**10.11** Na aula teórica apresentámos, sem demonstração, o resultado da adição de dois momentos angulares arbitrários (orbitais ou de spin). Recordamos aqui os resultados. Seja

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

onde os valores próprios de $J_i^2$ são $\hbar^2 j_i(j_i+1)$. Então temos os resultados seguintes:

1. Os valores próprios de $J^2$ são $\hbar^2 j(j+1)$. Os valores possíveis para $j$ são

$$j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|$$

2. Qualquer estado $|j, m\rangle$ se pode exprimir como uma combinação linear dos produtos dos estados $|j_1, m_1\rangle$ e $|j_2, m_2\rangle$ na seguinte forma

$$|j, m\rangle = \sum_{m=m_1+m_2} C(j, m; j_1, m_1, j_2, m_2)\, |j_1, m_1\rangle\, |j_2, m_2\rangle$$

onde $C(j, m; j_1, m_1, j_2, m_2)$ são os coeficientes de Clebsch-Gordon e estão dados na Fig. 4 para os valores mais baixos de $j_1$ e $j_2$.

a) Consulte a tabela para verificar que no caso de spin 1/2 se obtém os resultados demonstrados na aula.

$$\texttt{Singleto}:$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}\,|1/2, 1/2\rangle\,|1/2, -1/2\rangle - \frac{1}{\sqrt{2}}\,|1/2, -1/2\rangle\,|1/2, 1/2\rangle$$

$$\texttt{Tripleto}:$$

$$|1,1\rangle = |1/2, 1/2\rangle\,|1/2, 1/2\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\,|1/2, 1/2\rangle\,|1/2, -1/2\rangle + \frac{1}{\sqrt{2}}\,|1/2, -1/2\rangle\,|1/2, 1/2\rangle$$

$$|1,-1\rangle = |1/2, -1/2\rangle\,|1/2, -1/2\rangle$$

b) Mostre que a multiplicidade total é $(2j_1+1)(2j_2+1)$ como deveria ser. Para isso considere $j_1 > j_2$ e mostre que

$$[2(j_1+j_2)+1] + [2(j_1+j_2-1)+1] + \cdots + [2(j_1-j_2)+1] = (2j_1+1)(2j_2+1)$$

Verifique, ver alínea a), que no caso de spin 1/2 temos 4 estados.

c) Para o problema do átomo de hidrogénio com spin vamos precisar da adição do momento angular orbital $L$ com o spin do eletrão. Os valores possíveis para $\vec{J} = \vec{L} + \vec{S}$ são $j = l \pm 1/2$ (para $l \geq 1$ claro, doutra forma para $l = 0$ temos $\vec{J} = \vec{S}$). Os resultados necessários são (Gasiorowicz, Eq. 10.82, a menos dum sinal global, na definição da segunda relação para estar de acordo com a tabela dos coeficientes Clebsch-Gordon),

$$\begin{aligned}
\psi_{l+1/2,m_j} &= \sqrt{\frac{l+m_j+1/2}{2l+1}}Y_{l,m_j-1/2}\;\chi^+ + \sqrt{\frac{l+1/2-m_j}{2l+1}}Y_{l,m_j+1/2}\;\chi^- \\
\psi_{l-1/2,m_j} &= -\sqrt{\frac{l+1/2-m_j}{2l+1}}Y_{l,m_j-1/2}\;\chi^+ + \sqrt{\frac{l+m_j+1/2}{2l+1}}Y_{l,m_j+1/2}\;\chi^-
\end{aligned}$$

Use a tabela da Fig. 4 para verificar este resultado para $l = 1, 2$. Verifique que as multiplicidades estão corretas.

# 35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND $d$ FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

| | | $J$ | $J$ | ... |
|---|---|---|---|---|
| | | $M$ | $M$ | ... |
| $m_1$ | $m_2$ | Coefficients | | |
| $m_1$ | $m_2$ | | | |
| . | . | | | |

## 1/2×1/2

| $m_1$ | $m_2$ | J=1, M=+1 |
|---|---|---|
| +1/2 | +1/2 | 1 |

| $m_1$ | $m_2$ | J=1, M=0 | J=0, M=0 |
|---|---|---|---|
| +1/2 | −1/2 | 1/2 | 1/2 |
| −1/2 | +1/2 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | J=1, M=−1 |
|---|---|---|
| −1/2 | −1/2 | 1 |

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

## 2×1/2

| $m_1$ | $m_2$ | J=5/2, M=+5/2 |
|---|---|---|
| +2 | +1/2 | 1 |

| $m_1$ | $m_2$ | J=5/2, M=+3/2 | J=3/2, M=+3/2 |
|---|---|---|---|
| +2 | −1/2 | 1/5 | 4/5 |
| +1 | +1/2 | 4/5 | −1/5 |

| $m_1$ | $m_2$ | J=5/2, M=+1/2 | J=3/2, M=+1/2 |
|---|---|---|---|
| +1 | −1/2 | 2/5 | 3/5 |
| 0 | +1/2 | 3/5 | −2/5 |

| $m_1$ | $m_2$ | J=5/2, M=−1/2 | J=3/2, M=−1/2 |
|---|---|---|---|
| 0 | −1/2 | 3/5 | 2/5 |
| −1 | +1/2 | 2/5 | −3/5 |

| $m_1$ | $m_2$ | J=5/2, M=−3/2 | J=3/2, M=−3/2 |
|---|---|---|---|
| −1 | −1/2 | 4/5 | 1/5 |
| −2 | +1/2 | 1/5 | −4/5 |

| $m_1$ | $m_2$ | J=5/2, M=−5/2 |
|---|---|---|
| −2 | −1/2 | 1 |

## 1×1/2

| $m_1$ | $m_2$ | J=3/2, M=+3/2 |
|---|---|---|
| +1 | +1/2 | 1 |

| $m_1$ | $m_2$ | J=3/2, M=+1/2 | J=1/2, M=+1/2 |
|---|---|---|---|
| +1 | −1/2 | 1/3 | 2/3 |
| 0 | +1/2 | 2/3 | −1/3 |

| $m_1$ | $m_2$ | J=3/2, M=−1/2 | J=1/2, M=−1/2 |
|---|---|---|---|
| 0 | −1/2 | 2/3 | 1/3 |
| −1 | +1/2 | 1/3 | −2/3 |

| $m_1$ | $m_2$ | J=3/2, M=−3/2 |
|---|---|---|
| −1 | −1/2 | 1 |

## 3/2×1/2

| $m_1$ | $m_2$ | J=2, M=+2 |
|---|---|---|
| +3/2 | +1/2 | 1 |

| $m_1$ | $m_2$ | J=2, M=+1 | J=1, M=+1 |
|---|---|---|---|
| +3/2 | −1/2 | 1/4 | 3/4 |
| +1/2 | +1/2 | 3/4 | −1/4 |

| $m_1$ | $m_2$ | J=2, M=0 | J=1, M=0 |
|---|---|---|---|
| +1/2 | −1/2 | 1/2 | 1/2 |
| −1/2 | +1/2 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | J=2, M=−1 | J=1, M=−1 |
|---|---|---|---|
| −1/2 | −1/2 | 3/4 | 1/4 |
| −3/2 | +1/2 | 1/4 | −3/4 |

| $m_1$ | $m_2$ | J=2, M=−2 |
|---|---|---|
| −3/2 | −1/2 | 1 |

## 2×1

| $m_1$ | $m_2$ | J=3, M=+3 |
|---|---|---|
| +2 | +1 | 1 |

| $m_1$ | $m_2$ | J=3, M=+2 | J=2, M=+2 |
|---|---|---|---|
| +2 | 0 | 1/3 | 2/3 |
| +1 | +1 | 2/3 | −1/3 |

| $m_1$ | $m_2$ | J=3, M=+1 | J=2, M=+1 | J=1, M=+1 |
|---|---|---|---|---|
| +2 | −1 | 1/15 | 1/3 | 3/5 |
| +1 | 0 | 8/15 | 1/6 | −3/10 |
| 0 | +1 | 2/5 | −1/2 | 1/10 |

| $m_1$ | $m_2$ | J=3, M=0 | J=2, M=0 | J=1, M=0 |
|---|---|---|---|---|
| +1 | −1 | 1/5 | 1/2 | 3/10 |
| 0 | 0 | 3/5 | 0 | −2/5 |
| −1 | +1 | 1/5 | −1/2 | 3/10 |

| $m_1$ | $m_2$ | J=3, M=−1 | J=2, M=−1 | J=1, M=−1 |
|---|---|---|---|---|
| 0 | −1 | 2/5 | 1/2 | 1/10 |
| −1 | 0 | 8/15 | −1/6 | −3/10 |
| −2 | +1 | 1/15 | −1/3 | 3/5 |

| $m_1$ | $m_2$ | J=3, M=−2 | J=2, M=−2 |
|---|---|---|---|
| −1 | −1 | 2/3 | 1/3 |
| −2 | 0 | 1/3 | −2/3 |

| $m_1$ | $m_2$ | J=3, M=−3 |
|---|---|---|
| −2 | −1 | 1 |

## 3/2×1

| $m_1$ | $m_2$ | J=5/2, M=+5/2 |
|---|---|---|
| +3/2 | +1 | 1 |

| $m_1$ | $m_2$ | J=5/2, M=+3/2 | J=3/2, M=+3/2 |
|---|---|---|---|
| +3/2 | 0 | 2/5 | 3/5 |
| +1/2 | +1 | 3/5 | −2/5 |

| $m_1$ | $m_2$ | J=5/2, M=+1/2 | J=3/2, M=+1/2 | J=1/2, M=+1/2 |
|---|---|---|---|---|
| +3/2 | −1 | 1/10 | 2/5 | 1/2 |
| +1/2 | 0 | 3/5 | 1/15 | −1/3 |
| −1/2 | +1 | 3/10 | −8/15 | 1/6 |

| $m_1$ | $m_2$ | J=5/2, M=−1/2 | J=3/2, M=−1/2 | J=1/2, M=−1/2 |
|---|---|---|---|---|
| +1/2 | −1 | 3/10 | 8/15 | 1/6 |
| −1/2 | 0 | 3/5 | −1/15 | −1/3 |
| −3/2 | +1 | 1/10 | −2/5 | 1/2 |

| $m_1$ | $m_2$ | J=5/2, M=−3/2 | J=3/2, M=−3/2 |
|---|---|---|---|
| −1/2 | −1 | 3/5 | 2/5 |
| −3/2 | 0 | 2/5 | −3/5 |

| $m_1$ | $m_2$ | J=5/2, M=−5/2 |
|---|---|---|
| −3/2 | −1 | 1 |

## 1×1

| $m_1$ | $m_2$ | J=2, M=+2 |
|---|---|---|
| +1 | +1 | 1 |

| $m_1$ | $m_2$ | J=2, M=+1 | J=1, M=+1 |
|---|---|---|---|
| +1 | 0 | 1/2 | 1/2 |
| 0 | +1 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | J=2, M=0 | J=1, M=0 | J=0, M=0 |
|---|---|---|---|---|
| +1 | −1 | 1/6 | 1/2 | 1/3 |
| 0 | 0 | 2/3 | 0 | −1/3 |
| −1 | +1 | 1/6 | −1/2 | 1/3 |

| $m_1$ | $m_2$ | J=2, M=−1 | J=1, M=−1 |
|---|---|---|---|
| 0 | −1 | 1/2 | 1/2 |
| −1 | 0 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | J=2, M=−2 |
|---|---|---|
| −1 | −1 | 1 |

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

## 3/2×3/2

| $m_1$ | $m_2$ | J=3, M=+3 |
|---|---|---|
| +3/2 | +3/2 | 1 |

| $m_1$ | $m_2$ | J=3, M=+2 | J=2, M=+2 |
|---|---|---|---|
| +3/2 | +1/2 | 1/2 | 1/2 |
| +1/2 | +3/2 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | J=3, M=+1 | J=2, M=+1 | J=1, M=+1 |
|---|---|---|---|---|
| +3/2 | −1/2 | 1/5 | 1/2 | 3/10 |
| +1/2 | +1/2 | 3/5 | 0 | −2/5 |
| −1/2 | +3/2 | 1/5 | −1/2 | 3/10 |

| $m_1$ | $m_2$ | J=3, M=0 | J=2, M=0 | J=1, M=0 | J=0, M=0 |
|---|---|---|---|---|---|
| +3/2 | −3/2 | 1/20 | 1/4 | 9/20 | 1/4 |
| +1/2 | −1/2 | 9/20 | 1/4 | −1/20 | −1/4 |
| −1/2 | +1/2 | 9/20 | −1/4 | −1/20 | 1/4 |
| −3/2 | +3/2 | 1/20 | −1/4 | 9/20 | −1/4 |

| $m_1$ | $m_2$ | J=3, M=−1 | J=2, M=−1 | J=1, M=−1 |
|---|---|---|---|---|
| +1/2 | −3/2 | 1/5 | 1/2 | 3/10 |
| −1/2 | −1/2 | 3/5 | 0 | −2/5 |
| −3/2 | +1/2 | 1/5 | −1/2 | 3/10 |

| $m_1$ | $m_2$ | J=3, M=−2 | J=2, M=−2 |
|---|---|---|---|
| −1/2 | −3/2 | 1/2 | 1/2 |
| −3/2 | −1/2 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | J=3, M=−3 |
|---|---|---|
| −3/2 | −3/2 | 1 |

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

## 2×3/2

| $m_1$ | $m_2$ | J=7/2, M=+7/2 |
|---|---|---|
| +2 | +3/2 | 1 |

| $m_1$ | $m_2$ | J=7/2, M=+5/2 | J=5/2, M=+5/2 |
|---|---|---|---|
| +2 | +1/2 | 3/7 | 4/7 |
| +1 | +3/2 | 4/7 | −3/7 |

| $m_1$ | $m_2$ | J=7/2, M=+3/2 | J=5/2, M=+3/2 | J=3/2, M=+3/2 |
|---|---|---|---|---|
| +2 | −1/2 | 1/7 | 16/35 | 2/5 |
| +1 | +1/2 | 4/7 | 1/35 | −2/5 |
| 0 | +3/2 | 2/7 | −18/35 | 1/5 |

| $m_1$ | $m_2$ | J=7/2, M=+1/2 | J=5/2, M=+1/2 | J=3/2, M=+1/2 | J=1/2, M=+1/2 |
|---|---|---|---|---|---|
| +2 | −3/2 | 1/35 | 6/35 | 2/5 | 2/5 |
| +1 | −1/2 | 12/35 | 5/14 | 0 | −3/10 |
| 0 | +1/2 | 18/35 | −3/35 | −1/5 | 1/5 |
| −1 | +3/2 | 4/35 | −27/70 | 2/5 | −1/10 |

| $m_1$ | $m_2$ | J=7/2, M=−1/2 | J=5/2, M=−1/2 | J=3/2, M=−1/2 | J=1/2, M=−1/2 |
|---|---|---|---|---|---|
| +1 | −3/2 | 4/35 | 27/70 | 2/5 | 1/10 |
| 0 | −1/2 | 18/35 | 3/35 | −1/5 | −1/5 |
| −1 | +1/2 | 12/35 | −5/14 | 0 | 3/10 |
| −2 | +3/2 | 1/35 | −6/35 | 2/5 | −2/5 |

| $m_1$ | $m_2$ | J=7/2, M=−3/2 | J=5/2, M=−3/2 | J=3/2, M=−3/2 |
|---|---|---|---|---|
| 0 | −3/2 | 2/7 | 18/35 | 1/5 |
| −1 | −1/2 | 4/7 | −1/35 | −2/5 |
| −2 | +1/2 | 1/7 | −16/35 | 2/5 |

| $m_1$ | $m_2$ | J=7/2, M=−5/2 | J=5/2, M=−5/2 |
|---|---|---|---|
| −1 | −3/2 | 4/7 | 3/7 |
| −2 | −1/2 | 3/7 | −4/7 |

| $m_1$ | $m_2$ | J=7/2, M=−7/2 |
|---|---|---|
| −2 | −3/2 | 1 |

## 2×2

| $m_1$ | $m_2$ | J=4, M=+4 |
|---|---|---|
| +2 | +2 | 1 |

| $m_1$ | $m_2$ | J=4, M=+3 | J=3, M=+3 |
|---|---|---|---|
| +2 | +1 | 1/2 | 1/2 |
| +1 | +2 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | J=4, M=+2 | J=3, M=+2 | J=2, M=+2 |
|---|---|---|---|---|
| +2 | 0 | 3/14 | 1/2 | 2/7 |
| +1 | +1 | 4/7 | 0 | −3/7 |
| 0 | +2 | 3/14 | −1/2 | 2/7 |

| $m_1$ | $m_2$ | J=4, M=+1 | J=3, M=+1 | J=2, M=+1 | J=1, M=+1 |
|---|---|---|---|---|---|
| +2 | −1 | 1/14 | 3/10 | 3/7 | 1/5 |
| +1 | 0 | 3/7 | 1/5 | −1/14 | −3/10 |
| 0 | +1 | 3/7 | −1/5 | −1/14 | 3/10 |
| −1 | +2 | 1/14 | −3/10 | 3/7 | −1/5 |

| $m_1$ | $m_2$ | J=4, M=0 | J=3, M=0 | J=2, M=0 | J=1, M=0 | J=0, M=0 |
|---|---|---|---|---|---|---|
| +2 | −2 | 1/70 | 1/10 | 2/7 | 2/5 | 1/5 |
| +1 | −1 | 8/35 | 2/5 | 1/14 | −1/10 | −1/5 |
| 0 | 0 | 18/35 | 0 | −2/7 | 0 | 1/5 |
| −1 | +1 | 8/35 | −2/5 | 1/14 | 1/10 | −1/5 |
| −2 | +2 | 1/70 | −1/10 | 2/7 | −2/5 | 1/5 |

| $m_1$ | $m_2$ | J=4, M=−1 | J=3, M=−1 | J=2, M=−1 | J=1, M=−1 |
|---|---|---|---|---|---|
| +1 | −2 | 1/14 | 3/10 | 3/7 | 1/5 |
| 0 | −1 | 3/7 | 1/5 | −1/14 | −3/10 |
| −1 | 0 | 3/7 | −1/5 | −1/14 | 3/10 |
| −2 | +1 | 1/14 | −3/10 | 3/7 | −1/5 |

| $m_1$ | $m_2$ | J=4, M=−2 | J=3, M=−2 | J=2, M=−2 |
|---|---|---|---|---|
| 0 | −2 | 3/14 | 1/2 | 2/7 |
| −1 | −1 | 4/7 | 0 | −3/7 |
| −2 | 0 | 3/14 | −1/2 | 2/7 |

| $m_1$ | $m_2$ | J=4, M=−3 | J=3, M=−3 |
|---|---|---|---|
| −1 | −2 | 1/2 | 1/2 |
| −2 | −1 | 1/2 | −1/2 |

| $m_1$ | $m_2$ | J=4, M=−4 |
|---|---|---|
| −2 | −2 | 1 |

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

**Figure 35.1**: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

Figura 4: Coeficientes de Clebsch-Gordon para $j_i = 1/2, 1, 3/2, 2$. Fonte: Particle Data Group web page, `http://pdg.lbl.gov/2007/reviews/clebrpp.pdf`