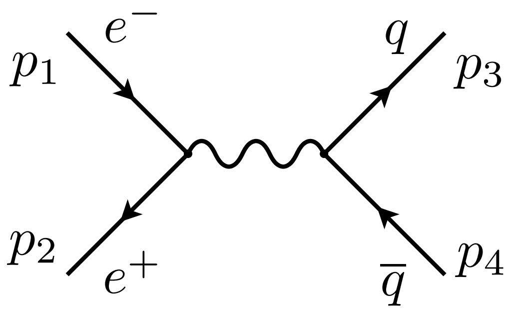
$$J_{v_1v_4}(\downarrow,\downarrow) = \sqrt{s}\ \left(\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}, i\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)$$

$$J_{v_2v_3}(\uparrow,\uparrow) = \sqrt{s}\ \left(\sin\frac{\theta}{2}, \cos\frac{\theta}{2}, -i\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right)$$

$$J_{v_2v_3}(\downarrow,\downarrow) = \sqrt{s}\ \left(-\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}, -i\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)$$

Hadron production in $e^- + e^+$ scattering

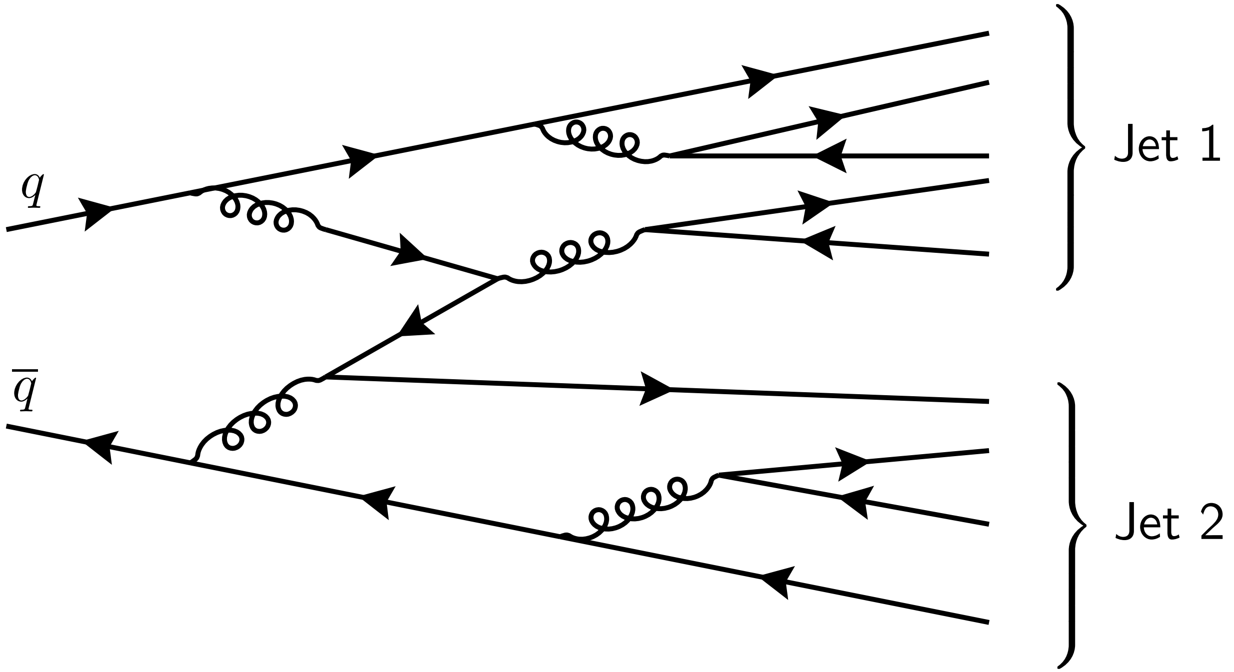
- In the scattering $e^- + e^+$ we can produce a variety of final states $e^- + e^+$ (Bhabha), $\mu^- + \mu^+$, $\gamma + \gamma$ and in general any fermion pair $f\overline{f}$.
- Therefore we can also have the pair production of quark-antiquark pairs, $e^- + e^+ \to q + \overline{q}$. If the energies are much below M_Z we can, as we will see later, neglect the other interaction and consider only QED, where we have the diagram similar to $e^- + e^+ \to \mu^- + \mu^+$



- In fact things are more complicated than with muons as the quarks are not free (confinement). When they are at distances of the order of the dimensions of the hadrons (1 fm $= 10^{-15}$ m) the strong interaction produces new $q\overline{q}$ pairs and gluons that finally will recombine to produce hadrons that will be detected. We call this process hadronization

Hadronization

- We show in the figure what we just described as hadronization



- When these events are observed at the detectors they keep the memory of how they were produced and appear as two jets of particles in opposite directions (*back-to-back*) and pointing to the original quarks from where they originate

Hadronization

- We show real events with two and three jets

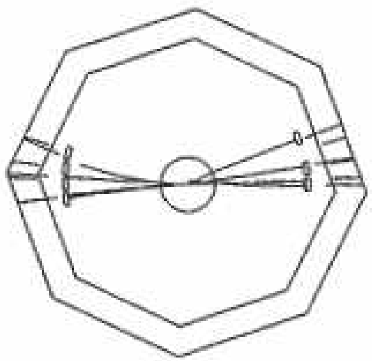


Fig. 8.2 A typical two-jet event. (*Source:* J. Dorfan, SLAC.)

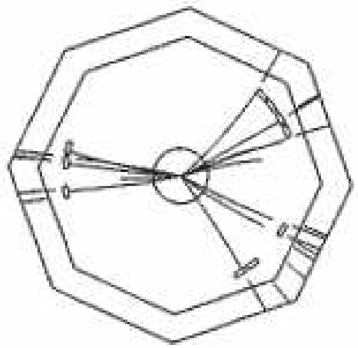
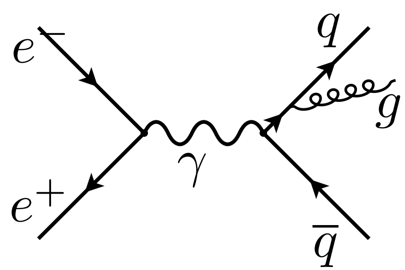


Fig. 8.3 A three-jet event. (*Source:* J. Dorfan, SLAC.)

- The events with three jets are interpreted as one of the jets coming from the hadronization of a gluon, a higher order process



- The observation of such events are the experimental evidence for the existence of gluons, the carriers of the strong interaction in the Quantum Chromodynamics.

Elementary process $e^- + e^+ \to q + \overline{q}$

- Besides all these complications the elementary process at the base of all these considerations is quite simple and can be obtained easily from our previous calculations

$$e^- + e^+ \to q + \overline{q}$$

- The amplitude is then

$$\mathcal{M} = -\frac{Q_q e^2}{s}\left[\overline{v}(p_2)\gamma^\mu u(p_1)\right]\left[\overline{u}(p_3)\gamma^\mu v(p_4)\right]$$

where Q_q is the charge of quark in units of e, that is, $Q_u = 2/3$, $Q_d = -1/3$.

- Using the results of our calculation for $e^- + e^+ \to \mu^- + \mu^+$ we get (neglecting the masses)

$$\left\langle |\mathcal{M}|^2 \right\rangle = e^4 Q_q^2 \left(1 + \cos^2\theta\right)$$

- Therefore

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\left\langle |\mathcal{M}|^2 \right\rangle = \frac{\alpha^2 Q_q^2}{4s}\left(1+\cos^2\theta\right) \Rightarrow \sigma = \frac{4\pi\alpha^2 Q_q^2}{3s}$$

The ratio R

- If we start from very low energy and start to increase the energy, we will go through the thresholds for the production of different species of leptons and quarks
- This can be described in a convenient from defining the Ratio R,

$$R \equiv \frac{\sigma(e^- + e^+ \to \text{hadrons})}{\sigma(e^- + e^+ \to \mu^- + \mu^+)}$$

- If we use the approximate expression given before, we should obtain

$$R(\sqrt{s}) = 3 \sum_i Q_i^2$$

where the sum is over all the quarks such that $\sqrt{s} > 2m_q$. The factor 3 (color factor) appears because each quark has three color possibilities (QCD).

- If we are at an energy such that we can only produce the u, d, s quarks we have

$$R = 3 \left[\left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 \right] = 2$$

The ratio R

- Above the threshold for the production of the c quarks we have

$$R = 2 + 3\left(\frac{2}{3}\right)^2 = \frac{10}{3} = 3.33$$

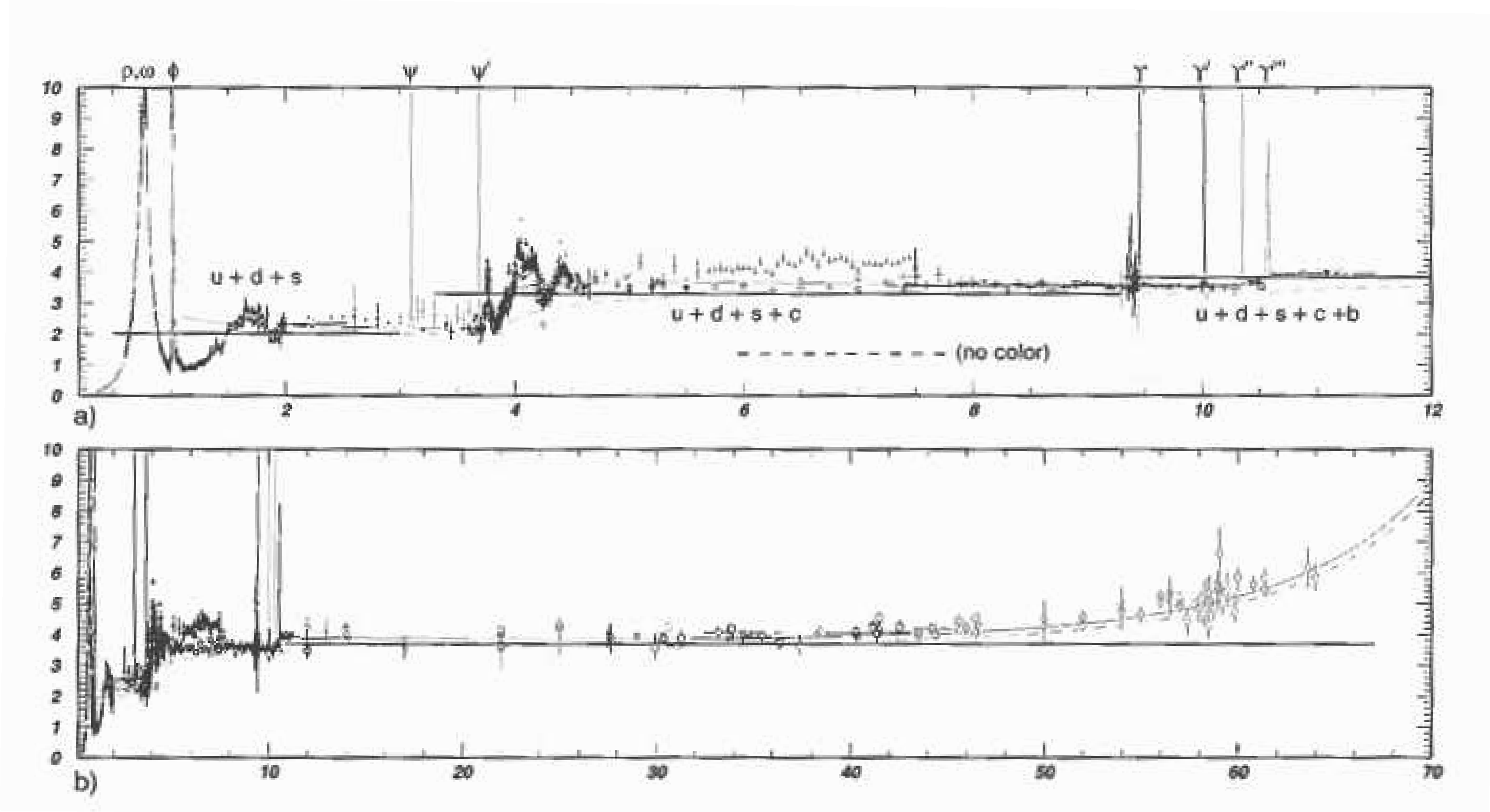
- Above the threshold for the b

$$R = \frac{10}{3} + 3\left(\frac{-1}{3}\right)^2 = \frac{11}{3} = 3.67$$

- If we had energy for the production of the top quark we would have $R = 5$. We have therefore a staircase effect, as the energy grows, R also grows in steps.

The ratio R

- How does this compare with experiment? We see in the figure



the plot of R based in experimental data. We see that the staircase behaviour is present, including the color factor of 3

The ratio R

- The resonances shown are not explained by the above argument
- When the reaction has exactly enough energy to produce bound states quark-antiquark then these resonances appear like in the figure: $\rho, \omega, \phi, \psi, \cdots$.
- But if we exclude these resonant behaviour the general plot confirms the existence of color triplets the basis for the construction of Quantum Chromodynamics, the theory of strong interactions. We will come back to this after we discuss gauge theories in one of the following chapters.