# Solução Teste Exemplo MQ 2013/2014



## I

1) falsa: $[x, p_x] \neq 0$

2) falsa: $x \psi^*_{n+1} \psi_n$ é par

3) falsa: $[H, p] \neq 0$

4) Verdadeira: $p = \alpha A + \beta A^+$ e $A^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$\Rightarrow \langle n+1|p|n\rangle = \beta\sqrt{n+1}\,\langle n+1|n+1\rangle = \beta\sqrt{n+1} \neq 0$$

## II

1) $\psi(x,0) = \sum_n A_n u_n(x) \Rightarrow A_1 = -\frac{1}{\sqrt{2}}\,;\ A_4 = B\,;\ A_n = 0\ n\neq 1,4$

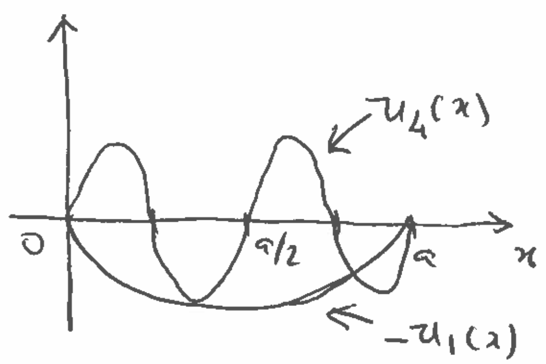
$$\sum_n |A_n|^2 = 1 \Rightarrow |A_1|^2 + |A_4|^2 = 1 \Rightarrow \frac{1}{2} + |B|^2 = 1$$

Portanto $B = \frac{1}{\sqrt{2}}$ (real e positivo)

2) $\langle E\rangle = \sum_n |A_n|^2 E_n = \frac{1}{2}E_1 + \frac{1}{2}E_4 = \frac{1}{2}\frac{\pi^2\hbar^2}{2ma^2}(1+16)$

$$= \frac{17}{4}\frac{\pi^2\hbar^2}{ma^2}$$

3) Método Gráfico




Para $0<x<a/2$ há uma interferência construtiva e para $a/2<x<a$ destrutiva. Logo

$$P(0<x<a/2) > 1/2$$

Método Analítico



$$P(0<x<a/2) = \frac{1}{2}\int_0^{a/2} dx\, u_1^2(x) + \frac{1}{2}\int_0^{a/2} dx\, u_4^2(x) - \int_0^{a/2} dx\, u_1(x)\, u_4(x)$$

$$\int_0^{a/2} dx\, u_1^2(x) = \frac{2}{a}\int_0^{a/2} dx\, \sin^2\left(\frac{\pi x}{a}\right) = \frac{2}{a}\frac{a}{\pi}\int_0^{\pi/2} dy\, \sin^2 y$$

$$= \frac{2}{\pi}\left[\frac{1}{2}y - \frac{1}{4}\sin 2y\right]_0^{\pi/2} = \frac{1}{2}$$

$$\int_0^{a/2} dx\, u_4^2(x) = \frac{2}{a}\int_0^{a/2} dx\, \sin^2\left(\frac{4\pi x}{a}\right) = \frac{2}{a}\frac{a}{4\pi}\int_0^{2\pi} dy\, \sin^2 y$$

$$= \frac{1}{2\pi}\left[\frac{1}{2}y - \frac{1}{4}\sin 2y\right]_0^{2\pi} = \frac{1}{2}$$

$$\int_0^{a/2} dx\, u_1(x) u_4(x) = \frac{2}{a}\int_0^{a/2} dx\, \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{4\pi x}{a}\right)$$

$$= \frac{2}{a}\frac{a}{\pi}\int_0^{\pi/2} dy\, \sin y\, \sin 4y$$

$$= \frac{2}{\pi}\left[\frac{1}{6}\sin(3y) - \frac{1}{10}\sin(5y)\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left(-\frac{1}{6} - \frac{1}{10}\right) = -\frac{8}{15\pi}$$

Logo

$$P(0<x<a/2) = \frac{1}{2} + \frac{8}{15\pi} > 1/2$$

4) $\psi(x,t) = -\frac{1}{\sqrt{2}} u_1(x) e^{-\frac{i}{\hbar}E_1 t} + \frac{1}{\sqrt{2}} u_4(x) e^{-\frac{i}{\hbar}E_4 t}$

Como $E_4 = 16 E_1$ se $\frac{E_1 T}{\hbar} = 2\pi \Rightarrow \frac{E_4 T}{\hbar} = 38\pi$

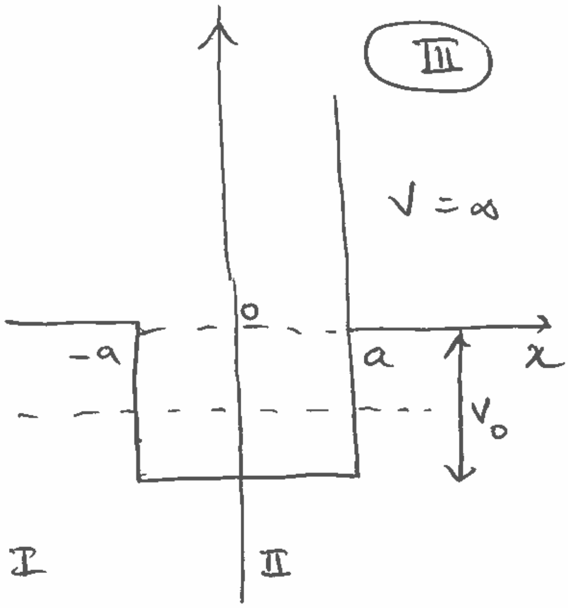
e portanto

$$\psi(x,T) = \psi(x,0)$$

Logo $T = \frac{2\pi\hbar}{E_1}$

III

1)




Em I: $\frac{d^2u}{dx^2} - \alpha^2 u = 0$

$$\alpha^2 = \frac{2m|E|}{\hbar^2}$$

Em II: $\frac{d^2u}{dx^2} + q^2 u = 0$

$$q^2 = \frac{2m}{\hbar^2}(V_0 - |E|)$$

Logo

$$\begin{cases} u_I(x) = A\, e^{+\alpha x} & x < -a \\ u_{II}(x) = B \sin[q(x-a)] & -a < x < a \end{cases}$$

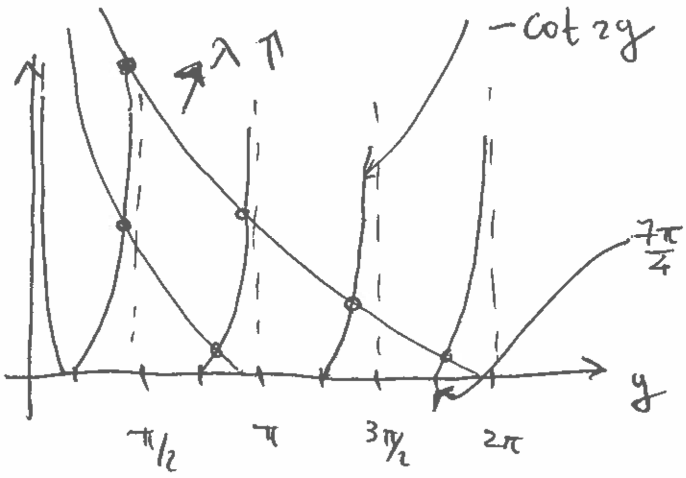
pois $u(a) = 0$. Igualando $\frac{1}{u}\frac{du}{dx}$ em $x = -a$ vem

$$+\alpha = q\cot(q(-2a))$$

ou ainda $\boxed{-\cot 2y = \frac{\sqrt{\lambda - y^2}}{y}}$ $\quad \lambda = \frac{2mV_0 a^2}{\hbar^2}$

$y = qa$

2) Graficamente




Há portanto um valor mínimo de $\lambda > \left(\pi/4\right)^2 = \frac{\pi^2}{16}$

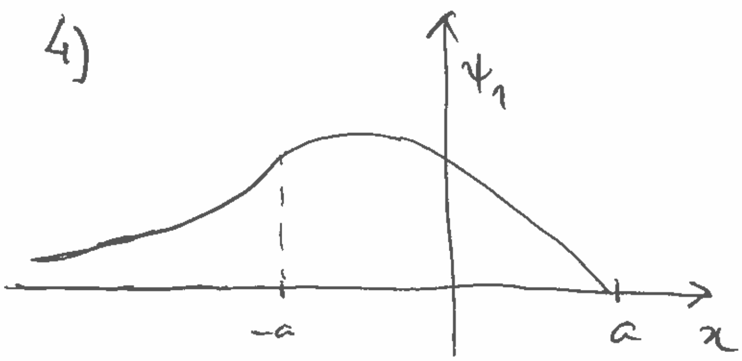
Logo
$$\frac{2mV_0a^2}{\hbar^2} > \frac{\pi^2}{16} \Rightarrow V > \frac{\pi^2\hbar^2}{32ma^2}$$

3)
$$\lambda = \frac{2mV_0a^2}{\hbar^2} = \frac{2ma^2}{\hbar^2}\,\frac{16\hbar^2}{ma^2} = 32$$

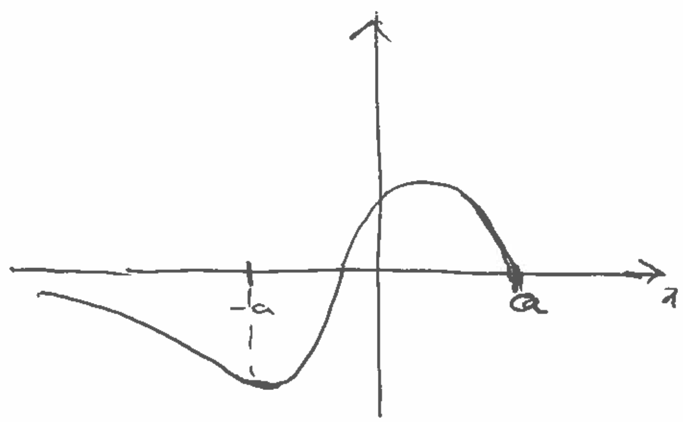
e
$$\left(\frac{7\pi}{4}\right)^2 < \lambda < (2\pi)^2$$

Pelo que há 4 estados ligados (ver figura)

4)




Estado fundamental




1º Estado Excitado (1 nó)

5) $E > 0$ $\begin{cases} u_I(x) = e^{ikx} + R e^{-ikx} \\ u_{II}(x) = A \sin[q(x-a)] \; ; \quad q^2 = \dfrac{2m}{\hbar^2}(V_0 + E) \end{cases}$

$u_{II}(a) = 0$. Em $x = -a$ a continuidade de $\frac{1}{u}\frac{du}{dx}$ dá

$$\frac{ik\left(e^{-ika} - R e^{ika}\right)}{e^{-ika} + R e^{ika}} = -q \cot(2qa)$$

$$R \, e^{ika}\left(q \cot(2qa) - ik\right) = e^{-ika}\left(-q \cot(2qa) - ik\right)$$

e portanto

$$R = - e^{-2ika} \frac{q \cot(2qa) + ik}{q \cot(2qa) - ik} \Rightarrow |R|^2 = 1$$

Temos

$$j_I(x) = \frac{\hbar k}{m}\left(1 - |R|^2\right) = 0$$

$$j_{II}(x) = 0 \quad (\text{a função é real})$$

$$\boxed{j_I(x) = j_{II}(x)}$$

Conclusão: Tudo o que incide é reflectido!