Teoria de Campo – Problem Series 1

Curso de Engenharia Física Tecnológica – 2015/2016

`Due on the 8/4/2016`

`Version of 15/02/2016`

1.1 Suppose that for a particle at rest the polarization vector is

$$s^\mu = (0, \vec{\eta}) \qquad \text{com} \qquad \vec{\eta}\cdot\vec{\eta} = 1$$

a) Show that in the reference frame where the particle moves with velocity $\vec{\beta}$ the polarization vector is given by

$$s^\mu = \left(\gamma\vec{\eta}\cdot\vec{\beta},\ \vec{\eta} + \frac{\gamma^2\vec{\beta}\,(\vec{\eta}\cdot\vec{\beta})}{\gamma+1}\right)$$

b) Show that it satisfies $s^2 = -1$ and $s\cdot p = 0$ with $p = m(\gamma, \gamma\vec{\beta})$.

c) Show that the longitudinal polarization vector, that is, $\vec{s}_L \parallel \vec{\beta}$, is given by,

$$s_L^\mu = \left(\gamma\beta, \gamma\vec{\beta}/\beta\right) \tag{1}$$

d) Show that

$$s^\mu = \frac{1}{2m}\overline{u}(p,\lambda)\gamma^\mu\gamma_5 u(p,\lambda)$$

where $u(p,\lambda)$ is a spinor with mass m and polarization λ, is a good polarization vector, that is, $s^2 = -1$ and $s\cdot p = 0$.

Hint: For the difficult part ($s^2 = -1$) consider the helicity basis, that is, take the helicity spinor $u(p,h)$ with helicity h moving in an arbitrary direction as we have seen in class.

1.2 For the scattering $1+2 \to 3+4$ we can define in center of mass frame (`CM`),

$$P_{\text{CM}} = (\sqrt{s}, \vec{0}) = p_1 + p_2 = p_3 + p_4$$

where $\sqrt{s}$ is the total energy in the `CM`. Show that,

$$p^0_{1\text{CM}} = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, \qquad p^0_{2\text{CM}} = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}$$
$$p^0_{3\text{CM}} = \frac{s + m_3^2 - m_4^2}{2\sqrt{s}}, \qquad p^0_{4\text{CM}} = \frac{s + m_4^2 - m_3^2}{2\sqrt{s}}$$
$$|\vec{p}_1|_{\text{CM}} = \frac{\lambda(\sqrt{s}, m_1, m_2)}{2\sqrt{s}}, \quad |\vec{p}_3|_{\text{CM}} = \frac{\lambda(\sqrt{s}, m_3, m_4)}{2\sqrt{s}}$$

where

$$\lambda(x,y,z) = \sqrt{(x^2 - y^2 - z^2)^2 - 4y^2z^2}$$

Note: This a very important problem as we will be using these results quite often.

1.3 Consider the scattering $1+2 \to 3+4$ in center of mass frame (CM). Do not neglect the mass but consider $m_1 = m_2, m_3 = m_4$. Consider the quantity

$$P(h,s) = \frac{1 + h\gamma_5 \not{s}}{2}$$

where $h = \pm$, $s = (\gamma\beta, \gamma\vec{\beta}/\beta)$ for a particle with velocity $\vec{\beta}$ (see Eq. (1))

a) Show that it satisfies the requirements to be a projector, that is,

$$P(+,s) + P(-,s) = 1, P(+,s)P(-,s) = 0, P(\pm,s)P(\pm,s) = P(\pm,s)$$

b) Use the helicity spinors for particle 1 ($\theta = 0, \phi = 0$) and particle 3 ($\theta, \phi = 0$), to show explicitly that the quantity

$$P(h_i, s_i) = \frac{1 + h_i\gamma_5 \not{s}_i}{2}$$

where $s_i = (\gamma_i\beta_i, \gamma_i\vec{\beta}_i/\beta_i)$ and $i = 1, 3$ for particle 1 and 3, respectively, is a projector for the helicity of those particles, that is,

$$P(+,s_1)u_\uparrow(p_1) = u_\uparrow(p_1), P(-,s_1)u_\uparrow(p_1) = 0$$
$$P(+,s_1)u_\downarrow(p_1) = 0, P(-,s_1)u_\downarrow(p_1) = u_\downarrow(p_1)$$

and similarly for particle 3.

1.4 Fill in the entries of the *multiplication table* for the γ matrices as indicated in Table 1. This is a very useful table in actual calculations. To establish the Table we should note that any product of matrices γ can be written in terms of the 16 independent matrices we discussed in class. Also note that our conventions imply:

$$\varepsilon^{0123} = +1\,,$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma_0\gamma_1\gamma_2\gamma_3\,,$$

$$\varepsilon_{\alpha\beta\gamma\delta_1}\varepsilon^{\alpha\beta\gamma\delta_2} = -6g^{\delta_2}_{\delta_1}\,.$$

$$\varepsilon_{\alpha\beta_1\gamma_1\delta_1}\varepsilon^{\alpha\beta_2\gamma_2\delta_2} = -\sum_P (-1)^P g^{P[\beta_2}_{\beta_1} g^{\gamma_2}_{\gamma_1} g^{\delta_2]}_{\delta_1}$$

$$\varepsilon_{\alpha\beta\gamma_1\delta_1}\varepsilon^{\alpha\beta\gamma_2\delta_2} = -2\left(g^{\gamma_2}_{\gamma_1}g^{\delta_2}_{\delta_1} - g^{\delta_2}_{\gamma_1}g^{\gamma_2}_{\delta_1}\right)$$

	1	γ_5	γ^μ	$\gamma_5\gamma^\mu$	$\sigma^{\mu\nu}$
1	1				
γ_5					
γ^α					
$\gamma_5\gamma^\alpha$					
$\sigma^{\alpha\beta}$					

Table 1: Multiplication table for γ matrices.

1.5 Starting from the definition

$$S_{fi} = \lim_{t \to \varepsilon_f \infty} \int d^3x \; \psi_f^\dagger(x) \Psi_i(x)$$

obtain the central result of Chapter 2, Eq. (2.50),

$$S_{fi} = \delta_{fi} - ieQ_e\varepsilon_f \int d^4y \; \overline{\psi}_f(y) \not{A}(y) \Psi_i(y) \,. \tag{2}$$

where $e > 0$ e $Q_e = -1$. This proof has some subtleties, therefore we go step by step.

a) First show that (Eq. (2.40))

$$S_F(x' - x) = \theta(t' - t) \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{2} \psi_p^r(x') \overline{\psi}_p^r(x) - \theta(t - t') \int \frac{d^3p}{(2\pi)^3} \sum_{r=3}^{4} \psi_p^r(x') \overline{\psi}_p^r(x)$$

where

$$\psi_p^r(x) = \frac{1}{\sqrt{2E}} \; w^r(\vec{p}) \; e^{-i\varepsilon_r p\cdot x}$$

b) Now derive Eqs. (2.47) and (2.48),

$$\lim_{t \to +\infty} \Psi(x) - \psi(x) = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{2} \psi_p^r(x) \left[-ieQ_e \int d^4y \; \overline{\psi}_p^r(y) \not{A}(y) \Psi(y) \right]$$

$$\lim_{t \to -\infty} \Psi(x) - \psi(x) = \int \frac{d^3p}{(2\pi)^3} \sum_{r=3}^{4} \psi_p^r(x) \left[+ieQ_e \int d^4y \; \overline{\psi}_p^r(y) \not{A}(y) \Psi(y) \right]$$

c) Finally use these results to show Eq. (2).