Teoria de Campo – Problem Series 1

Curso de Engenharia Física Tecnológica – 2017/2018

`Due on the 6/4/2018`

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1.1 The Poincaré group consists of the Lorentz group plus the translations. If $J_{\mu\nu}$ denote the generators of the Lorentz group and P_μ the generators of the translations the commutation relations are,

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\left(g_{\nu\rho}J_{\mu\sigma} - g_{\nu\sigma}J_{\mu\rho} - g_{\mu\rho}J_{\nu\sigma} + g_{\mu\sigma}J_{\nu\rho}\right) \tag{1}$$

$$[P_\alpha, J_{\mu\nu}] = i\left(g_{\mu\alpha}P_\nu - g_{\nu\alpha}P_\mu\right) \tag{2}$$

$$[P_\mu, P_\nu] = 0$$

Show that

$$\left[P^2, J_{\mu\nu}\right] = \left[P^2, P_\mu\right] = 0 \tag{3}$$

$$\left[W^2, J_{\mu\nu}\right] = \left[W^2, P_\mu\right] = \left[W^2, P^2\right] = 0$$

where

$$W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\; J^{\nu\rho}\; P^\sigma$$

is the Pauli-Lubanski vector operator.

1.2 Consider finite *boosts*. Define

$$\omega^i \equiv \omega^{0i} \quad \text{and} \quad \tanh\omega = |\vec{\beta}|$$

where $\vec{\beta}$ is the relative velocity between the two frames.

a) Show that

$$S_L = e^{-\frac{1}{2}\vec{\omega}\cdot\vec{\alpha}}$$

b) For finite rotations show that this can be written as

$$S_L(\vec{\omega}) = \cosh\frac{\omega}{2} - \hat{\omega}\cdot\vec{\alpha}\,\sinh\frac{\omega}{2}$$

where $\hat{\omega} = \frac{\vec{\beta}}{|\vec{\beta}|}$.

c) Consider now a Lorentz boost along the x axis with relative velocity β_0. Show explicitly that

$$S_L(\beta_0)\gamma^\mu S_L^{-1}(\beta_0)\; a^\nu{}_\mu = \gamma^\nu$$

for finite Lorentz boosts.

1.3 Fill in the entries of the *multiplication table* for the γ matrices as indicated in Table 1. This is a very useful table in actual calculations. To establish the Table we should note that any product of matrices γ can be written in terms of the 16 independent matrices we discussed in class. Also note that our conventions imply:

$$\varepsilon^{0123} = +1\,, \qquad \varepsilon_{\alpha\beta_1\gamma_1\delta_1}\varepsilon^{\alpha\beta_2\gamma_2\delta_2} = -\sum_P (-1)^P g^{P[\beta_2}_{\beta_1} g^{\gamma_2}_{\gamma_1} g^{\delta_2]}_{\delta_1}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma_0\gamma_1\gamma_2\gamma_3\,, \qquad \varepsilon_{\alpha\beta\gamma_1\delta_1}\varepsilon^{\alpha\beta\gamma_2\delta_2} = -2\left(g^{\gamma_2}_{\gamma_1}g^{\delta_2}_{\delta_1} - g^{\delta_2}_{\gamma_1}g^{\gamma_2}_{\delta_1}\right)$$

$$\varepsilon_{\alpha\beta\gamma\delta_1}\varepsilon^{\alpha\beta\gamma\delta_2} = -6g^{\delta_2}_{\delta_1}\,.$$

	1	γ_5	γ^μ	$\gamma_5\gamma^\mu$	$\sigma^{\mu\nu}$
1	1				
γ_5					
γ^α					
$\gamma_5\gamma^\alpha$					
$\sigma^{\alpha\beta}$					

Table 1: Multiplication table for γ matrices.

1.4 Starting from the definition

$$S_{fi} = \lim_{t\to\varepsilon_f\infty} \int d^3x\ \psi_f^\dagger(x)\Psi_i(x)$$

obtain the central result of Chapter 2, Eq. (2.50),

$$S_{fi} = \delta_{fi} - ieQ_e\varepsilon_f \int d^4y\ \overline{\psi}_f(y)\not{A}(y)\Psi_i(y)\,. \tag{4}$$

where $e > 0$ e $Q_e = -1$. This proof has some subtleties, therefore we go step by step.

a) First show that (Eq. (2.40))

$$S_F(x'-x) = \theta(t'-t)\int \frac{d^3p}{(2\pi)^3}\sum_{r=1}^{2}\psi_p^r(x')\overline{\psi}_p^r(x) - \theta(t-t')\int \frac{d^3p}{(2\pi)^3}\sum_{r=3}^{4}\psi_p^r(x')\overline{\psi}_p^r(x)$$

where

$$\psi_p^r(x) = \frac{1}{\sqrt{2E}}\ w^r(\vec{p})\ e^{-i\varepsilon_r p\cdot x}$$

b) Now derive Eqs. (2.47) and (2.48),

$$\lim_{t\to+\infty}\Psi(x) - \psi(x) = \int \frac{d^3p}{(2\pi)^3}\sum_{r=1}^{2}\psi_p^r(x)\left[-ieQ_e\int d^4y\ \overline{\psi}_p^r(y)\not{A}(y)\Psi(y)\right]$$

$$\lim_{t\to-\infty}\Psi(x) - \psi(x) = \int \frac{d^3p}{(2\pi)^3}\sum_{r=3}^{4}\psi_p^r(x)\left[+ieQ_e\int d^4y\ \overline{\psi}_p^r(y)\not{A}(y)\Psi(y)\right]$$

c) Finally use these results to show Eq. (4).