Teoria de Campo – Problem Series 1

Curso de Engenharia Física Tecnológica – 2015/2016

`Due on the 7/4/2017`

`Version of 22/02/2017`

1.1 Consider spinors obeying the Dirac equation and take the Dirac representation for the γ matrices.

a) Show that (we define $\hat{p} = \vec{p}/|\vec{p}|$):

$$\left(1 - \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right) = \left(1 - \frac{|\vec{p}|}{E+m}\right)\frac{1+\vec{\sigma}\cdot\hat{p}}{2} + \left(1 + \frac{|\vec{p}|}{E+m}\right)\frac{1-\vec{\sigma}\cdot\hat{p}}{2}$$

b) Consider a fermion field with mass with left chirality, given by

$$\psi_L = \frac{1-\gamma_5}{2}\psi$$

where ψ is a positive energy spinor. Show that we can write

$$\psi_L = N \begin{bmatrix} \left(\alpha_P \dfrac{1+\vec{\sigma}\cdot\hat{p}}{2} + \alpha_N \dfrac{1-\vec{\sigma}\cdot\hat{p}}{2}\right)\chi \\ -\left(\alpha_P \dfrac{1+\vec{\sigma}\cdot\hat{p}}{2} + \alpha_N \dfrac{1-\vec{\sigma}\cdot\hat{p}}{2}\right)\chi \end{bmatrix} e^{-ip\cdot x}$$

where N is the normalization and χ a two component spinor. Determine α_P and α_N (modulo the normalization). What is the meaning of these coefficients?

c) We define the polarization of the chiral fermion ψ_L as

$$P = \frac{|\alpha_P|^2 - |\alpha_N|^2}{|\alpha_P|^2 + |\alpha_N|^2}$$

Show that $P = -|\vec{p}|/E = -\beta$. Discuss the limit when $|\vec{p}| \gg m$. Comment the result.

1.2 Consider finite rotations. Define

$$(\theta^1, \theta^2, \theta^3) \equiv (\omega^2{}_3, \omega^3{}_1, \omega^1{}_2) \quad \text{and} \quad (\Sigma^1, \Sigma^2, \Sigma^3) \equiv (\sigma^{23}, \sigma^{31}, \sigma^{12})$$

a) Show that

$$S_R = e^{\frac{i}{2}\vec{\theta}\cdot\vec{\Sigma}}$$

b) For finite rotations show that this can be written as

$$S_R(\vec{\theta}) = \cos\frac{\theta}{2} + i\,\hat{\theta}\cdot\vec{\Sigma}\,\sin\frac{\theta}{2}$$

where $\theta = \sqrt{\vec{\theta}\cdot\vec{\theta}}$ and $\hat{\theta} = \frac{\vec{\theta}}{\theta}$.

c) Consider now a rotation around the z axis by a finite angle θ_0. Show explicitly that

$$S_R(\theta_0)\gamma^\mu S_R^{-1}(\theta_0)\, a^\nu{}_\mu = \gamma^\nu$$

1.3 Consider the scattering $1+2 \to 3+4$ in center of mass frame (CM). Do not neglect the mass but consider $m_1 = m_2, m_3 = m_4$. Consider the quantity

$$P(h,s) = \frac{1 + h\gamma_5 \not{s}}{2}$$

where $h = \pm$ and $s = (\gamma\beta, \gamma\vec{\beta}/\beta)$ is the spin 4-vector for a particle with velocity $\vec{\beta}$.

a) Show that it satisfies the requirements to be a projector, that is,

$$P(+,s) + P(-,s) = 1, P(+,s)P(-,s) = 0, P(\pm,s)P(\pm,s) = P(\pm,s)$$

b) Use the helicity spinors for particle 1 ($\theta = 0, \phi = 0$) and particle 4 ($\theta \to \pi - \theta, \phi = \pi$), to show explicitly that the quantity

$$P(h_i, s_i) = \frac{1 + h_i\gamma_5 \not{s}_i}{2}$$

where $s_i = (\gamma_i\beta_i, \gamma_i\vec{\beta}_i/\beta_i)$ and $i = 1,4$ for particle 1 and 4, respectively, is a projector for the helicity of those particles, that is,

$$P(+,s_1)u_\uparrow(p_1) = u_\uparrow(p_1), P(-,s_1)u_\uparrow(p_1) = 0$$
$$P(+,s_1)u_\downarrow(p_1) = 0, P(-,s_1)u_\downarrow(p_1) = u_\downarrow(p_1)$$

and similarly for particle 4.

1.4 Fill in the entries of the *multiplication table* for the γ matrices as indicated in Table 1. This is a very useful table in actual calculations. To establish the Table we should note that any product of matrices γ can be written in terms of the 16 independent matrices we discussed in class. Also note that our conventions imply:

$$\varepsilon^{0123} = +1\,, \qquad \varepsilon_{\alpha\beta_1\gamma_1\delta_1}\varepsilon^{\alpha\beta_2\gamma_2\delta_2} = -\sum_P (-1)^P g_{\beta_1}^{P[\beta_2} g_{\gamma_1}^{\gamma_2} g_{\delta_1}^{\delta_2]}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma_0\gamma_1\gamma_2\gamma_3\,, \qquad \varepsilon_{\alpha\beta\gamma_1\delta_1}\varepsilon^{\alpha\beta\gamma_2\delta_2} = -2\left(g_{\gamma_1}^{\gamma_2} g_{\delta_1}^{\delta_2} - g_{\gamma_1}^{\delta_2} g_{\delta_1}^{\gamma_2}\right)$$

$$\varepsilon_{\alpha\beta\gamma\delta_1}\varepsilon^{\alpha\beta\gamma\delta_2} = -6 g_{\delta_1}^{\delta_2}\,.$$

	1	γ_5	γ^μ	$\gamma_5\gamma^\mu$	$\sigma^{\mu\nu}$
1	1				
γ_5					
γ^α					
$\gamma_5\gamma^\alpha$					
$\sigma^{\alpha\beta}$					

Table 1: Multiplication table for γ matrices.

1.5 Starting from the definition

$$S_{fi} = \lim_{t \to \varepsilon_f \infty} \int d^3x \; \psi_f^\dagger(x) \Psi_i(x)$$

obtain the central result of Chapter 2, Eq. (2.50),

$$S_{fi} = \delta_{fi} - ieQ_e \varepsilon_f \int d^4y \; \overline{\psi}_f(y) \not{A}(y) \Psi_i(y) \, . \tag{1}$$

where $e > 0$ e $Q_e = -1$. This proof has some subtleties, therefore we go step by step.

a) First show that (Eq. (2.40))

$$S_F(x'-x) = \theta(t'-t) \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{2} \psi_p^r(x') \overline{\psi}_p^r(x) - \theta(t-t') \int \frac{d^3p}{(2\pi)^3} \sum_{r=3}^{4} \psi_p^r(x') \overline{\psi}_p^r(x)$$

where

$$\psi_p^r(x) = \frac{1}{\sqrt{2E}} \; w^r(\vec{p}) \; e^{-i\varepsilon_r p \cdot x}$$

b) Now derive Eqs. (2.47) and (2.48),

$$\lim_{t \to +\infty} \Psi(x) - \psi(x) = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{2} \psi_p^r(x) \left[-ieQ_e \int d^4y \; \overline{\psi}_p^r(y) \not{A}(y) \Psi(y) \right]$$

$$\lim_{t \to -\infty} \Psi(x) - \psi(x) = \int \frac{d^3p}{(2\pi)^3} \sum_{r=3}^{4} \psi_p^r(x) \left[+ieQ_e \int d^4y \; \overline{\psi}_p^r(y) \not{A}(y) \Psi(y) \right]$$

c) Finally use these results to show Eq. (1).