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GAUGE THEORIES*

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Introduction

The four-point Fermi theory of the weak interactions in the V–A form, together with the conserved vector current hypothesis, has long been known to be an incomplete theory. Even though it describes well μ and beta decay, it is not a renormalizable theory, and higher-order effects cannot be calculated. Physicists have long felt that mediating the interaction by vector boson exchanges would solve the problem, but until a few years ago have been unsuccessful at doing so.

Perhaps the most significant development in weak-interaction theory in the last few years, both from a purely theoretical viewpoint and for its possible impact on future experiments, has been the construction of renormalizable models of weak interactions based on the notion of spontaneously broken gauge symmetry. The basic strategy of this construction appeared in 1967 and 1968 in papers by Weinberg and by Salam. In these papers, the weak and electromagnetic interactions are unified in a Yang-Mills gauge theory with the intermediate vector bosons $W^{\pm}$ and the photons as gauge bosons. The idea itself was not new. What was new in the Weinberg-Salam strategy was to attribute the observed dissimilarities between weak and electromagnetic interactions to a spontaneous breakdown of gauge symmetry.

This mechanism has been studied by Higgs, Brout, Englert, Kibble, Guralnik, Hagen, and others since 1964. It takes place in a gauge theory in which the stable vacuum is not invariant under gauge transformations. In the absence of gauge bosons, non-invariance of the vacuum under a continuous symmetry implies the existence of massless scalar bosons, according to the Goldstone theorem. In a gauge theory, these would-be Goldstone bosons combine with the would-be massless gauge bosons (with two transverse polarizations) to produce a set of massive vector bosons (with three polarizations). In fact, the number of vector mesons which acquire mass exactly equals the number of Goldstone scalar mesons which disappear.

There are two attractive features of the model of Weinberg and Salam. The first is their elegant unifications of the electromagnetic and weak interactions. The second is the suggestion, stressed by these authors, that a theory of this kind might be renormalizable because the equations of motion are identical to those of an unbroken gauge theory. Not much was known about the renormalizability of these theories, and so the development of the Weinberg-Salam theory lay dormant for some years.

Two developments were responsible for the resurgence of interest in these models in 1971. The first was the quantization and renormalization of the Yang-Mills theory. After the pioneering works of Feynman, deWitt, Mandelstam, and Fadeev and Popov, vigorous studies on the renormalizability and the connection between massive and massless gauge theories were carried out by Boulware, Fadeev, Fradkin, Slavnov, J.C. Taylor, Tuytin, Van Dam and Veltman, among others. The second is the detailed study of the σ-model, which is the simplest field theory which exhibits spontaneous breakdown of symmetry. We learned from this study that the divergences of the theory were not affected by the spontaneous breakdown of symmetry so that the same renormalization counterterms remove the divergences from the theory whether or not the symmetry is spontaneously broken.

In 1971, G. 't Hooft presented a very important paper on manifestly renormalizable formulations of massive Yang-Mills theories wherein the masses of the gauge bosons arise from spontaneous breakdown of the gauge symmetry. His formulation takes explicit advantage of the gauge freedom afforded in such a theory.

Since then, there has been an explosion of interest in the subject, and the study of spontaneously broken gauge theories has become a major industry among theorists. Many models have been proposed and their implications explored. These models all predict new heavy vector mesons or heavy leptons, with interesting experimental implications. The fact that each model has a specific prediction for the properties of weak neutral currents has stimulated experimental interest in trying to detect them.

One of the most difficult problems has been to include hadrons naturally into the scheme. There have been many proposals, some of them very complicated. Surely this is an important subject for further research.

On the other hand, because the models are renormalizable, all higher order corrections are now calculable. There have been many calculations of radiative corrections to the muon anomalous magnetic moment and to weak decay rates, and, in some models, of electromagnetic mass differences. The possibility of doing such calculations has rendered the old "cutoff" methods obsolete.

In the fall of 1972, B.W. Lee gave a series of lectures on these subjects at the State University of New York at Stony Brook. What follows is based on these lectures as expanded and elaborated by both of us afterwards. They are divided into two parts. Part I describes the construction of models with spontaneously broken gauge symmetries, and some of their phenomenological implications. Part II describes the path-integral formulation of quantum field theory, and its application to the question of the renormalizability of these theories.

Part I begins by reviewing the theoretical tools needed to construct the models. Section 1 describes local gauge invariance and its application to non-Abelian gauge groups. Section 2 explains the spontaneous symmetry breaking mechanism and the origin of Goldstone bosons. In section 3 this idea is applied to locally gauge invariant theories, where instead of massless Goldstone bosons one obtains automatically massive vector gauge mesons, without introducing explicitly a symmetry-breaking mass term in the Lagrangian.

The next three sections are a brief review of the phenomenology of weak interactions and conventional theoretical ideas about them. They are far from a complete review of the subject; rather, their purpose was to make the series of lectures self-contained. The subjects covered include a few basic phenomena, the V – A theory, intermediate vector bosons, Cabibbo theory, and a few special topics which will be useful in later lectures.

Section 7 describes the original model of Weinberg and Salam in some detail. Section 8 discusses some experimental implications of this model, the inclusion of hadrons, and the question of neutral currents. Section 9 discusses a class of models with heavy leptons, and describes in some detail the model of Georgi and Glashow. Several other models are briefly described in section 10.

Part II is more mathematical. Its subject is the development of techniques for calculating higher order corrections to scattering amplitudes in spontaneously broken gauge theories, and, ultimately, to show why they are renormalizable. The subject is formulated in the language of path-integral quantization. Since this language is not very familiar to many physicists, we begin by reviewing it in detail.

Section 11 develops the integral-over-paths expression for the time-translation operator, following Feynman. In section 12, the method is extended to quantum field theory, and a general expression for the Green's functions is obtained. Using this principle, in section 13 we obtain the rules for calculating the Green's functions for the Yang-Mills theory in the Coulomb gauge.

The Coulomb gauge is the easiest to quantize in from first principles; what is really needed is the rule for calculating Green's functions and the S-matrix in any gauge. In section 14, the elegant, though somewhat intuitive, prescription for doing this, due principally to Fadeev and Popov, is described. Section 15 contains a formal proof that the Landau and Coulomb gauges give the same renormalized S-matrix.

In section 16, the generating functionals for the proper vertices are obtained and the idea of a superpotential is introduced. The σ-model is discussed in section 17, as an example of the usefulness of this approach in renormalizing theories with spontaneously broken symmetries. In section 18, we outline the renormalization scheme of Bogoliubov, Parasiuk, Hepp, and Zimmerman, whose topological analysis forms the basis for renormalizing gauge theories. The renormalization scheme of 't Hooft and Veltman is described in section 19, and the general application of all these methods to the renormalization of spontaneously broken gauge theories is discussed in section 20. Renormalization is done there in the Landau gauge, and the Feynman rules are derived. A more general class of gauges, called the R_ξ gauges, are derived in section 21, and in section 22 it is proved that the S-matrix is the same in all these gauges, and that the Goldstone bosons really do disappear in all gauges. As an illustration, the muon anomalous magnetic moment is computed in the last section, and shown explicitly to be gauge independent.

In the second half of Part II we fail to give a comprehensive review of all the work done by others (among them, notably, 't Hooft and Veltman; Ross and J.C. Taylor) towards proving the renormalizability and physical acceptability of spontaneously broken gauge theories. For the moment we are not equipped to do so. We apologize to our colleagues and the reader for presenting only our views and strategy. It would be presumptuous to assert that the renormalizability has been proved completely by us here or elsewhere. There are still some loose ends in our arguments for that. We do hope, however, to have marshalled sufficiently strong arguments for it, so that serious students of spontaneously broken gauge theories can accept their renormalizability as something more than just a working hypothesis.

These sections are not a final report on a closed subject. Rather, they are a reasonably self-contained course of study about a beautiful idea. Indeed the mathematical elegance and aesthetic apeal of this scheme for constructing models of weak interactions is what convinces many physicists that it may contain a germ of truth. The fact that some of the phenomenological implications of the various models may be tested in the near future is very exciting.

We have benifited greatly for our education in this field, from discussions and correspondence with many of our colleagues, among them: C. Albright, T. Appelquist, W.A. Bardeen, J.D. Bjorken, S. Coleman, C.G. Callan, H.H. Chen, R.R. Dashen, L.D. Fadeev, D.Z. Freedman, P. Freund, D. Fujikawa, H. Georgi, S. Glashow, D.J. Gross, R. Jackiw, W. Lee, Y. Nambu, A. Pais, E. Paschos, J. Primack, H. Quinn, A.I. Sanda, G. 't Hooft, S.B. Treiman, M. Veltman, S. Weinberg, M. Weinstein, L. Wolfenstein, C.N. Yang, J. Zinn-Justin, and B. Zumino. We would like to record our gratitude to them. One of us (ESA) would like to thank Professor C.N. Yang for the hospitality of the Institute of Theoretical Physics. We would like to thank Mrs. Dorothy DeHart and Mrs. Hannah Schlowsky for typing the difficult manuscript.

PART I

GAUGE MODELS OF WEAK AND ELECTROMAGNETIC INTERACTIONS

1. Gauge invariance in classical field theories

Now if we adopt the view that this arbitrary convention
should be independently chosen at every space-time point,
then we are naturally led to the concept of gauge fields.

C.N. Yang

In field theories one takes as the basic object the Lagrangian density $\mathcal{L}$ which is a function of all the fields $\phi_i(x)$ in the theory, and their gradients $\partial_\mu \phi_i(x)$. The Lagrangian L itself is the space integral of $\mathcal{L}$, and the integral over all space and time is called the action S:

$$S = \int_{-\infty}^{\infty} L(t)\, \mathrm{d}t = \int \mathrm{d}^4x\, \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)). \tag{1.1}$$

The equations of motion follow from Hamilton's principle,

$$\delta \int_{t_1}^{t_2} L(t) \mathrm{d}t = 0 \tag{1.2}$$

for any t_1 and t_2, where the variations of the fields must vanish at t_1 and t_2. Hamilton's principle implies that the fields satisfy Euler's equations:

$$\frac{\delta \mathcal{L}}{\delta \phi_i} = \frac{\partial}{\partial x^\mu} \frac{\delta \mathcal{L}}{\delta(\partial \phi_i / \partial x^\mu)}. \tag{1.3}$$

The idea of gauge transformations stems from the old observation that to every continuous symmetry of the Lagrangian there corresponds a conservation law. For example, suppose $\mathcal{L}$ has no explicit time dependence: the form of $\mathcal{L}$ is independent of the time x^0. Under an infinitesimal time translation, each of the fields ϕ_i is changed by

$$\delta \phi_i(x^0, \boldsymbol{x}) = \phi_i(x^0 + \epsilon, \boldsymbol{x}) - \phi_i(x) = \epsilon \partial \phi_i / \partial x^0 \quad \text{and} \quad \delta(\partial_\mu \phi_i) = \epsilon \partial_\mu [\partial \phi_i / \partial x^0]. \tag{1.4}$$

Similarly, $\delta \mathcal{L} = \epsilon \partial \mathcal{L} / \partial x^0$:

$$\epsilon \frac{\partial \mathcal{L}}{\partial x^0} = \sum_i \left[\frac{\delta \mathcal{L}}{\delta \phi_i} \delta \phi_i + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \delta(\partial_\mu \phi_i) \right]. \tag{1.5}$$

Using the equation of motion in the first term, one gets

$$\epsilon \frac{\partial \mathcal{L}}{\partial x^0} = \epsilon \sum_i \left[\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \right) \frac{\partial \phi_i}{\partial x^0} + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \partial_\mu \left(\frac{\partial \phi_i}{\partial x_0} \right) \right] \tag{1.6}$$

or

$$\frac{\partial \mathcal{L}}{\partial x^0} = \partial_\mu \sum_i \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \frac{\partial \phi_i}{\partial x^0} \tag{1.7}$$

which can be rewritten

$$\frac{\partial}{\partial x^0} \left[\mathcal{L} - \frac{\delta \mathcal{L}}{\delta(\partial \phi_i/\partial x^0)} \frac{\partial \phi_i}{\partial x^0}\right] = \nabla \cdot \sum_i \frac{\delta \mathcal{L}}{\delta(\nabla \phi_i)} \frac{\partial \phi_i}{\partial x^0}. \tag{1.8}$$

The bracket on the left-hand side is the Hamiltonian density $\mathcal{H}(x)$. Since the fields are required to vanish sufficiently rapidly for large $|\boldsymbol{x}|$,

$$\partial H/\partial t = 0 \tag{1.9}$$

where $H = \int \mathrm{d}^3x\, \mathcal{H}(x)$ is the Hamiltonian.

Continuing along these lines it is easy to see that in a Lorentz invariant theory, the energy, momentum and angular momentum can be defined and are conserved. In order for the equations of motion to be covariant, $\mathcal{L}$ must be a Lorentz scalar density. This is one of the reasons that it is useful to work with $\mathcal{L}$ instead of H in a relativistic field theory.

Here we will be interested in conservation laws that are *not* consequences of classical space-time symmetries. For every conserved quantum number one can *construct* a transformation on the fields which leaves $\mathcal{L}$ invariant. The simplest example is electric charge. Suppose each field ϕ_i has charge q_i (in units of e). Then define a group of transformations on the fields by

$$\phi_i(x) \to \exp(-\mathrm{i}q_i\theta)\phi_i(x). \tag{1.10}$$

The group is the group of unitary transformations in one dimension, U(1). It is not hard to see that $\mathcal{L}$ must be invariant under these transformations. Every term in $\mathcal{L}$ is a product of fields $\phi_1 \ldots \phi_n$. Under the transformation above,

$$\phi_1(x) \ldots \phi_n(x) \to \exp\{-\mathrm{i}(q_1 + q_2 + \ldots q_n)\phi\}\phi_1(x) \ldots \phi_n(x).$$

Charge conservation requires that $\mathcal{L}$ be neutral; therefore the sum $q_1 + q_2 + \ldots q_n$ must vanish.

Some terms in $\mathcal{L}$ contain gradients of the fields as well as the fields themselves. But since θ is independent of x, $\partial_\mu\phi_i \to \exp(-\mathrm{i}q_i\theta)\partial_\mu\phi_i$ as well, so these terms are also invariant. A transformation like (1.10) is called a gauge transformation, or more properly, a gauge transformation of the first kind. The invariance of $\mathcal{L}$ under the gauge group is called gauge invariance of the first kind, or sometimes global gauge invariance (because θ is independent of x).

The infinitesimal form of (1.10) is

$$\delta\phi_i = -\mathrm{i}\epsilon q_i \phi_i \tag{1.11}$$

where in (1.11) ϵ is an infinitesimal parameter. Global gauge invariance can be succintly stated:

$$\delta \mathcal{L} = 0. \tag{1.12}$$

If $\mathcal{L}$ depends only on ϕ_i and on $\partial_\mu\phi_i$, then eq. (1.12) gives

$$0 = \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi_i}\,\delta\phi_i + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)}\,\delta(\partial_\mu\phi_i) = -\mathrm{i}\epsilon\,\frac{\partial\mathcal{L}}{\partial x_\mu}\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)}\,q_i\phi_i\right] .$$

Thus for the operation (1.11) which leaves the Lagrangian invariant there is a conserved current J^μ

$$\partial J^\mu(x;q)/\partial x^\mu = 0$$

with

$$J^\mu = \mathrm{i}\,\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)}\,q_i\phi_i .$$

The gauge group has an infinitesimal generator Q. The q_i are just the eigenvalues of Q, and $\exp(-\mathrm{i}q_i\theta)$ is a one dimensional representation of U(1) generated by Q. In quantized theory the operator Q

$$Q = \int \mathrm{d}^3 J_0(x,\,t)$$

is the charge operator, and

$$\delta\phi_i = -\mathrm{i}\epsilon[Q,\,\phi_i] = -\mathrm{i}\epsilon\; q_i\phi_i .$$

A theory may contain more than one conserved quantity, and be invariant under a more complicated group of transformations than U(1). The simplest non-Abelian example is isospin. In a theory with isospin symmetry, the fields will come in multiplets which form a basis for representations of the isospin group SU(2). Then we can define a gauge transformation by

$$\phi \to \exp(-\mathrm{i}\,\mathbf{L}\cdot\boldsymbol{\theta})\phi \tag{1.13}$$

where ϕ is a column vector and $\mathbf{L}$ is the appropriate matrix representation of SU(2). For a doublet, for example, $\mathbf{L} = \frac{1}{2}\boldsymbol{\tau}$ ($\boldsymbol{\tau}$ are the Pauli matrices). For a triplet

$$L^i_{jk} = -\mathrm{i}\epsilon^{ijk} .$$

Since the generators, T_i of the group satisfy

$$[T_i,\,T_j] = \mathrm{i}\epsilon^{ijk}T_k ,$$

the representation matrices satisfy the same rule : $[L^i,\,L^j] = \mathrm{i}\epsilon^{ijk}L^k$. The Lagrangian $\mathcal{L}$ will be invariant under any of the transformations of the group.

Under an infinitesimal transformation,

$$\delta\phi = -\mathrm{i}\mathbf{L}\cdot\boldsymbol{\epsilon}\phi \tag{1.14}$$

where we may think of $\boldsymbol{\epsilon}$ as three independent infinitesimal parameters.

Thus if ϕ is a two component isospinor,

$$\delta\phi = -\frac{\mathrm{i}}{2}\,\boldsymbol{\tau}\cdot\boldsymbol{\epsilon}\,\phi ,$$

and if ϕ_i are the components of an isovector,

$$\delta\phi_i = \epsilon^{ijk}\epsilon^j\phi_k .$$

Isospin invariance requires $\delta\mathcal{L} = 0$ for all ϵ^j.

The idea is easily generalized to any internal symmetry Lie group G. Let T_i be the group generators, and c_{ijk} the structure constants:

$$[T_i, T_j] = \mathrm{i}c_{ijk}T_k . \tag{1.15}$$

The fields ϕ_i will transform according to some (generally reducible) representation of G. The T_i are represented by the matrices L_i. A finite gauge transformation is

$$\phi \to \exp(-\mathrm{i}\mathbf{L}\cdot\boldsymbol{\theta})\phi \tag{1.16}$$

the corresponding infinitesimal one is

$$\delta\phi = -\mathrm{i}\mathbf{L}\cdot\boldsymbol{\epsilon}\,\phi \tag{1.17}$$

where the number of independent parameters θ^j is the dimension of the group. The Lagrangian $\mathcal{L}$ is invariant under the group: $\delta\mathcal{L} = 0$.

It is well known that electrodynamics possesses a formal symmetry larger than gauge transformations of the first kind. The gauge transformation can depend on the space-time point which is the argument of the field:

$$\phi_i(x) \to \phi_i'(x) = \exp\{-\mathrm{i}q_i\theta(x)\}\phi_i(x). \tag{1.18}$$

This is called a gauge transformation of the second kind, or *local* gauge transformation. The infinitesimal form of (1.18) is

$$\delta\phi_i(x) = -\mathrm{i}q_i\theta(x)\phi_i(x). \tag{1.19}$$

Here $\theta(x)$ is an arbitrary infinitesimal function of x. Terms in the Lagrangian which depend only on the fields are obviously invariant under (1.18). Terms with field gradients, such as the kinetic energy term, need more care. The reason is that, from (1.18)

$$\partial_\mu\phi_i \to \exp\{-\mathrm{i}q_i\theta(x)\}\,\partial_\mu\phi_i(x) - \mathrm{i}q_i[\partial_\mu\theta(x)]\,\exp\{-\mathrm{i}q_i\theta(x)\}\,\phi_i(x). \tag{1.20}$$

The second term is the difference between the way $\partial_\mu\phi_i$ and ϕ_i transform; but the Lagrangian will be invariant only if it is a product of terms all of which transform like (1.10), with the sum of the q_i vanishing.

Electrodynamics is made invariant by introducing the photon field according to the following rule, usually called minimal coupling: A gradient of a charged field, $\partial_\mu\phi_i$, is allowed to appear in $\mathcal{L}$ only in conjunction with the photon field, A_μ, in the combination $(\partial_\mu - \mathrm{i}eq_iA_\mu)\phi_i$. A_μ is the field of a spin-one meson – the photon – which is our first example of a gauge boson. We require it to transform under local gauge transformations in a special way, so that the combination $(\partial_\mu - \mathrm{i}eq_iA_\mu)\phi_i(x)$ transforms like $\phi_i(x)$ in (1.10). That is,

$$(\partial_\mu - \mathrm{i}eq_iA_\mu')\phi_i'(x) = \exp\{-\mathrm{i}q_i\theta(x)\}(\partial_\mu - \mathrm{i}eq_iA_\mu)\phi_i(x). \tag{1.21}$$

Then $\mathcal{L}$ will be invariant under local gauge transformations as well. Putting in what we know for $\partial_\mu \phi_i'(x)$, we get

$$\exp\{-iq_i\theta(x)\}\partial_\mu\phi_i(x) - iq_i[\partial_\mu\theta(x)]\exp\{-iq_i\theta(x)\}\phi_i(x) - ieq_iA_\mu'(x)\exp\{-iq_i\theta(x)\}\phi_i(x)$$
$$= \exp\{-iq_i\theta(x)\}\,\partial_\mu\phi_i(x) - ieq_iA_\mu(x)\exp\{-iq_i\theta(x)\}\phi_i(x). \quad (1.22$$

The solution to (1.22) is

$$A_\mu'(x) = -\frac{1}{e}\partial_\mu\theta(x) + A_\mu(x) \quad (1.23$$

or

$$\delta A_\mu(x) = -\frac{1}{e}\partial_\mu\theta(x). \quad (1.24$$

In addition to terms coupling the photon field to the charged particle fields, there could be quadratic kinetic energy and mass terms coupling A_μ only to itself. The solution is well-known. Define the field-strength tensor $F_{\mu\nu}$:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (1.2$$

Then $\delta F_{\mu\nu} = 0$ under (1.23), and therefore the photon kinetic energy term, will be gauge invaria if it is constructed out of $F_{\mu\nu}$:

$$\mathcal{L}_{\mathrm{EM}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (1.2$$

The coefficient $-\frac{1}{4}$ is dictated by the requirement that the Euler-Lagrange equations result in Maxwell's equations with the conventional normalization of the electric charge e.

A photon mass term would have the form $-\frac{1}{2}m^2A_\mu A^\mu$, which obviously violates local gauge in variance. The conclusion is that local gauge invariance is impossible unless the photon is massles

It is supererogatory to observe that the photon was not discovered by requiring local gauge in variance. Rather, gauge transformations were discovered as a useful property of Maxwell's equations. However, in quantum electrodynamics, gauge invariance allows one to derive the Ward–Takahashi identities which in turn allow one to prove many theorems, including, most importa as we shall see, the theory's renormalizability.

The generalization of local gauge invariance to non-Abelian groups was first studied by Yang and Mills, for the case of isotopic spin, SU(2). It is elementary to generalize their idea to any internal symmetry group.

Let the group have generators T_i as before:

$$[T_i, T_j] = ic_{ijk}T_k. \quad (1.2$$

A collection of fields transforms according to

$$\phi(x) \to \phi'(x) = \exp\{-i\mathbf{L}\cdot\boldsymbol{\theta}\}\phi(x) \equiv U(\theta)\phi(x) \quad (1.$$

where $\phi(x)$ is a column vector and L^i is a matrix representation of the generators of the group. The Lagrangian $\mathcal{L}$ is assumed to be invariant under transformation with constant θ^i. The probl is to construct a theory which is invariant under local gauge transformations $\theta^i(x)$ as well, by in troducing vector fields $A^i(x)$ in analogy with electrodynamics.

Under a local gauge transformation

$$\phi(x) \to U(\theta)\phi(x) \quad (1.$$

and therefore

$$\partial_\mu\phi(x) \to U(\theta)\partial_\mu\phi(x) + (\partial_\mu U(\theta))\phi(x). \qquad (1.30)$$

The idea is to introduce a covariant derivative $D_\mu\phi(x)$ which transforms like $\phi(x)$:

$$D_\mu\phi(x) \to U(\theta)D_\mu\phi(x). \qquad (1.31)$$

Then, if $\partial_\mu\phi(x)$ appears in $\mathcal{L}$ only as a part of $D_\mu\phi(x)$, $\mathcal{L}$ will be invariant under local gauge transformations.

The covariant derivative $D_\mu\phi(x)$ is constructed by introducing a vector field $A^j_\mu(x)$ for each dimension of the Lie algebra, and defining

$$D_\mu\phi(x) = (\partial_\mu - ig\mathbf{L}\cdot\mathbf{A}_\mu(x))\phi(x). \qquad (1.32)$$

The coupling constant g, analogous to e, is arbitrary.

How do the A^j_μ transform in order to ensure (1.31)? That is, A'^j_μ must be defined so that the quantity

$$\begin{aligned} D'_\mu\phi' &= \partial_\mu\phi' - igA'^j_\mu L^j\phi' \\ &= (\partial_\mu U(\theta))\phi(x) + U(\theta)\partial_\mu\phi - ig\mathbf{A}'_\mu\cdot\mathbf{L}\,U(\theta)\phi, \end{aligned} \qquad (1.33)$$

is equal to

$$U(\theta)(\partial_\mu - ig\mathbf{A}_\mu\cdot\mathbf{L})\phi. \qquad (1.34)$$

The solution is

$$-ig\mathbf{A}'_\mu\cdot\mathbf{L}\,U(\theta)\phi = -igU(\theta)\mathbf{A}_\mu\cdot\mathbf{L}\phi - (\partial_\mu U(\theta))\phi, \qquad (1.35)$$

or, since (1.35) must hold for all ϕ,

$$\begin{aligned} \mathbf{A}'_\mu\cdot\mathbf{L} &= U(\theta)\,\mathbf{A}_\mu\cdot\mathbf{L}\,U^{-1}(\theta) - \frac{i}{g}(\partial_\mu U(\theta))U^{-1}(\theta) \\ &= U(\theta)\,[\mathbf{A}_\mu\cdot\mathbf{L} - \frac{i}{g}U^{-1}(\theta)\partial_\mu U(\theta)]\,U^{-1}(\theta). \end{aligned} \qquad (1.36)$$

We leave it as an exercise to show that the transformations form a group: in particular, if

$$\mathbf{L}\cdot\mathbf{A}'_\mu = U(\theta)[\mathbf{A}_\mu\cdot\mathbf{L} - \frac{i}{g}U^{-1}(\theta)\partial_\mu U(\theta)]\,U^{-1}(\theta)$$

and

$$\mathbf{L}\cdot\mathbf{A}''_\mu = U(\theta')[\mathbf{A}'_\mu\cdot\mathbf{L} - \frac{i}{g}U^{-1}(\theta')\partial_\mu U(\theta')]\,U^{-1}(\theta'),$$

then

$$\mathbf{L}\cdot\mathbf{A}''_\mu = U(\theta'')\,[\mathbf{A}_\mu\cdot\mathbf{L} - \frac{i}{g}U^{-1}(\theta'')\partial_\mu U(\theta'')]\,U^{-1}(\theta'')$$

where

$$U(\theta'') = U(\theta')U(\theta).$$

This transformation rule appears to depend on the representation, but in fact depends only on the commutators $[L^i, L^j]$ whose form is representation-independent. This fact becomes apparent from the infinitesimal transformation:

$$L^j\delta A^j_\mu = -\frac{1}{g}L^j\partial_\mu\theta^j + \mathrm{i}L^iA^i_\mu\theta^jL^j - \mathrm{i}\theta^jL^jA^i_\mu L^i$$

$$= -\frac{1}{g}L^j\partial_\mu\theta^j + \mathrm{i}\theta^jA^i_\mu[L^i, L^j] = -\frac{1}{g}L^j\partial_\mu\theta^j - \theta^jA^i_\mu c_{ijk}L^k. \tag{1.37}$$

Since the L^j are linearly independent,

$$\delta A^i_\mu = -\frac{1}{g}\partial_\mu\theta^i + c_{ijk}\theta^jA^k_\mu. \tag{1.38}$$

The transformation properties of A^i_μ do not depend on the representation L^j.

Next we must construct the analog of the kinetic energy term, i.e. the term $\mathcal{L}_0$ which contains only the fields A^i_μ and their derivatives. Because these fields do not all carry zero quantum numbers under all the T_i (unlike the photon, which is electrically neutral), $\mathcal{L}_0$ cannot have the simple form it has in electrodynamics. In fact, from (1.38), it is easy to see that

$$\delta[\partial_\mu A^i_\nu - \partial_\nu A^i_\mu] = c_{ijk}\theta_j(\partial_\mu A^k_\nu - \partial_\nu A^k_\mu) + c_{ijk}[(\partial_\mu\theta^j)A^k_\nu - (\partial_\nu\theta^j)A^k_\mu]. \tag{1.39}$$

$\mathcal{L}_0$ will be invariant if it is constructed out of a tensors $F^i_{\mu\nu}$ according to

$$\mathcal{L}_0 = -\tfrac{1}{4}F^i_{\mu\nu}F^{\mu\nu i} \tag{1.40}$$

provided the $F^i_{\mu\nu}$ transform covariantly like a set of fields in the regular (adjoint) representation of G. Therefore we must add something to $\partial_\mu A^i_\nu - \partial_\nu A^i_\mu$ to cancel the unwanted terms in (1.39). Now from (1.38)

$$c_{ijk}\delta[A^j_\mu A^k_\nu] = -\frac{c_{ijk}}{g}[(\partial_\mu\theta^j)A^k_\nu - (\partial_\nu\theta^j)A^k_\mu] + c_{ijk}c_{jlm}\,\theta^lA^m_\mu A^k_\nu + c_{ijk}c_{klm}\theta^lA^m_\nu A^j_\mu. \tag{1.41}$$

The first terms (times g) can just cancel the unwanted piece of (1.39). The last two terms can be rewritten, using the antisymmetry of the structure constants, as

$$[c_{imk}c_{kjl} - c_{ijk}c_{kml}]\,\theta^lA^j_\mu A^m_\nu. \tag{1.42}$$

Let T^i stand also for the regular representation matrices. Then $(T^i)_{jk} = -\mathrm{i}\,c_{ijk}$, and the bracket in (1.42) is

$$c_{imk}c_{lkj} - c_{lmk}c_{ikj} = [T^l, T^i]_{mj} = \mathrm{i}\,c_{lik}(T^k)_{mj} = c_{ilk}c_{kjm}.$$

Therefore,

$$c_{ijk}\delta[A_{\mu j}A_{\nu k}] = -\frac{c_{ijk}}{g}[(\partial_\mu\theta^j)A^k_\nu - (\partial_\nu\theta^j)A^k_\mu] + c_{ilk}\theta^l c_{kjm}A_{\mu j}A_{\nu m}.$$

So define

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\mu A^i_\nu + g\, c_{ijk}A_{\mu j}A_{\nu k} \tag{1.43}$$

then

$$\delta F^i_{\mu\nu} = c_{ijk}\theta^j F^k_{\mu\nu} \tag{1.44}$$

and $\mathcal{L}_o = -\frac{1}{4}F^i_{\mu\nu}F^{\mu\nu i}$ is invariant. Under finite gauge transformations, $U(\theta) = \exp(-iL^i\theta^i)$, $F^i_{\mu\nu}$ transforms as $\mathbf{F}_{\mu\nu}\cdot\mathbf{L} \to U(\theta)\mathbf{F}_{\mu\nu}\cdot\mathbf{L}U^{-1}(\theta)$ so that $\mathrm{Tr}(\mathbf{F}_{\mu\nu}\cdot\mathbf{L})^2 \sim \mathbf{F}_{\mu\nu}\cdot\mathbf{F}^{\mu\nu}$ is invariant.

Again, a mass-term of the form $\frac{1}{2}m^2\,\mathbf{A}_\mu\mathbf{A}^\mu$ would violate the local gauge invariance.

We conclude by summarizing the construction of local gauge theories with non-Abelian symmetries. Start with a Lagrangian $\mathcal{L}_1(\phi_i, \partial_\mu\phi_i)$ invariant under a Lie group G with generators T_i and structure constants c_{ijk}. The fields transform according to some representation $\exp(-iL\cdot\boldsymbol{\theta})$ of the group, with constant θ^i. Add to the theory a set of vector fields A^i_μ, one for each T^i. The full Lagrangian is

$$\mathcal{L} = \mathcal{L}_o + \mathcal{L}_1(\phi_i, (\partial_\mu - ig\mathbf{A}_\mu\cdot\mathbf{L})\phi_i). \tag{1.45}$$

The first term is

$$\mathcal{L}_o = -\tfrac{1}{4}\mathbf{F}_{\mu\nu}\cdot\mathbf{F}^{\mu\nu} \tag{1.46}$$

where

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\, c_{ijk}A^j_\mu A^k_\nu. \tag{1.47}$$

The transformation rule for the gauge bosons is

$$\mathbf{L}'\cdot\mathbf{A}_\mu = U(\theta)\mathbf{L}\cdot\mathbf{A}_\mu U^{-1}(\theta) - \frac{1}{g}(\partial_\mu U(\theta))U^{-1}(\theta). \tag{1.48}$$

where here θ^i is a function of x.

One final note. If G is a direct product of two or more subgroups, the coupling constants g associated with each subgroup need not be the same.

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We shall discuss the use of these identities in gauge theories extensively in Part II. In the generalized sense, these identities are the precise mathematical statements about the effects of gauge invariance (or other symmetries) of the Lagrangian on Green's functions.

2. Spontaneously broken symmetries

> If my view is correct, the universe may have a kind of domain structure. In one part of the universe you may have one preferred direction of the axis; in another part, the direction of the axis may be different.
>
> Y. Nambu

Nature seems to possess useful symmetries which, unlike electric charge conservation, are not exact symmetries of the S-matrix. Familiar examples are isospin, strangeness and SU(3). A traditional way of thinking about them is to imagine that the Lagrangian possesses a part which is exactly symmetric and another, in some sense "small", term which violates the symmetries. This idea is behind our conventional picture of a "hierarchy" of interactions – strong, electromagnetics and weak – in which the stronger interactions possess more symmetry than the weaker ones. Another type of symmetry is PCAC, which even in the exact symmetry limit is not a symmetry of the physical spectrum, that is, particles do not occur in equal-mass multiplets which can be assigned to a representation of the group (in this case SU(2) × SU(2)). Nevertheless the Ward–Takahashi identities and current-algebra predictions of SU(2) × SU(2) symmetry are physically useful.

By now it is well-known that the second kind of symmetry can be obtained from an exactly symmetric Lagrangian, provided that the physical vacuum is not invariant under the symmetry group. Such a symmetry is popularly called a "spontaneously broken symmetry". The mechanics of how this works is the subject of this section. Then we will go on to see what wonderful things happen when the symmetry of the Lagrangian is made into a local gauge symmetry of the kind described in the first lecture.

It is instructive to begin by understanding how a field theory is like a collection of anharmonic oscillators. A simple Lagrangian density with only a single scalar field is

$$\mathcal{L} = \tfrac{1}{2}(\partial^\mu\phi\partial_\mu\phi) - \tfrac{1}{2}\mu^2\phi^2 - \tfrac{1}{4}\lambda\phi^4 . \tag{2.1}$$

For simplicity, let there be only one space dimension. Then the Lagrangian is

$$L = \int_{-\infty}^{\infty} \mathcal{L}(x,t)\,\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \mathrm{d}x\left[\tfrac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \tfrac{1}{2}\mu^2\phi^2 - \tfrac{1}{4}\lambda\phi^4\right] . \tag{2.2}$$

We may think of $\phi(x,t)$ as being a canonical coordinate at each x. Divide space into unit cells of length ϵ labeled by the coordinate x_i: $x_i - x_{i-1} = \epsilon$. Then we may replace the integral defining L by a discrete sum. The discrete coordinates are $q_i(t) = \phi(x_i, t)$, and L becomes

$$L = \sum_{i=-\infty}^{\infty}\left[\tfrac{1}{2}\left(\frac{\mathrm{d}q_i}{\mathrm{d}t}\right)^2 - \frac{1}{2\epsilon^2}(q_i - q_{i-1})^2 - \tfrac{1}{2}\mu^2 q_i^2 - \tfrac{1}{4}\lambda_i^4\right]. \tag{2.3}$$

The second term represents a coupling between coordinates at adjacent points, and the last term makes the potential anharmonic. The canonical momentum is

$$p_i = dq_i/dt$$

and if we define

$$V(z) = \tfrac{1}{2}\mu^2 z^2 + \tfrac{1}{4}\lambda z^4$$

the Hamiltonian is

$$H = \sum_{i=-\infty}^{\infty} \left[\tfrac{1}{2} p_i^2 + \frac{1}{2\epsilon^2}(q_i - q_{i-1})^2 + V(q_i) \right] . \tag{2.4}$$

Field oscillations are bounded only if $\lambda \geqslant 0$, which we therefore require. In the usual case $\mu^2 > 0$ also. To do any kind of perturbation calculation, we must find the minimum of the potential,

$$\sum_i \left[\frac{1}{2\epsilon^2}(q_i - q_{i-1})^2 + V(q_i^2) \right]$$

and start with the unperturbed harmonic oscillator solutions as the zeroth approximation (these are the "free field" solutions of field theory). Whatever V is, we must have $q_i = q_{i-1}$ at the minimum of the potential; i.e., all the q_i are equal. If $\mu^2 > 0$, the function V looks like fig. 2.1 and the minimum occurs at $q_i = 0$. On the other hand, if $\mu^2 < 0$, the potential looks like fig. 2.2. Now $q = 0$ is not a minimum. There are two symmetric minima at $q = \pm[-\mu^2/\lambda]^{1/2}$.

In field theory, the ground state is the vacuum. What we have shown in an admittedly heuristic manner is that if $\mu^2 < 0$ the vacuum expectation value of the field is not zero; rather, it is independent of x ($q_i = q_{i-1}$) and has the value $\pm[-\mu^2/\lambda]^{1/2}$ to zeroth order in perturbation theory.

Let v be the vacuum expectation value of the field:

$$\langle\phi\rangle_0 = v = \pm[-\mu^2/\lambda]^{1/2} . \tag{2.5}$$

Either value of v may be chosen, but not both. We may by convention choose the plus sign, since L is invariant under $\phi \to -\phi$.

The only symmetry this simple Lagrangian possesses is reflection invariance: $\phi \to -\phi$. Clearly the new vacuum is not an eigenstate of this operation, since $v \neq -v$. In this way the symmetry is "spontaneously" broken. Define a new field ϕ' by

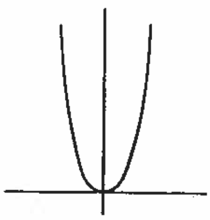

Fig. 2.1. The potential function for positive μ^2.

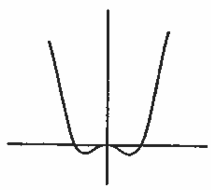

Fig. 2.2. The potential function for negative μ^2.

$$\phi' = \phi - v$$

then

$$\langle \phi' \rangle_o = 0$$

so we can do ordinary perturbation theory in ϕ'. In terms of ϕ' (up to a constant)

$$\mathcal{L} = \tfrac{1}{2}(\partial^\mu \phi' \partial_\mu \phi') + \mu^2 \phi'^2 - \lambda v \phi'^3 - \tfrac{1}{4}\lambda \phi'^4 . \tag{2.6}$$

The bare states have (positive) mass $-2\mu^2$, but do not exhibit the symmetry of the Lagrangian in an obvious way.

A slightly more complicated model has two fields, which we may call σ and π:

$$\mathcal{L} = \tfrac{1}{2}[\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \pi \partial^\mu \pi] - V(\sigma^2 + \pi^2) \tag{2.7}$$

where

$$V = \tfrac{1}{2}\mu^2(\sigma^2 + \pi^2) + \tfrac{1}{4}\lambda(\sigma^2 + \pi^2)^2 . \tag{2.8}$$

$\mathcal{L}$ is obviously invariant under O(2)[=U(1)]:

$$\begin{pmatrix} \sigma' \\ \pi' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix} . \tag{2.9}$$

The minimum occurs when

$$\partial V/\partial \sigma = 0 = \sigma[\mu^2 + \lambda(\sigma^2 + \pi^2)] \tag{2.10a}$$

$$\partial V/\partial \pi = 0 = \pi[\mu^2 + \lambda(\sigma^2 + \pi^2)] . \tag{2.10b}$$

Clearly when $\mu^2 < 0$, the absolute minimum occurs on the circle $\sqrt{\sigma^2 + \pi^2} = [-\mu^2/\lambda]^{1/2}$. We can always define the axes in the $\sigma - \pi$ plane so that

$$\langle \sigma \rangle_o = [-\mu^2/\lambda]^{1/2}, \qquad \langle \pi \rangle_o = 0.$$

[Another approach is to add explicitly a small symmetry-breaking term $c\sigma$ to V, as in the σ-model of Gell-Mann and Lévy. Then the minimum occurs when

$$\sigma[\mu^2 + \lambda(\sigma^2 + \pi^2)] = c, \qquad \pi[\mu^2 + \lambda(\sigma^2 + \pi^2)] = 0.$$

The term $c\sigma$ picks out the particular direction in (σ, π) space. There is no solution to these equations except $\pi = 0$, and $\sigma[\mu^2 + \lambda\sigma^2] = c$; in the limit $c \to 0$, either $\sigma = 0$ or $\sigma = [-\mu^2/\lambda]^{1/2}$. The first solution is a minimum when $\mu^2 > 0$, the second when $\mu^2 < 0$.]

As before, when $\mu^2 < 0$, define

$$s = \sigma - \langle \sigma \rangle_o$$

and rewrite $\mathcal{L}$ in terms of s and π instead of σ and π:

$$\mathcal{L} = \tfrac{1}{2}[\partial_\mu s \partial^\mu s + \partial_\mu \pi \partial^\mu \pi] + \mu^2 s^2 - \lambda \langle \sigma \rangle_o s(s^2 + \pi^2) - \tfrac{1}{4}\lambda(s^2 + \pi^2)^2 . \tag{2.11}$$

Evidently, s is the field of a particle with positive mass $-2\mu^2$ while the π-field is massless. This is our first example of Goldstone's theorem. If a theory has a symmetry of the Lagrangian which is not a symmetry of the vacuum, there must be a massless boson.

Here is a more general example. Let ϕ be an n-component real field, with Lagrange density

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi^i \partial^\mu \phi^i) - \tfrac{1}{2}\mu^2 \phi^i\phi^i - \tfrac{1}{4}\lambda(\phi^i\phi^i)^2. \tag{2.12}$$

$\mathcal{L}$ is obviously invariant under the orthogonal group in n dimensions, O(n). If $\mu^2 < 0$, the potential has a ring of minima at $v = [-\mu^2/\lambda]^{1/2}$ i.e. there is a minimum whenever $\phi^i\phi^i = -\mu^2/\lambda$. Let us choose the nth component of ϕ to be the one which develops a vacuum expectation value. That is to say, considering ϕ as an n-component column vector,

$$\langle\phi\rangle_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ v \end{pmatrix}.$$

The original symmetry group, O(n), has $\frac{1}{2}n(n-1)$ generators. The new feature in this example is that there is still a non-trivial group which leaves the vacuum invariant. This is the subgroup of O(n) which does not mix up the nth component with the others; it is O($n-1$), with $\frac{1}{2}(n-1)(n-2)$ generators.

Let L_{ij} be the $\frac{1}{2}n(n-1)$ independent matrices generating O(n). Let l_{ij} be the subset which form the surviving symmetry O($n-1$) [$l_{ij} = L_{ij}$ for $i, j \neq n$]. Then call the rest k_i [$k_i = L_{in}$]. These are $n-1$ independent k_i. Instead of simply subtracting the vacuum-expectation value of the field to define new fields as before, we can parametize the n field in a way which will be more useful later. Define η and ξ_i, $1 \leqslant i \leqslant n-1$, by

$$\phi = \exp(\mathrm{i}\xi_i k_i/v) \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ v+\eta \end{pmatrix}$$

Since, in general, $(L_{ij})_{kl} = -\mathrm{i}[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}]$, the k_i have matrix elements

$$(k_i)_{kl} = (L_{in})_{kl} = -\mathrm{i}[\delta_{ik}\delta_{nl} - \delta_{il}\delta_{nk}]$$

and so k_i operating on the column vector $v_i = v\delta_{in}$ is the vector

$$(k_i v)_j = v(k_i)_{jl}\delta_{ln} = v(k_i)_{jn} = -\mathrm{i}v\delta_{ij}.$$

Thus, in lowest order this definition is equivalent to our previous procedure; up to terms quadratic in the fields. $\phi_i = \xi_i (i < n)$ and $\phi_n = v + \eta$. [We will show in Part II that such a redefinition of the fields leaves the renormalized S-matrix invariant, but not the Green's functions.]

In terms of the new fields ξ_i and η, the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\left[\partial^\mu \eta \partial_\mu \eta + \partial^\mu \xi_i \partial_\mu \xi_i\right] + \text{higher order terms with derivative couplings}$$
$$-\tfrac{1}{2}\mu^2(v+\eta)^2 - \tfrac{1}{4}\lambda(v+\eta)^4. \tag{2.13}$$

The η field has bare mass $-2\mu^2 (> 0)$, and the $n-1$ ξ_i fields are massless. Thus to each generator of the original group which leaves the vacuum invariant, there corresponds a massless Goldstone boson.

The fact that the number of massless bosons is the same as the number of broken generators seems to be an accident of our example, the n-dimensional representation of O(n). But it is in fact general. Write any Lagrangian in terms of the n *real* scalar fields ϕ_i, which form an n-component vector ϕ (a complex representation can always be turned into a real one by doubling the number of basis vectors)

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi \cdot \partial^\mu \phi - V(\phi). \tag{2.14}$$

Of course, $\mathcal{L}$ may contain other fields (e.g. spinor fields) which couple to each other and to ϕ, but these terms are not relevant here. $V(\phi)$ is a polynomial in ϕ which is invariant under some group G (and not under a larger group containing G). G has N generators T_α, and ϕ transforms according to an n-dimensional (in general reducible) representation L^α: $\delta\phi = -\mathrm{i}\theta^\alpha L^\alpha \phi$.

Because the representation is real, $\mathrm{i}L^\alpha$ must be a real matrix; so L^α is an imaginary matrix, and because it is Hermitean it is antisymmetric. Because V is invariant under G, its response to an infinitesimal group transformation (specified by θ^α) is

$$0 = \delta V = \frac{\partial V}{\partial \phi_i}\delta\phi_i = -\mathrm{i}\frac{\partial V}{\partial \phi^i}\theta^\alpha L^\alpha_{ij}\phi_j. \tag{2.15}$$

Since θ^α are arbitrary, we obtain N equations

$$\frac{\partial V}{\partial \phi_i} L^\alpha_{ij}\phi_j = 0 \tag{2.16}$$

for all α. Differentiating again, we get

$$\frac{\partial^2 V}{\partial \phi_i \partial \phi_k} L^\alpha_{ij}\phi_j + \frac{\partial V}{\partial \phi_i} L^\alpha_{ik} = 0. \tag{2.17}$$

Now evaluate (2.17) at $\phi = v$, the value of ϕ which minimizes V:

$$(\partial V/\partial \phi_i)_{\phi = v} = 0.$$

The result is

$$(\partial^2 V/\partial\phi_i\partial\phi_k)_{\phi=v}\, L^\alpha_{ij} v_j = 0. \tag{2.18}$$

If V is expanded about v, there are no linear terms, and the constant term is irrelevant:

$$V = -\tfrac{1}{2} M^2_{ij} (\phi - v)_i (\phi - v)_j + \text{higher order terms}. \tag{2.19}$$

Therefore $\partial^2 V/\partial\phi_i\partial\phi_j$ evaluated at $\phi = v$ is just $-M^2_{ij}$ where M^2_{ij} is the mass matrix, and so

$$(M^2)_{ij} L^\alpha_{jk} v_k = 0 \tag{2.20}$$

for each α.

Let S be the M-dimensional subgroup of G which remains a symmetry of the vacuum. If L^α is a generator of S, $L^\alpha v = 0$, and so (2.20) contains no information about M^2. For each of the $N - M$ vectors $L^\alpha_{jk} v_k$ which are not zero, (2.20) says that M^2 has a zero eigenvalue. If the vectors $L^\alpha v$ in fact span an $N - M$ dimensional space, we have demonstrated that there are $N - M$ massless (Goldstone) bosons in the theory.

This fact is almost obvious from our examples. To construct a formal proof, define $A^{\alpha\beta} = (L^\alpha v, L^\beta v)$. [$(a, b)$ means $\Sigma_i a_i^* b_i$, even though we have a real vector space.] Since L^α is Hermitean, $A^{\alpha\beta} = (v, L^\alpha L^\beta v)$. Then

$$A^{\alpha\beta} - A^{\beta\alpha} = (v, [L^\alpha, L^\beta] v) = \mathrm{i} c_{\alpha\beta\gamma} (v, L^\gamma v) = 0, \tag{2.21}$$

the last equality following again because L is antisymmetric. Therefore let $\widetilde{A}$ be the $(N - M) \times (N - M)$ matrix obtained by restricting α and β to those values for which $L^\alpha v \neq 0$. $\widetilde{A}$ is symmetric, and can be diagonalized. Then let O be the $(N - M) \times (N - M)$ orthogonal matrix which diagonalizes $\widetilde{A}$:

$$\widetilde{A}'^{\alpha\beta} = (O\widetilde{A}O^T)^{\alpha\beta} = (O^{\alpha\gamma} L^\gamma v, O^{\beta\delta} L^\delta v). \tag{2.22}$$

Now $O^{\alpha\gamma} L^\gamma$ cannot annihilate v, since then it would be in S, which it manifestly isn't. Thus $O^{\alpha\gamma} L^\gamma \neq 0$, and the diagonal elements of $\widetilde{A}'$ are all positive, and the space spanned by the $O^{\alpha\gamma} L^\gamma$, or equivalently the L^α, is $N - M$ dimensional. The L^α which do not annihilate v are independent, which completes the proof that M^2 has $N - M$ non-zero eigenvalues. The matrix $A^{\alpha\beta}$ will play a fundamental role in the next section.

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3. The Higgs mechanism

In this section we shall discuss Lagrangians with spontaneously broken symmetries which also possess the kind of local gauge invariance which we described in the first section. The combination leads to an exception to Goldstone's theorem which provides the basis for a class of renormalizable models of the weak and electromagnetic interactions.

The simplest example is constructed from a single self-interacting charged field ϕ with Lagrangian

$$\mathcal{L} = (\partial_\mu \phi^* \partial_\mu \phi) - \mu^2 \phi^* \phi - \lambda(\phi^*\phi)^2. \tag{3.1}$$

This Lagrangian is invariant under a U(1) group of transformations:

$$\phi \to \phi' = e^{-i\theta}\phi. \tag{3.2}$$

Next we introduce a gauge field A_μ and construct a Lagrangian invariant under local gauge transformations. Following the prescription derived in the first section, we obtain

$$\mathcal{L} = [(\partial_\mu + ieA_\mu)\phi^*(\partial^\mu - ieA^\mu)\phi] - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3.3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Under local gauge transformations,

$$\begin{aligned} \phi(x) &\to \phi'(x) = \exp\{-i\theta(x)\}\phi(x) \\ \phi^*(x) &\to \phi^{*\prime}(x) = \exp\{i\theta(x)\}\phi(x) \\ A_\mu &\to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\theta(x) \end{aligned} \tag{3.4}$$

and $\mathcal{L}$ is invariant under the transformations (3.4).

If $\mu^2 > 0$, (3.3) is just the Lagrangian for charged scalar electrodynamics. If $\mu^2 < 0$, we must shift the fields to write $\mathcal{L}$ in terms of those with vanishing vacuum expectation values.

The Lagrangian (3.3) possesses the same O(2) symmetry as the (σ, π) model discussed in eq. (2.7), transforming according to (2.9). The correspondence is $\sigma/\sqrt{2} \leftrightarrow \mathrm{Re}\,\phi$, $\pi/\sqrt{2} \leftrightarrow \mathrm{Im}\,\phi$. Just as σ could always be chosen to develop a vacuum expectation value, we can assume, without loss of generality, that

$$\langle\phi\rangle_0 = v/\sqrt{2}$$

where v is real.

Instead of shifting ϕ by subtracting $\langle\phi\rangle_0$ from it, we will parametrize ϕ exponentially, as we did with the real n-vectors in section 2: The new real fields are ξ and η, defined by

$$\phi = \exp(i\xi/v)(v+\eta)/\sqrt{2}$$
$$= \frac{1}{\sqrt{2}}[v + \eta + i\xi + \text{quadratic and higher order terms}]. \tag{3.5}$$

The field ξ is associated with the spontaneously broken U(1) symmetry. In the absence of the gauge field A_μ, we could conclude that the ξ field was massless, because when (3.3) is written in terms of ξ and η there is no term quadratic in ξ. This argument no longer works. Let us write (3.3) in terms of A_μ, ξ, and η:

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\partial^\mu\eta\,\partial_\mu\eta + \tfrac{1}{2}\partial_\mu\xi\,\partial^\mu\xi$$
$$+ \tfrac{1}{2}e^2v^2A_\mu A^\mu - \sqrt{2}\,ev\,A_\mu\,\partial^\mu\xi + \mu^2\eta^2 + \text{cubic and higher order terms} \tag{3.6}$$

where we have the relation $v^2 = -\mu^2/\lambda$. The η field has mass $-2\mu^2$, but the fields A_μ and ξ have gotten mixed up in a way whose interpretation is not immediately apparent. Without the term $-\sqrt{2}\,ev\,A_\mu\,\partial^\mu\xi$ in (3.6), we could have concluded that the vector field has mass $\mu^2 = e^2v^2$ and that the ξ field is massless. A correct procedure would be to compute the combined propagator for the A_μ and ξ fields, find the Feynman rules, and examine the poles of the S-matrix. We'll do some of this in later lectures, but there is an easier way to discover the particle spectrum. Recall tha[illegible]e Lagrangian (3.3) is invariant under local gauge transformations (3.4). Choose the gauge function to be $\xi(x)/v$. Then

$$\phi \to \phi' = \exp\{-i\xi(x)/v\}\phi = (v+\eta)/\sqrt{2} \tag{3.7a}$$

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{ev}\partial_\mu\xi. \tag{3.7b}$$

Since $\mathcal{L}$ is invariant under these transformations,

$$\mathcal{L} = \tfrac{1}{2}[(\partial_\mu + ieA'_\mu)(v+\eta)][(\partial^\mu - ieA'^\mu)(v+\eta)] - \tfrac{1}{2}\mu^2(v+\eta)^2 - \tfrac{1}{4}\lambda(v+\eta)^4 - \tfrac{1}{4}F'_{\mu\nu}F'^{\mu\nu} \tag{3.8}$$

where $F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$.

Eq. (3.8) can be expanded as follows:

$$\mathcal{L} = -\tfrac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \tfrac{1}{2}\partial_\mu\eta\partial^\mu\eta + \tfrac{1}{2}e^2v^2A'_\mu A'^\mu$$
$$+ \tfrac{1}{2}e^2A'^2_\mu\eta(2v+\eta) - \tfrac{1}{2}\eta^2(3\lambda v^2 + \mu^2) - \lambda v\eta^3 - \tfrac{1}{4}\lambda\,\eta^4. \tag{3.9}$$

In this gauge there are no terms coupling different particles, so that the (bare) spectrum can be simply read off the quadratic terms. There is a scalar η-meson, with mass $3\lambda^2v^2 + \mu^2$ (which in zeroth order is $-2\mu^2$), a *massive* vector meson A'_μ, with mass ev, and *no* particle corresponding to ξ. In fact, the ξ-field has disappeared altogether! It has been "gauged away".

Where has it gone? From eq. (3.7b), we can see that it is responsible for the longitudinal component of the vector field in the new gauge. It's clear that there are the same number of actual particle states as there were before we redefined the fields in eq. (3.5). Originally, there were two real scalar fields and a massless photon with two possible polarizations. For positive μ^2, this is th correct collection of particles. When $\mu^2 < 0$, we have just seen that the theory describes one scalar particle (1 helicity state) and one massive vector particle (3 helicity states) so the total nu ber of degrees of freedom — in the sense of particles with fixed polarizations — is the same in ea case.

In the gauge (3.9) $\mathcal{L}$ looks like an ordinary field theory of particles, each decoupled from eac other in second order, and therefore is manifestly unitary order by order in perturbation theory. This gauge is frequently called the unitary gauge, or U-gauge. (The U-gauge is "manifestly unitar in the sense that the fictitious particles, whose Green's functions have singularities that apparent violate unitarity, are manifestly absent. We do not mean to imply that the unitarity of the S-matrix, or even the correct Feynman rules, are obvious in this gauge.) However since it contai a massive vector meson, whose propagator for large k grows as $k_\mu k_\nu/m^2k^2$ instead of $-(g_{\mu\nu} - k_\mu k_\nu/k^2)/k^2$ characteristic of massless vector fields, this model is not obviously renormalizable in the U-gauge.

In the original Lagrangian (3.3), the fields admit gauge transformations (3.4) and it is necessar to choose a condition, such as $\partial^\mu A_\mu = 0$, which fixes the gauges. In Part II, we will show that the theory is renormalizable in such a gauge, which we shall call a renormalizable gauge, or R-gauge. In R-gauges, there are spurious poles in the vector and ξ propagators, which must cancel in all S-matrix elements since they are absent in the U-gauge. The R-gauge formulation is not manifest unitary.

For a non-Abelian example, we let the symmetry group be SU(2), and put the scalar mesons i the triplet representation. The fields transform according to

$$\delta\phi_i = -\mathrm{i}\epsilon^j L^j_{ik}\phi_k = \epsilon^j\epsilon^{ijk}\phi_k .$$

The part of the Lagrangian containing ϕ is

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi_i + g\epsilon^{ijk}A^j_\mu\phi_k)(\partial^\mu\phi_i + g\epsilon^{ij'k'}A^{\mu j'}\phi_{k'}) - V(\phi^2) \tag{3.10}$$

where V is an SU(2) invariant quartic polynomial in ϕ.

When $\phi = 0$ is a minimum of V, (3.10) is an ordinary, isospin conserving gauge invariant, Yang-Mills type theory. Our interest is in the spontaneous symmetry-breaking case: If V has a non-zero minimum, we can always perform an isospin rotation so that it is the third component which acquires a vacuum-expectation value:

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} .$$

The vacuum is no longer invariant under T_1 and T_2, but T_3 remains a good symmetry: there is on conserved quantum number, T_3 or electric charge.

We parametrize ϕ as in the previous lecture:

$$\phi = \exp\{\frac{1}{v}(\xi_1 L^1 + \xi_2 L^2)\}\begin{pmatrix} 0 \\ 0 \\ v+\eta \end{pmatrix} = \langle\phi\rangle_0 + \begin{pmatrix} \xi_2 \\ \xi_1 \\ \eta \end{pmatrix} + \text{higher orders.} \tag{3.11}$$

The fields ξ_1 and ξ_2 are the would-be Goldstone bosons associated with the two broken degrees of freedom. Since the Lagrangian (3.10) is invariant under local SU(2) transformations, we may make the following gauge transformation:

$$\phi' = \exp\{-\frac{i}{v}(\xi_1 L^1 + \xi_2 L^2)\}\phi$$

$$L\cdot A'_\mu = \exp\{-\frac{i}{v}(\xi_1 L^1 + \xi_2 L^2)\} L\cdot A \exp\{\frac{i}{v}(\xi_1 L^1 + \xi_2 L^2)\} - \frac{1}{g}[\partial_\mu \exp\{-\frac{i}{v}(\xi_1 L^1 + \xi_2 L^2)\}]$$
$$\times \exp\{\frac{i}{v}(\xi_1 L^1 + \xi_2 L^2)\}. \qquad (3.12)$$

Again, since $\phi' = \begin{bmatrix} 0 \\ 0 \\ v+\eta \end{bmatrix}$, the fields ξ_1 and ξ_2 completely disappears when the Lagrangian is written in the new gauge:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \eta \partial^\mu \eta + \tfrac{1}{2} g^2 v^2 \epsilon^{ij3}\epsilon^{i'j'3} A^j_\mu A^{\mu j'} - V[(v+\eta)^2]$$
$$+ \text{higher order terms} + \text{terms independent of } \phi. \qquad (3.13)$$

The term in (3.13) quadratic in the vector fields is

$$\tfrac{1}{2}M^2[A^2_\mu A^{2\mu} + A^1_\mu A^{\mu 1}] \qquad (3.14)$$

where $M^2 = g^2 v^2$. The vector mesons corresponding to the broken symmetry generators have acquired a mass $M = gv$. Since the T_3 symmetry survives, there remains one massless vector meson, A^3_μ.

The general features of a spontaneously-broken gauge model should now be clear. We start out with a Lagrangian $\mathcal{L}$ invariant under local gauge transformations of some group G. There are n scalar fields which transform under an n-dimensional representation. There are N gauge mesons, A^α_μ. The Lagrangian is given by

$$\mathcal{L} = -\tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu\alpha} + \tfrac{1}{2}[(\partial_\mu - ig^\alpha L^\alpha A^\alpha_\mu)\phi(\partial^\mu - ig^\beta L^\beta A^{\mu\beta})\phi]$$
$$- V(\phi) + \text{terms with other fields}. \qquad (3.15)$$

Here, $F^\alpha_{\mu\nu}$ is given by (1.44); the g^α are independent of α within any simple subgroup of G; and $V(\phi)$ is a fourth-order G-invariant polynomial in ϕ which is minimized by setting $\phi = v$.

Now we suppose the symmetry-breaking leaves the vacuum invariant under an M-dimensional subgroup S of G. There are M generators L^α satisfying $L^\alpha v = 0$. There remain $(N-M)L^\alpha$ for which $L^\alpha v \neq 0$. We showed in section 2 that the $L^\alpha v$ span an $N-M$ dimensional space, and that in the absence of the gauge mesons there would be $N-M$ massless scalar particles.

We can parametrize ϕ by

$$\phi = \exp(\Sigma i \xi_\alpha L^\alpha / v)(v+\eta). \qquad (3.16)$$

The sum is over those $(N-M)L^\alpha$ which do not annihilate v. The vector η represents as many independent fields as there are dimensions in the part of the n-dimensional representation space orthogonal to all $L^\alpha v$. $(= n - N + M)$

Next we make the gauge transformation defined by

$$\phi' = \exp(-i\xi^\alpha L^\alpha/v)\,\phi,$$

and

$$A'^{\gamma}_{\mu} L^\gamma = \exp\left(-\frac{i}{v} L^\alpha \xi^\alpha\right)\left\{A^\gamma_\mu L^\gamma - \exp\left(\frac{i}{v} L^\beta \xi^\beta\right)\frac{1}{g}\,\partial_\mu \exp\left(-\frac{i}{v} L^\beta \xi^\beta\right)\right\}\exp\left(\frac{i}{v} L^\alpha \xi^\alpha\right). \qquad (3.17)$$

In the new gauge, $\mathcal{L}$ depends only on the η_i and the gauge fields A'^{α}_{μ}, $N - M$ of which are now massive.

The term in $\mathcal{L}$ responsible for the vector meson masses is

$$\tfrac{1}{2}(g^\alpha L^\alpha v, g^\beta L^\beta v) A^\alpha_\mu A^{\mu\beta}. \qquad (3.18)$$

Since we may always restrict ourselves to real representations of G, so that L, being Hermitean, is antisymmetric, the vector meson mass matrix,

$$(M^2)^{\alpha\beta} = g^\alpha g^\beta (v, L^\alpha L^\beta v) \qquad \text{(no sum over } \alpha, \beta) \qquad (3.19)$$

is symmetric and positive definite, with α, β restricted to values for which $L^\alpha v \neq 0$. Except for the coupling constants g^α, $(M^2)^{\alpha\beta}$ is just the matrix $A^{\alpha\beta}$ we defined in section 2.

Thus, the $N - M$ Goldstone bosons are not physical massless particles, but are absorbed into the longitudinal components of the $N - M$ massive vector bosons: as can be seen from eq. (3.17),

$$A'^{\alpha}_{\mu} = A^\alpha_\mu - \frac{1}{gv}\,\partial_\mu \xi^\alpha + O(\xi^2).$$

The number of the independent degrees of freedom for a given momentum remains the same. The masses of the physical vector mesons are the eigenvalues of (M^2). The remaining M vector mesons remain massless, corresponding to the surviving M-dimensional symmetry subgroup S.

Weinberg has discovered an elegant proof that the unitary gauge always exists. In that gauge, $\phi'(x)$ has no components in the subspace spanned by the Goldstone bosons, which we know is the space spanned by $L^\alpha v$. Therefore

$$(L^\alpha v, \phi'(x)) = 0 \qquad (3.20)$$

defines the unitary gauge. (This definition is just as good as the more familiar definition of a gauge by imposing a condition on the vector fields.) Therefore, if $\phi(x)$ is the scalar field in any gauge, there is a unitary gauge provided there exists a local gauge transformation $O(x) = \exp\{-i\xi^\alpha(x) L^\alpha\}$ such that

$$(L^\alpha v, O(x)\phi(x)) = 0 \qquad (3.21)$$

for all α and all x. For any x, O may be any element of the representation of G defined at x. We have chosen only real representations, so O is orthogonal. Consider the scalar product

$$(v, O\phi). \qquad (3.22)$$

For fixed ϕ, the scalar product (3.22) is a real number which depends on O. As long as the Lie group (of which O in an n-dimensional real representation) is compact, (3.22) maps the group into a compact portion of the real line, and therefore has a maximum and a minimum. Let O_ϕ be a matrix O which is an extremum of (3.22). For any O, if we vary O slightly

$$\delta O = O - \exp\{-i\epsilon^\alpha(x)L^\alpha\}O = -i\epsilon^\alpha(x)L^\alpha O.$$

Since O_ϕ makes (3.22) take on an extreme value,

$$0 = \delta(v, O\phi)_{O=O_\phi} = (v, \delta O\phi)_{O=O_\phi}$$

$$= -i\,\epsilon^\alpha(v, L^\alpha O_\phi\phi) = -i\epsilon^\alpha(L^\alpha v, O_\phi\phi). \quad (3.23)$$

Since ϵ^α is arbitrary,

$$(L^\alpha v, O_\phi\phi) = 0$$

for all α, so $\phi' = O_\phi\phi$ satisfies the unitary gauge condition. Therefore the unitary gauge always exists: it can be obtained by making a gauge transformation O from an arbitrary ϕ which extremizes $(v, O\phi)$ at each point x. If G is simply connected, the real numbers $(v, O\phi)$ form a compact segment of the real line, and therefore have two extrema, a maximum and a minimum. Generally, if (ϕ', v) is one extremum, $(-\phi', v)$ is the other, and the physics of the two gauges are the same.

In nature, M is apparently 1; the only conservation law associated with a massless vector meson is charge conservation. Nevertheless, it is instructive to consider the more general case.

In the next sections we will consider the application of these ideas to models of weak and electromagnetic interaction.

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4. Review of weak interaction phenomenology

In later sections we will discuss a class of models for weak and electromagnetic interactions which utilize the idea of spontaneously broken gauge symmetry. One constraint on these models is that they reproduce the known phenomenology of weak interaction. We will review some important features in this section.

Our notation will be that of the textbook by Bjorken and Drell. The Dirac γ-matrices are

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

The γ^μ anticommute according to $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, and $\{\gamma^\mu, \gamma_5\} = 0$. The spin matrix is $\boldsymbol{\sigma} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$. It is easy to see that $\sigma^i = \frac{1}{2} i\, \epsilon^{ijk}\gamma^j\gamma^k$. The Lagrangian for a free Dirac spinor with mass m is

$$\mathcal{L} = \bar{\psi}(x)(i\gamma\cdot\partial - m)\psi(x) \tag{4.1}$$

which leads to the equation of motion [see (1.3)]

$$[i\gamma\cdot\partial - m]\psi = 0. \tag{4.2}$$

Most of our information about weak interactions comes from spontaneous decay processes, in which the energies and momenta transferred are small compared to the high energies available to study strong and electromagnetic interactions in particle accelerators. Therefore there is no reason to expect that a phenomenological description of known decays will be correct at higher energies, but nevertheless a more complete theory must agree with what we know at low energies.

The only known leptons are the muon, the electron and the neutrinos. All known experiments are consistent with lepton number conservation if we assign "lepton number" +1 to μ^-, e^-, and ν, and -1 to μ^+, e^+, and $\bar{\nu}$. Furthermore, the decays $\mu^- \to e^- + \gamma$ or $\mu^- \to e^- + e^- + e^+$ are not seen, even though they conserve lepton number. Apparently there is also a conserved "muon number" which forbids these processes. The neutrino associated with the muon is different from the neutrino associated with the electron. Experimentally, neutrinos produced in the decay $\pi^- \to \mu^- + \bar{\nu}$ are not seen to produce electrons by inverse beta decay $\nu + n \to p + e^-$. Therefore, we believe there are two doublets of leptons, (μ^-, ν_μ) and (e^-, ν_e), which are distinguished by a quantum number. It is possible that the muon quantum number is multiplicative (like parity), but there is at present no particular evidence for this unattractive idea.

The mass of the muon is 105.6594 ± 0.0004 MeV. It has a lifetime of $(2.994 \pm 0.0006) \times 10^{-6}$ seconds, decaying almost always into $e^- + \nu_\mu + \bar{\nu}_e$. Other modes, if they exist, are very rare. The electron mass is (0.511004 ± 0.000002) MeV and it has a lifetime of at least 6×10^{28} seconds. As far as is known, the muon and the electron are identical in all properties except for their masses, the large difference between which is a major puzzle. Perhaps the empirical relation

$$m_e = \tfrac{3}{2}\alpha m_\mu \tag{4.3}$$

which is accurate to better than one-percent, provides a clue.

The neutrinos appear to be massless although experimentally it is not possible to put such fantastic upper limits on their masses as are known for the photon. The electron-neutrino certainly weighs less than 6×10^{-5} MeV, but the muon neutrino may have a mass of an MeV or more. Nevertheless it is attractive – and consistent with experiment – to assume both are exactly massless, as we shall see below.

It is important in gauge theories that the photon be exactly massless. From the fact that the earth's magnetic field has been detected tens of thousands of miles away, one concludes that the Compton wavelength of the photon must be of this order at least, corresponding to a mass less than 10^{-21} MeV.

If the neutrino has a finite mass, it must occur in both helicity states, since a positive helicity state can be transformed into a negative one by Lorentz transformation. If the neutrino is exactly massless, either helicity state provides a complete representation of the Poincaré group, and only parity conservation would require both to occur. Formally, it is easy to see that under the transformation $\psi \to -\gamma_5\psi$, the kinetic energy term in (4.1) is invariant while the mass term is not. If the mass is zero, the free Lagrangian is invariant under this transformation. The interaction part of the Lagrangian will be invariant provided the neutrino field occurs only in the combination $(1-\gamma_5)\psi$.

Let us introduce here a notation which will be useful later. On any spinor field, let $P_L = \frac{1}{2}(1-\gamma_5)$, and $P_R = \frac{1}{2}(1+\gamma_5)$. P_L and P_R are projection operators, in the sense that $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L P_R = P_R P_L = 0$, and $P_L + P_R = 1$. Any spinor field ψ can be broken up using P_L and P_R:

$$\psi = \psi_L + \psi_R = P_L\psi + P_R\psi = \tfrac{1}{2}(1-\gamma_5)\psi + \tfrac{1}{2}(1+\gamma_5)\psi. \tag{4.4}$$

The free Lagrangian becomes

$$\mathcal{L} = \mathrm{i}\bar{\psi}_L\gamma\cdot\partial\psi_L + \mathrm{i}\bar{\psi}_R\gamma\cdot\partial\psi_R - m[\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L]. \tag{4.5}$$

If $m \neq 0$, the breakup (4.4) has no Lorentz invariant meaning. If $m = 0$, ψ_R is a solution to (4.2) with spin analog the direction of the momentum, ψ_L a solution with spin in the opposite direction. They have positive and negative helicities, respectively. These facts are easily obtained from the massless Dirac equation,

$$(\gamma^0 - \boldsymbol{\gamma}\cdot\hat{n})\psi = 0,$$

where $\hat{n}$ is a unit vector in the direction of the neutrino's momentum, and from the definition of γ_5 and the spin operator σ.

The decay spectra in those weak decays for which there are the most data, namely $n \to p + e^- + \bar{\nu}_e$, $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$, and $\pi^- \to \mu^- + \bar{\nu}_\mu$, plus a large collection of nuclear decays, show no sign of right-handed (positive helicity) neutrinos or left-handed (negative helicity) anti-neutrinos. We will assume that there exist only left-handed neutrinos, which is possible only if the neutrinos are exactly massless.

Finally, it is easy to show that if only the left-handed neutrino field, $\psi_L(\nu)$, appears in the Lagrangian, the neutrino remains massless to all orders in perturbation theory. Under the operation $\psi(x) \to -\gamma_5\psi(x)$, $\psi^\dagger(x) \to -\psi^\dagger(x)\gamma_5$, and therefore $\bar{\psi}(x) \to \bar{\psi}(x)\gamma_5$. These rules hold for the interacting fields. Therefore the neutrino propagator (including interactions) is

$$S'(p) = \int \mathrm{d}^4x\, \exp(\mathrm{i}p\cdot x)\langle|T(\psi(x)\bar{\psi}(0))|\rangle$$
$$= -\int \mathrm{d}^4x\, \exp(\mathrm{i}p\cdot x)\gamma_5\langle|T(\psi(x)\bar{\psi}(0))|\rangle\gamma_5. \tag{4.6}$$

Therefore

$$S(p)\gamma_5 = -\gamma_5 S(p)$$

and

$$\gamma_5 S^{-1}(p) = -S^{-1}(p)\gamma_5. \tag{4.7}$$

In general, $S^{-1}(p) = \gamma\cdot p + \delta m + O(\gamma\cdot p)^2$. From (4.7), the term in δm is forbidden, and $S^{-1}(0) = 0$; the full propagator $S(p)$ has a role at $\gamma\cdot p = 0$.

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5. Weak interaction phenomenology (continued)

The idea that all known weak decays can be described by a local four-point interaction is due to Fermi, and such interactions are called Fermi couplings. Following the discovery that weak decays violate parity conservation, Feynman and Gell-Mann proposed that the correct form for the Fermi interaction is

$$G[J_\mu(x)J^{\mu\dagger}(x)]/\sqrt{2}. \tag{5.1}$$

Here $J_\mu(x)$ is a charged current, which has a lepton part and a hadron part:

$$J_\mu(x) = l_\mu(x) + h_\mu(x). \tag{5.2}$$

The lepton part of the current is

$$l_\mu(x) = \overline{\psi}_e(x)\gamma_\mu(1-\gamma_5)\gamma_{\nu_e}(x) + \overline{\psi}_\mu\gamma_\mu(1-\gamma_5)\psi_{\nu_\mu}(x). \tag{5.3}$$

From (5.2) and (5.3) the μ-decay spectrum can be calculated, and seems to be in agreement with experiment. The rate for μ-decay comes out to be

$$\Gamma(\mu^- \to e^- + \bar{\nu}_e + \nu_\mu) = G^2 m_\mu^5/192\pi^3. \tag{5.4}$$

From (5.4) and the known rate of μ-decay, the Fermi coupling constant G which appears in (5.1) can be evaluated. An easy formula to remember is

$$G = 1.01 \times 10^{-5}\, m_p^{-2} \tag{5.5}$$

where m_p is the proton mass.
The lepton current can be written

$$l^\mu = 2\overline{\psi}_L(e)\gamma^\mu\psi_L(\nu_e) + 2\overline{\psi}_L(\mu)\gamma^\mu\psi_L(\nu_\mu). \tag{5.6}$$

It is entirely a left-handed current. We define a leptonic left-handed isospin by grouping $\binom{\nu_e}{e}_L$ into a doublet $\chi_L(e^-)$, and $\binom{\nu_\mu}{\mu}_L$ into a doublet $\chi_L(\mu)$. Then

$$l^\mu = 2[\chi_L^\dagger(e)\gamma^\mu\tau^-\chi_L(e) + \chi_L^\dagger(\mu)\gamma^\mu\tau^-\chi_L(\mu)] \tag{5.7}$$

where $\tau^- = \frac{1}{2}[\tau^1 - i\tau^2] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

We define a "left-handed isospin" current for leptons by

$$j^i_{L\mu}(x) = \tfrac{1}{2}\left[\chi^\dagger_L(e)\gamma^\mu\tau^i\chi_L(e) + \chi^\dagger_L(\mu)\gamma^\mu\tau^i\chi_L(\mu)\right] \tag{5.8}$$

and the corresponding charges by

$$T^i_L = \int j^i_{Lo}(x)\,d^3x. \tag{5.9}$$

The T^i_L generate an $SU(2)_L$ algebra:

$$[T^i_L, T^j_L] = i\,\epsilon^{ijk}T^k_L. \tag{5.10}$$

It is convenient to introduce

$$T^\pm_L = [T^1_L \pm iT^2_L]/\sqrt{2}. \tag{5.11}$$

Then

$$[T^3_L, T^\pm_L] = \pm T^\pm_L, \qquad [T^+_L, T^-_L] = T^3_L \tag{5.12}$$

and there is an analogous definition for $j^{\mu\pm}_L$.

Evidently,

$$l^\mu = 2\sqrt{2}\,(j^-_L)^\mu.$$

The leptonic part of the weak interactions in (5.1) is not invariant under $SU(2)_L$, since there is no term $j^3_{L\mu}j^{\mu 3}_L$ there. The existence and magnitude of a neutral leptonic current is an open experimental question.

The decay of the neutron $n \to p + e^- + \nu_e$ is well described by assuming that the hadronic current h_μ in (5.2) has a term

$$\bar{\psi}(n)\gamma_\mu(g_V - g_A\gamma_5)\psi(p). \tag{5.13}$$

The vector coupling constant is strikingly close to 1, while g_A is about 1.24. An explanation of the fact that $g_V \approx 1$ was first suggested by Feynman and Gell-Mann, and by Gerschtein and Zel'dovich. Their hypothesis is that the strangeness conserving part of the h_μ has the form

$$(V^1_\mu - iV^2_\mu) - (A^1_\mu - iA^2_\mu) \tag{5.14}$$

where V^i_μ is a vector current and A^i_μ is an axial vector current; and further that V^1_μ and V^2_μ are the first and second components of the isospin current. That is, that

$$T^i = \int V^i_o(x)\,d^3x$$

are the isospin generators, conserved by the strong interactions. This rule is called the conserved vector current (CVC) hypothesis. Since the T^i form a Lie group, of which the proton and neutron form the basis of an irreducible representation, their matrix elements are fixed to be the Clebsch-Gordan coefficients, and g_V is predicted to be 1.

It is important to know whether g_V is really exactly one. The measured decay rates, both of the neutron and the muon, include electromagnetic corrections to the term obtained simply by replacing the fields in (5.13) with free wave functions. The radiative corrections to μ decay were calculated long ago, and turn out to be finite. The decay rate is corrected by about 4%. The radiative corrections to β-decay are logarithmically divergent. Putting in a cutoff of a few GeV, one can conclude that, even taking these corrections into account, there remains a discrepancy with the simple CVC prediction $g_V = 1$.

The calculation is unsatisfactory because the radiative corrections to neutron decay involve strong interaction corrections, and because of the difficulty in distinguishing the vector from the axial-vector part. The latter difficulty is overcome by considering β-decays of spin-zero particles, to which only the vector current contributes. The rate predicted for $\pi^+ \to \pi^0 + e^+ + \nu$ is in good agreement with CVC, but since the branching ratio of this mode to the principal mode, $\pi^+ \to \mu^+ + \nu$, is 10^{-8}, the uncertainty is about 7%, which is too large for us to start worrying about radiative corrections. Decays of spin-zero heavy nuclei provide the best tests, because their rates can be accurately measured; but these calculations are plagued with nuclear physics complications. These have been estimated carefully for nine low-mass, spin zero nuclei. The result depends only on two parameters, the cutoff Λ and a model dependent number $\overline{Q}$ which depends on the underlying field theory. (In the quark model, $\overline{Q} = \frac{1}{6}$.) For a wide range of Λ and $\overline{Q}$, g_V is the same within experimental uncertainty for all nine nuclei. For $\overline{Q} = \frac{1}{6}$ and $\Lambda = 30$ GeV, comparison with μ-decay gives

$$g_V = 0.976 = \cos(0.22). \tag{5.15}$$

No reasonable values of the parameters give $g_V = 1$.

Thus, all our knowledge of non-strange β-decays is consistent with h_μ containing a term

$$g_V[(V^1_\mu - iV^2_\mu) - (A^1_\mu - iA^2_\mu)] \tag{5.16}$$

with $1 - g_V \sim 0.02$. Since $g_V \neq 1$, these vector currents alone do not generate a SU(2) group as the lepton currents do.

The decays of strange hadrons are consistent with the idea that h_μ contains a strangeness-changing vector and axial current term. From the observed absence of decays like $\Xi^0 \to p + e + \bar{\nu}$, we conclude that this term changes hypercharge by no more than one unit. From the absence of decays like $\Sigma^+ \to n + e^+ + \nu$ or $\Xi^0 \to \Sigma^- + e^+ + \nu$. one concludes that the strangeness-changing current changes the hyperchange (strangeness) and the electric charge by the same sign. This is known as the $\Delta S = \Delta Q$ rule. As a consequence, the change in T^3 is always $\pm\frac{1}{2}$, suggesting that this current has $T = \frac{1}{2}$.

Let us write h_μ as a sum of a $\Delta S = 0$ and a $\Delta S = 1$ part

$$h_\mu = g_V h^{(0)}_\mu + g_S h^{(1)}_\mu. \tag{5.17}$$

$h^{(0)}_\mu$ has the form (5.14), and is the third component of an isotopic triplet. It is natural to extend this idea to SU(3), and assume that $h^{(1)}_\mu$ is the charged $\Delta S = \Delta Q$, $T = \frac{1}{2}$, member of an octet of currents (i.e., the one that transforms like K^-). By comparing a large number of decays, there is rather striking evidence that this is indeed the case. Therefore we can assume – since it is not in contradiction with experiment – that

$$h^{(1)}_\mu = (V^4_\mu - iV^5_\mu) - (A^4_\mu - iA^5_\mu) \tag{5.18}$$

where F^i

$$F^i = \int V_0^i(x)\mathrm{d}^3x \tag{5.19}$$

are the generators of SU(3) and the $A_\mu^i(x)$ are an octet of axial currents.

Although SU(3) is not an exact symmetry, the matrix elements can still be estimated. The conclusion is that (5.18) does not disagree with experiment, but that g_S is nowhere near 1. The best fit is

$$g_S/g_V \sim 0.25. \tag{5.20}$$

In 1963 Cabibbo observed that within experimental error,

$$g_S^2 + g_V^2 = 1. \tag{5.21}$$

Therefore

$$\begin{aligned} h_\mu &= \cos\theta\ h_\mu^{(0)} + \sin\theta\ h_\mu^{(1)} \\ &= \exp(2\mathrm{i}\theta F^7)h_\mu^{(0)}\exp(-2\mathrm{i}\theta F^7), \end{aligned} \tag{5.22}$$

where F^7 is the 7th generator of SU(3).

In this way universality can be recovered, and the discrepancy between g_V and 1 understood. That is, if (5.22) is correct h_μ is a correctly normalized component of a multiplet of currents which generate an SU(2) group. The angle θ is called the Cabibbo angle, and is somewhere around 0... ~ 0.25. Its origin is unknown, and a plausible explanation would be very interesting.

...re there any neutral currents? We discussed leptonic neutral currents in the last section. The existence of the charged strangeness-conserving currents in (5.22) naturally suggests also neutral strangeness-conserving currents. Experimentally the existence of such currents is at this time an open question, which we shall return to in section 8.

By commuting $h_\mu^{(1)}$ with T_L^+ (which is the charge associated with $h_\mu^{(0)\dagger}$), one obtains a neutral, strangeness-changing current, transforming under SU(3) like K°. Experimentally, these currents do not seem to mediate leptonic weak interactions. Decays like $\Sigma^+ \to p + e^+ + e^-$ are never seen. Furthermore, the upper limits for branching ratios of $K^0 \to \mu^+ + \mu^-$ or $K^+ \to \pi^+ + \nu + \bar{\nu}$ are of the order 10^{-8}. Any model for weak decays must account for the absence or suppression of those currents.

Note that in writing (5.18) we tacitly assumed that $h_\mu^{(1)}$ is a left-handed current like $h_\mu^{(0)}$. If, in fact, it were right-handed—$V + A$ instead of $V - A$—it would commute with T_L^+, and no strangeness-changing neutral current would exist. This idea has been occasionally suggested, but seems contradicted by experiments.

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6. Unitarity bounds, W-mesons, PCAC

We conclude our tour of the weak interactions with these topics: unitarity bounds, W-mesons, and PCAC.

Although equation (5.1) adequately describes decays, it cannot be a complete theory. When the interaction (5.1) is used to describe scattering, the Born approximation must fail at some energy, since the amplitude cannot be strictly real. Unlike electrodynamics, the Fermi coupling (5.1) does not lead to a renormalizable theory, so it is not possible to make these higher-order corrections.

For any leptonic scattering, the cross sections are not proportional to the lepton mass. The only other dimensional parameter available is G. Since the cross-sections are proportional to G^2, they are dimensionally constrained to grow like

$$\sigma \sim G^2 s \tag{6.1}$$

neglecting the lepton masses. Because of the local form of (5.1), the cross-sections are restricted to a single partial wave, so there is a unitarity bound

$$\sigma \sim 1/s \tag{6.2}$$

which is violated by (6.1) when Gs is of the order 1.

For example, consider $\bar{\nu}_e + e^- \to \bar{\nu}_e + e^-$. Ignoring the electron mass, the spin-averaged cross-section is

$$\bar{\sigma} = G^2 s/3\pi . \tag{6.3}$$

Exercise.

Since the electron which interacts with the neutrino is left-handed in this limit, and $\bar{\nu}_e$ is right-handed, the total angular momentum along the direction of motion in the center of mass is 1, so the spin must be 1, not 0. Therefore scattering takes place in the spin-one state, if the electron mass, m_e, can be neglected. From the Jacob-Wick expansion for the scattering amplitude in the helicity representation, we have

$$T_{\mu_4\mu_3,\mu_2\mu_1}(s,\theta) = \frac{1}{\pi}\sum_j (2j+1) t^j_{\mu_4\mu_3,\mu_2\mu_1}(s) d^j_{(\mu_1-\mu_2),(\mu_3-\mu_4)}(\theta) \tag{6.4}$$

where μ_i are the helicities of the four particles, d^j are the j-dimensional representations of rotations about the y axis, and t^j is the partial wave, normalized so that $\mathrm{Im}\, j = (q/W)|t^j|^2$. ($q$ and W are the c.m. momentum and $\sqrt{s}$, respectively; $W \approx 2q$.) Since (6.1) is a point interaction, there is no orbital angular momentum, and the spin is one, so only $j = 1$ contributes to the sum in (6.4). There is only one helicity state for each particle, so

$$T_{1/2-1/2;1/2-1/2} = \frac{3}{\pi} t^1_{1/2-1/2;1/2-1/2} d^1(\theta) = \frac{3}{2\pi} t^1_{1/2-1/2;1/2-1/2}(1+\cos\theta).$$

From unitarity, $|t^j|^2$ is bounded by 2. So in the forward direction

$$|\mathrm{Im}\, T(s,0)| \leqslant 6/\pi$$

and from the optical theorem

$$\sigma = (4\pi^2/gW)\ \mathrm{Im}\, T(s,0) \leqslant 48\pi/s.$$

The spin averaged cross-section includes both electron helicity states, so

$$\bar{\sigma} = \tfrac{1}{2}\sigma \leqslant 24\pi/s. \tag{6.5}$$

Comparing (6.3) with (6.5), we learn that (6.3) violates the unitarity bound when

$$G^2s^2 = 72\pi^2 \qquad \text{or} \qquad s = (\pi/G)\sqrt{72} = 2.7 \times 10^6 m_p = 2.5 \times 10^6\ \mathrm{GeV}^2. \tag{6.6}$$

The smallest such bound is obtained for the inelastic process $\nu_\mu + e^- \to \nu_e + \mu^-$. The $V-A$ spin wavefunction is antisymmetric, so that this process has only $j = 0$. Since t^0 for this amplitude is an off-diagonal matrix element of $(W/q)\{\exp(2i\delta) - 1\}/2i$, $|t^0| \leqslant 1$. The total $\nu_\mu + e^- \to \nu_e + \mu^-$ cross-section is

$$\sigma = \frac{\pi^2}{s}\int |T|^2\, d\Omega \leqslant \frac{4\pi}{s} \tag{6.7}$$

and this spin-averaged cross-section is $\bar{\sigma} = 2\pi/s$. By direct calculation, $\bar{\sigma}$ is G^2s/π so that the Born approximation equals the unitarity bound when $s = \pi\sqrt{2}/G = 4.2 \times 10^5\ \mathrm{GeV}^2$.

The upshot of all this is that the form (5.1) for leptonic weak decays violates the unitarity bound at about 700 GeV total center-of-mass energy.

A popular modification of (5.1) is obtained by recognizing the analogy between (5.1) and second order electromagnetic interactions. The amplitude for electron-electron scattering can be calculated from the Feynman graph of fig. 6.1.

The contribution of fig. 6.1 to the amplitude T is

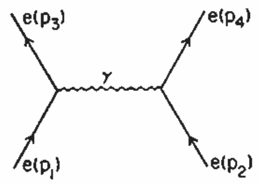

Fig. 6.1. Photon exchange graph in electron-electron scattering.

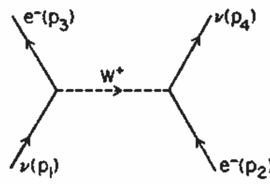

Fig. 6.2. W exchange graph in e⁻ν elastic scattering.

$$\frac{-ie^2}{2\pi^2}\,\bar{u}(p_3)\gamma^\mu u(p_1)g_{\mu\nu}\bar{u}(p_4)\gamma^\nu u(p_2)\,\frac{1}{k^2} \tag{6.8}$$

where the spinors are normalized so that $u^\dagger u = E$, and k is the momentum transfer. The numerator has a current-current form, just like (5.1). We introduce a charged vector meson W_μ, interacting with the weak current (5.2) according to

$$\mathcal{L}_I = g_W\,[J_\mu W^\mu + \text{h.c.}]\,; \tag{6.9}$$

W^μ is negatively charged. Then $\nu + e^- \to \nu + e^-$ is described by the graph in fig. 6.2

$$\frac{ig_W^2}{2\pi^2}\,\bar{u}(p_3)\gamma^\mu(1-\gamma_5)u(p_1)\bar{u}(p_4)\gamma^\nu(1-\gamma_5)u(p_2)\left[g_{\mu\nu}-\frac{k_\mu k_\nu}{M_W^2}\right]\frac{1}{k^2-M_W^2}\,. \tag{6.10}$$

In the Fermi theory (5.1), the amplitude for $\nu + e^- \to \nu + e^-$ is

$$\frac{iG}{2\sqrt{2}\pi^2}\,\bar{u}(p_3)\gamma^\mu(1-\gamma_5)u(p_1)\bar{u}(p_4)\gamma^\nu(1-\gamma_5)u(p_2)g_{\mu\nu}. \tag{6.11}$$

For low k, (6.11) and (6.10) are indistinguishable provided

$$g_W^2/M_W^2 = G/\sqrt{2}. \tag{6.12}$$

From the Dirac equation, $\gamma\cdot k$ can be replaced by m_e in (6.10), so that the second term in the propagator does not grow faster than the first. The amplitude is damped by a factor $1/k^2$ compared to Fermi point interaction, and doesn't come into glaring conflict with unitarity. Nevertheless, the theory is not renormalizable, as can easily be seen by calculating the amplitude for $\nu + \bar{\nu} \to W^+ + W^-$. In fact, if a renormalizable theory is constructed using W mesons coupled to charged currents, the theory must contain additional particles to cancel the divergences in graphs with $W^\pm$ mesons alone.

All the models we are about to describe contain charged W mesons to moderate the weak interactions. From (6.12), the sign of G is determined to be positive. In principle, this sign can be measured by looking for the parity-violating interference between a weak and electromagnetic or strong term in e.g., $p + p \to n + n$ or $e^+ + e^- \to \mu^+ + \mu^-$.

The radiative corrections to both μ and β decays in W-meson theories are ambiguous and depend on the method of computation. If one adopts the ξ-limiting procedure of Lee and Yang, the ratio of the rates for μ and β decays is finite. For a W mass > 2 GeV, one obtains $1 - g_V > 0.024$.

Finally we mention the success of the idea that the strong interactions are approximately invariant under $SU(2)_L \times SU(2)_R$. The generators are the T^i_L discussed above, whose charged components are the weak currents, and the T^i_R, constructed like T^i_L, replacing $V - A$ by $V + A$. Thus for nucleons,

$$T^i_L = \int d^3x\, \bar{\psi}(x) \frac{\tau^i}{2} \frac{(1-\gamma_5)}{2} \psi(x)$$

$$T^i_R = \int d^3x\, \bar{\psi}(x) \frac{\tau^i}{2} \frac{(1+\gamma_5)}{2} \psi(x). \tag{6.13}$$

From T^i_L and T^i_R we may construct

$$T^i = T^i_L + T^i_R \tag{6.14}$$

which are just the isotopic spin generators, and the axial charges

$$T^i_5 = T^i_R - T^i_L. \tag{6.15}$$

The group algebra is

$$[T^i_{L,R}, T^j_{L,R}] = i\epsilon^{ijk} T^k_{L,R}$$

$$[T^i_L, T^j_R] = 0 \tag{6.16}$$

or

$$[T^i, T^j] = i\epsilon^{ijk} T^k$$

$$[T^i, T^j_5] = i\epsilon^{ijk} T^k_5$$

$$[T^i_5, T^j_5] = i\epsilon^{ijk} T^k. \tag{6.17}$$

The charge T^i_5 is the space integral of the time component of the axial current.

The idea of PCAC (partially conserved axial current) is that $SU(2)_L \otimes SU(2)_R$ is an approximate symmetry of the strong interactions, realized in the Goldstone mode and that the pions a[he] Goldstone bosons in the symmetry limit, their mass being a measure of the symmetry breaking. Thus the Lagrangian has the form

$$\mathcal{L} = \mathcal{L}_{SYM} + \epsilon \mathcal{L}' \tag{6.18}$$

where ϵ is "small" of the order M^2_π / m^2_p and $\mathcal{L}_{SYM}$ is invariant under the group.

The matrix element of the axial vector current $A^i_\mu (= j^i_{R\mu} - j^i_{L\mu})$ between the vacuum and a one-pion state with momentum p is

$$\langle |A^i(x)| \pi^j(p)\rangle = \frac{iF_\pi}{(2\pi)^{3/2}} \frac{p_\mu \delta_{ij} \exp(-ip\cdot x)}{\sqrt{2p^0}}. \tag{6.19}$$

Except for the normalization constant F_π, the form of (6.19) is dicated by Lorentz invariance. The value of F_π can be determined from the decay $\pi^- \to \mu^- + \bar{\nu}_\mu$. From equations (5.1), (5.14) and (6.19) we calculate the total rate for π^- decay to be

$$\Gamma(\pi \to \mu + \nu) = G^2 M_\mu^2 F_\pi^2 (m_\mu^2 - m_\pi^2)^2 / 4\pi M_\pi^3. \tag{6.20}$$

From the measured pion life time 2.60×10^{-8} sec, and from the value of G obtained from $\mu \to e + \nu + \bar{\nu}$, the value of F_π is determined to be

$$F_\pi = 93 \text{ MeV}. \tag{6.21}$$

As a consequence of (6.19)

$$\langle |\partial^\mu A_\mu^i(x)| \pi^j(p)\rangle = \frac{\mathrm{i} F_\pi m_\pi^2}{(2\pi)^{3/2}\sqrt{2E_\pi}} \delta_{ij} \exp(-\mathrm{i}p\cdot x) = F_\pi m_\pi^2 \langle |\phi^i(x)| \pi^j(p)\rangle \tag{6.22}$$

where $\phi^i(x)$ is the renormalized pion field. In the symmetry limit ($\epsilon \to 0$), $m^2 = 0$ and $\partial^\mu A^i = 0$. In this limit, any matrix element of $A_\mu^i(x)$,

$$M_{\mathrm{ab}} = \langle \mathrm{b}|A_\mu^i(0)|\mathrm{a}\rangle \quad . \tag{6.23}$$

has a pole at $q^2 = 0$ ($q = p_\mathrm{b} - p_\mathrm{a}$) of the form

$$M_{\mathrm{ab}} = \frac{\mathrm{i} F_\pi q_\mu}{q^2} \langle \mathrm{b}|j_\pi^i(0)|\mathrm{a}\rangle \tag{6.24}$$

where $j_\pi^i(x) = \Box\phi^i(x)$ is the source of the pion field.

Low energy theorems in the unphysical world with an exactly conserved axial current can be obtained from (6.25) and its generalizations. The content of the PCAC assumption is that these are approximately true in the real world. Here are some examples.

Let a and b be nucleons. Then the most general form of (6.23) is

$$\frac{1}{(2\pi)^3} \bar{u}(p_0)[g_\mathrm{A}(q^2)\gamma_\mu\gamma_5 - q_\lambda\gamma_5 h(q^2)] \frac{\tau^i}{2} u(p_\mathrm{a}). \tag{6.25}$$

From the conservation of the axial current, we know that (6.25) multiplied by q^μ is zero, therefore

$$2Mg_\mathrm{A}(q^2) = q^2 h(q^2). \tag{6.26}$$

From (6.24) and the fact that

$$\langle \bar{N}_\mathrm{b} |j_\pi^i(0)| N_\mathrm{a}\rangle = \frac{g}{(2\pi)^3} \bar{u}(p_\mathrm{b})\tau^i\gamma_5 u(p_\mathrm{a}) \tag{6.27}$$

(where g is the pion-nucleon coupling) we obtain the Goldberger-Treiman relation

$$F_\pi = Mg_\mathrm{A}(0)/g \tag{6.28}$$

where $g_\mathrm{A}(0) = g_\mathrm{A}$.

Experimentally, $g^2/4\pi^2 \sim 14.6$, so $Mg_\mathrm{A}/g \approx 83$ MeV; comparing with the value 93 MeV obtained from π-decay, one gets an idea of the accuracy of PCAC.

Many other soft-pion theorems can be obtained from (6.24). The full power of the method becomes apparent when two or more soft pions are considered simultaneously, for then the commutators (6.17) of the SU(2) × SU(2) group enter the calculation.

For example, let

$$T^{\mu\nu}(q) = \int \langle \mathrm{b} | T(A_i^\mu(x) A_j^\nu(0)) | \mathrm{a} \rangle \exp(-\mathrm{i}q \cdot x) \mathrm{d}^4 x \tag{6.29}$$

where a and b are nucleon states with momentum p and isospin indices a and b. T has a double pole at $q^2 = 0$, whose residue is proportional to the forward πN scattering amplitude $T_{\pi\mathrm{N}}(q)$:

$$\begin{aligned} T^{\mu\nu} &= F_\pi^2 \frac{q^\mu q^\nu}{(q^2)^2} \int \langle \mathrm{b} | T(j_\pi^i(x) j_\pi^j(0)) | \mathrm{a} \rangle \exp(-\mathrm{i}q \cdot x) \mathrm{d}^4 x + \text{less singular terms} \\ &= \frac{F_\pi^2}{2\pi \mathrm{i} p^0} \frac{q^\mu q^\nu}{(q^2)^2} T_{\pi\mathrm{N}}(q) + \text{less singular terms at } q \to 0 \end{aligned} \tag{6.30}$$

where $T_{\pi\mathrm{N}}$ is normalized as in (6.4). Next we contract (6.30) by multiplying $T^{\mu\nu}$ with q_μ,

$$q_\mu T^{\mu\nu} = \frac{F_\pi^2}{2\pi \mathrm{i} p^0} \frac{q^\nu}{q^2} T_{\pi\mathrm{N}}, \quad \text{as } q \to 0. \tag{6.31}$$

On the other hand, from (6.29)

$$q_\mu T^{\mu\nu} = \mathrm{i} \int \left[\frac{\partial}{\partial x^\mu} \langle \mathrm{b} | T(A_i^\mu(x) A_j^\nu(0)) | \mathrm{a} \rangle \right] \exp(\mathrm{i}q \cdot x) \mathrm{d}^4 x .$$

Because $\partial_\mu A_i^\mu = 0$, only the equal time commutator remains.

$$\begin{aligned} q_\mu T^{\mu\nu} &= \mathrm{i} \int \langle \mathrm{b} | T(A_i^0(x), A_j^\nu(0)) | \mathrm{a} \rangle \delta(x^0) \exp(-\mathrm{i}q \cdot x) \, \mathrm{d}^4 x \\ &= -\mathrm{i}\epsilon^{ijk} \int \langle \mathrm{b} | V_j^\nu(0) | \mathrm{a} \rangle \exp(-\mathrm{i}q \cdot x) \delta^4(x) \mathrm{d}^4 x \end{aligned}$$

where we have used the local form of the second of equations (6.17),

$$[A_i^0(x), A_j^\nu(0)] \, \delta(x^0) = \mathrm{i}\epsilon^{ijk} V_k(0) \delta^4(x). \tag{6.32}$$

Since the vector currents are conserved,

$$\langle \mathrm{b} | V_k^\nu(0) | \mathrm{a} \rangle = \frac{1}{(2\pi)^3} \frac{p^\nu}{p^0} \frac{(\tau_k)_{\mathrm{ab}}}{2} . \tag{6.33}$$

Combining (6.33) with (6.31), we obtain

$$T_{\pi\mathrm{N}} = \frac{1}{8\pi^2 F_\pi^2} p \cdot q \, T_\pi \cdot \tau_{\mathrm{ab}} \tag{6.34}$$

where $(T_\pi^i)_{jk} = -\mathrm{i}\epsilon_{ijk}$ are the pion isospin matrices. Equation (6.34) is a threshold theorem for πN scattering in the symmetry limit. We can apply eq. (6.34) to the real world at the real threshold ($\nu = p \cdot q = Mm_\pi$), since the nucleon pole terms, being P-wave, do not contribute there. Important corrections to eq. (6.34) for the real world are symmetric in i and j, so we shall deal only with the antisymmetric part. The result is a formula for the difference between the $I = \frac{1}{2}$ and $I = \frac{3}{2}$ scattering lengths. Using the Goldberger-Treiman relation (6.28) for F_π, this difference is predicted to be

$$a_{I=3/2} - a_{I=1/2} = \frac{3g^2}{8\pi} \frac{1}{M^2 g_A^2} \frac{M m_\pi}{M + m_\pi} . \tag{6.35}$$

This is an equation for G_A in terms of measurable scattering lengths, as well as a relation between $a_{1/2}$ and $a_{3/2}$. Both are well-satisfied experimentally.

Assuming that $T_{\pi N}(q)$ satisfies an unsubtracted dispersion relation, one obtains a sum rule for g_A, ignoring terms of order m_π^2/M_N^2:

$$1 - \frac{1}{g_A^2} = \frac{2M^2}{g^2} \frac{1}{\pi} \int_{(M+m_\pi)^2}^{\infty} \frac{ds}{s^2 - M^2} [\sigma^+(s) - \sigma^-(s)] \tag{6.36}$$

which is the original form obtained by Adler and Weisberger. In eq. (6.36), $s = (p+q)^2$, and $\sigma^\pm$ is the total $\pi^\pm$ p cross section.

Many other soft-pion theorems can be found using similar methods. The reader is referred to the book by Adler and Dashen for a more complete treatment.

Finally we mention an example of a class of theorems which aren't true. Let

$$\epsilon_\mu^{(1)} \epsilon_\nu^{(2)} T^{\mu\nu\lambda} = \langle \gamma(\epsilon^{(1)}, k_1) \gamma(\epsilon^{(2)}, k_2) | A^\lambda(0) | \rangle \tag{6.37}$$

be the matrix element of the neutral axial current between two photons and the vacuum. Eq. (6.37) should contain a pole of the form

$$\frac{F_\pi q^\lambda}{q^2} \langle \gamma\gamma | j_\pi^0(0) | \rangle \tag{6.38}$$

where $q = k_1 + k_2$ and $\langle \gamma\gamma | j_\pi^0(0) | \rangle$ is proportional to the $\pi^0 \to \gamma\gamma$ amplitude:

$$T(\pi^0 \gamma\gamma) = -(2\pi)^{5/2} \sqrt{k_1^0 k_2^0} \, \langle \gamma\gamma | j_\pi^0(0) | \rangle . \tag{6.39}$$

Kinematically, $T(\pi^0 \to \gamma\gamma)$ must have the form

$$T = \epsilon_\mu^{(1)} \epsilon_\nu^{(2)} \epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta} f(q^2) . \tag{6.40}$$

Physically, $f(m_\pi^2)$ determines the π^0 lifetime. We assume $f(m_\pi^2) \approx f(0)$ to relate physical quantities to the predictions of PCAC.

The non-pole term in $T^{\mu\nu\lambda}$ must be a three-index pseudotensor. The only term first order in the momenta one can construct which is symmetric in the two photons is

$$\epsilon^{\mu\nu\lambda\alpha}(k_1 - k_2)_\alpha .$$

But this term violates electromagnetic gauge invariance, which requires $k_{1\mu} T^{\mu\nu\lambda} = k_{2\nu} T^{\mu\nu\lambda} = 0$. We conclude that

$$\epsilon_\mu^{(1)} \epsilon_\nu^{(2)} T^{\mu\nu\lambda} = -\frac{iF_\pi q^\lambda}{q^2} \frac{T(\pi^0 \to \gamma\gamma)}{(2\pi)^{5/2} \sqrt{k_1^0 k_2^0}} + \epsilon_\mu^{(1)} \epsilon_\nu^{(2)} T'^{\mu\nu\lambda} \tag{6.41}$$

when $T'^{\mu\nu\lambda}$ is at least second order in the momenta. We multiply (6.41) by q^λ and use the conservation of A^λ to obtain

$$0 = \frac{-\mathrm{i}F_\pi}{(2\pi)^{5/2}\sqrt{k_1^0 k_2^0}}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}f(q^2) + q_\lambda T'^{\mu\nu\lambda}. \tag{6.42}$$

Since $q_\lambda T^{\mu\nu\lambda}$ is at least third order in the momenta, it follows that $f(0) = 0$.

It has been shown that $f(0) = 0$ cannot in general be maintained in perturbation theory because of the singularities of the theory invalidate the formal arguments. Experimentally, $f(m_\pi^2) = O(m_\pi^2/M^2)$ predicts far too small a pion decay rate. Correct expressions in perturbation theory can be obtained if we set

$$\partial_\mu A^\mu = \frac{2\overline{Q}\alpha}{4\pi}F_{\mu\nu}(x)F^{\mu\nu}(x) \tag{6.43}$$

where F is the electromagnetic field tensor and $\overline{Q}$ is usually the same model-dependent number which entered our discussion of radiative corrections to β-decay. The value $\overline{Q} = \frac{1}{2}$, characteristic of a simple theory with one elementary charged fermion, like the proton, is in good agreement with experiment. The original Ward identity $\partial_\mu A^\mu = 0$, is recovered in models with an equal number of positive and negative fermion fields. Identities based on (6.43), which are correct in perturbation theory, are called anomalous Ward identities.

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7. The Weinberg-Salam model

In this section we will describe the first model, which was proposed about five years ago by Weinberg and Salam and which combines the weak and electromagnetic interaction through the use of the Higgs mechanism.

The idea is to put the $SU(2)_L$ group discussed in section 5 together with electromagnetic gauge group into a larger gauge symmetry. The charged gauge mesons become the $W^{\pm}$ intermediate vector bosons. There remains a heavy neutral vector meson, the photon, and one Higgs scalar. When no confusion can arise, we will use the name of a particle to stand for its field. In general, we follow Weinberg's notation.

In the simplest version, the only leptons are the electron e and its neutrino ν (we omit the subscript in ν_e for the moment). These may be grouped into a left-handed $SU(2)_L$ doublet

$$L = \begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix} \tag{7.1}$$

where $e_L = \frac{1}{2}(1 - \gamma_5)e$, and an $SU(2)_L$ singlet, $R = e_R = \frac{1}{2}(1 + \gamma_5)e$. We assign to the doublet a "hypercharge" $Y = -1$ and to the singlet e_R a "hypercharge" $Y = -2$, so that the rule

$$Q = T_L^3 + \tfrac{1}{2} Y \tag{7.2}$$

holds for all particles. Since all members of each irreducible multiplet of $SU(2)_L$ have the same hypercharge,

$$[T_L^i, Y] = 0. \tag{7.3}$$

The group generated by T_L^i and Y is $SU(2) \otimes U(1)$. We make this into the gauge symmetry of the model, introducing three gauge mesons A_μ^i associated with $SU(2)_L$ and a fourth B_μ associated with the U(1) subgroup. So far the model contains two pieces:

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{leptons}} \tag{7.4}$$

where, according to the prescription of the first lecture,

$$\mathcal{L}_{\text{gauge}} = -\tfrac{1}{4} F_{\mu\nu}^i F^{i\mu\nu} - \tfrac{1}{4} B_{\mu\nu} B^{\mu\nu}. \tag{7.5}$$

In (7.5)

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\epsilon^{ijk} A_\mu^j A_\nu^k$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{7.6}$$

The lepton part of $\mathcal{L}$ is

$$\mathcal{L}_{\text{leptons}} = \bar{R} i\gamma^\mu (\partial_\mu + ig' B_\mu) R + \bar{L} i\gamma^\mu \left(\partial_\mu + \frac{i}{2} g' B_\mu - ig \frac{\tau^i}{2} A_\mu^i\right) L. \tag{7.7}$$

Recall that if the symmetry group is a direct product, the coupling constants may differ for each factor. We take g to be associated with $SU(2)_L$, and $\frac{1}{2}g'$ with U(1). Notice that the $SU(2)_L$ invariance prohibits an electron mass term from appearing in (7.7).

We want to end up with three of the four vector mesons acquiring masses, since the final theory should have only one conserved quantity, the electric charge Q, and one massless meson, the photon.

To this end we introduce a doublet of (complex) Higgs scalars

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} . \tag{7.8}$$

The doublet ϕ transforms like L of eq. (7.1) under $SU(2)_L$, and has $Y = +1$ in order to maintain (7.2). It contributes a term to the Lagrangian

$$\mathcal{L}_{\text{scalars}} = \left(\partial_\mu\phi^\dagger + \frac{ig'}{2} B_\mu\phi^\dagger + i\frac{g}{2}\tau^i A^i_\mu\phi^\dagger\right)\left(\partial^\mu\phi - \frac{ig'}{2} B_\mu\phi - \frac{ig}{2}\tau^i A^{i\mu}\phi\right) - V(\phi^\dagger\phi). \tag{7.9}$$

The most general form for V is

$$V = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2. \tag{7.10}$$

There may also be an interaction term

$$\mathcal{L}_{\text{inter}} = -G_e[\bar{R}\phi^\dagger L + \bar{L}\phi R] \tag{7.11}$$

which is symmetric under the whole group as well as being Lorentz invariant.

Next we let μ^2 be negative so that one component, which we choose to be the neutral component of ϕ, develops a vacuum-expectation value,

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} /\sqrt{2}. \tag{7.12}$$

Notice that this breaks both the $SU(2)_L$ and the hypercharge U(1) symmetry. The surviving symmetry operator is the combination Q [eq. (7.2)]. We choose v to be real, as in the example in section 3. From (7.10),

$$v = [-\mu^2/\lambda]^{1/2}. \tag{7.13}$$

Next, we redefine the scalar fields, associating a new field with each broken generator. Actually, it is not necessary to find the generators orthogonal to Q; any three independent ones satisfying

$$T\begin{pmatrix} 0 \\ v \end{pmatrix} \neq 0$$

will do. Therefore, we define

$$U(\xi) = \exp(-i\xi\cdot\tau/2v)$$

and write

$$\phi = U^{-1}(\xi)\begin{pmatrix} 0 \\ (v+\eta)/\sqrt{2} \end{pmatrix} \tag{7.14}$$

replacing the four real components of ϕ by η and ξ^i.

Next we make a gauge transformation to the U-gauge, so that the particle content of the model becomes manifest:

$$\phi \to \phi' = U(\xi)\phi = \begin{pmatrix} 0 \\ v+\eta \end{pmatrix}/\sqrt{2}$$

$$L \to L' = U(\xi)L$$

$$\mathbf{A}_\mu \to \mathbf{A}'_\mu \tag{7.15}$$

where

$$\boldsymbol{\tau}\cdot\mathbf{A}'_\mu = U(\xi)\left[\boldsymbol{\tau}\cdot\mathbf{A}_\mu - \frac{i}{g}U^{-1}(\xi)\partial_\mu U(\xi)\right]U^{-1}(\xi) \tag{7.16}$$

and B_μ and R are unchanged. We will drop the primes on L' and A'_μ. The new fields are just as good as the old ones, since the gauge transformation is not singular

Now there are new terms quadratic in the new fields in both $\mathcal{L}_{\text{inter}}$ and $\mathcal{L}_{\text{scalars}}$. Eq. (7.11) becomes

$$\mathcal{L}_{\text{inter}} = -\frac{G_e v}{\sqrt{2}}[\bar{R}L + \bar{L}R] + \text{cubic and higher order terms} = -\frac{G_e v}{\sqrt{2}}\bar{e}e + \dots . \tag{7.17}$$

The electron has acquired a mass:

$$m_e = G_e v/\sqrt{2}. \tag{7.18}$$

The neutrino remains massless because there still are no right-handed neutrino fields. The part of the Lagrangian describing the ϕ field, eq. (7.9), has become

$$\mathcal{L}_{\text{scalar}} = \tfrac{1}{2}\,\partial^\mu\eta\partial_\mu\eta + \frac{(v+\eta)^2}{8}\chi_-^\dagger[(g'B_\mu + g\tau^i A^i_\mu)(g'B^\mu + g\tau^i A^{\mu i})]\chi_- - V\left[\left(\frac{v+\eta}{4}\right)^2\right] \tag{7.19}$$

where $\chi_- = \binom{0}{1}$.

The remaining scalar field η has a mass $-2\mu^2$. The quadratic term in the vector meson fields is

$$\tfrac{1}{8}v^2[(g'B_\mu - gA^3_\mu)(g'B^\mu - gA^{\mu 3}) + g^2((A^{\mu 1})^2 + (A^2_\mu)^2)]. \tag{7.20}$$

Define

$$W^\pm_\mu = (A^1_\mu \mp iA^2_\mu)/\sqrt{2}. \tag{7.21}$$

Evidently the charged fields $W^\pm_\mu$ have mass

$$M_W = \tfrac{1}{2}gv. \tag{7.22}$$

Define two neutral fields

$$Z_\mu = \frac{-gA^3_\mu + g'B_\mu}{\sqrt{g^2+g'^2}}, \qquad A_\mu = \frac{gB_\mu + g'A^3_\mu}{\sqrt{g^2+g'^2}}. \tag{7.23}$$

Z_μ and A_μ are eigenstates of the mass matrix, with masses

$$M_Z = \tfrac{1}{2} v\sqrt{g^2 + g'^2}, \qquad M_A = 0. \tag{7.24}$$

The single massless vector meson is the photon, corresponding to the surviving U(2) symmetry $\exp(-i\theta Q)$.

It's instructive to rewrite $\mathcal{L}_{\text{leptons}}$ – eq. (7.7) – in terms of the $W^\pm$, Z, and photon. From (7.21)

$$A_\mu^{(1)} = (W_\mu^- + W_\mu^+)/\sqrt{2}, \qquad A_\mu^{(2)} = (W_\mu^- - W_\mu^+)/i\sqrt{2}. \tag{7.21'}$$

Therefore the term in (7.7) containing $W^\pm$ is

$$\frac{g}{2}\bar{L}\,\gamma^\mu(\tau^1 A_\mu^1 + \tau^2 A_\mu^2)L = \frac{g}{2}\;(\bar{\nu}_L\gamma^\mu e_L + \bar{e}_L\gamma^\mu\nu_L)\frac{(W_\mu^- + W_\mu^+)}{\sqrt{2}} - i(\bar{\nu}_L\gamma^\mu e_L - \bar{e}_L\gamma^\mu\nu_L)\frac{(W_\mu^- - W_\mu^+)}{i\sqrt{2}}$$

$$= \frac{g}{\sqrt{2}}[\bar{\nu}_L\gamma^\mu e_L W_\mu^+ + \bar{e}_L\gamma^\mu\nu_L W_\mu^-]. \tag{7.25}$$

Comparing with eq. (6.12) we obtain

$$G/\sqrt{2} = g^2/8M_W^2 = 1/2v^2. \tag{7.26}$$

Next we examine the terms in (7.7) containing A_μ^3 and B_μ. Define an angle θ_W by

$$g'/g = \tan\theta_W. \tag{7.27}$$

Then from (7.23)

$$A_\mu = \cos\theta_W B_\mu + \sin\theta_W A_\mu^3, \qquad Z_\mu = \sin\theta_W B_\mu - \cos\theta_W A^3. \tag{7.28}$$

Inverting, we get

$$B_\mu = \cos\theta_W A_\mu + \sin\theta_W Z_\mu, \qquad A_\mu^3 = \sin\theta_W A_\mu - \cos\theta_W Z_\mu. \tag{7.29}$$

The terms in $\mathcal{L}_{\text{lepton}}$ coupling A_μ^3 and B_μ to the leptons are

$$-\frac{g}{2}[2\bar{e}_R\gamma^\mu e_R + \bar{e}_L\gamma^\mu e_L + \bar{\nu}_L\gamma^\mu\nu_L][\cos\theta_W A_\mu + \sin\theta_W Z_\mu] + \frac{g}{2}[\bar{e}_L\gamma^\mu e_L - \bar{\nu}_L\gamma^\mu\nu_L][\sin\theta_W A_\mu - \cos\theta_W Z_\mu]$$

$$= \frac{Z}{2\sqrt{g^2+g'^2}}[g'^2(2\bar{e}_R\gamma^\mu e_R + \bar{e}_L\gamma^\mu e_L + \bar{\nu}_L\gamma^\mu\nu_L) - g^2(\bar{e}_L\gamma^\mu e_L - \bar{\nu}_L\gamma^\mu\nu_L)] + \frac{gg'}{\sqrt{g^2+g'^2}} A_\mu[\bar{e}_R\gamma^\mu e_R + \bar{e}_L\gamma^\mu e_L]. \tag{7.30}$$

Thus the massless vector meson A_μ does couple to the electric current $\bar{e}\gamma^\mu e$, and we can identify the electron's charge $-e$:

$$e = gg'/\sqrt{g^2 + g'^2}. \tag{7.31}$$

Finally, we verify that local gauge invariance still holds for the local U(1) group corresponding to Q, with the photon field A_μ being the gauge meson. Under an infinitesimal transformation generated by $Q = \frac{1}{2}Y + T_L^3$,

$$\delta A^3_\mu = \frac{1}{g}\partial_\mu \epsilon(x)$$

$$\delta B_\mu = \frac{2}{g'}\partial_\mu \tfrac{1}{2}\epsilon(x) = \frac{1}{g'}\partial_\mu \epsilon(x)$$

so, from (7.28)

$$\delta Z_\mu = \left(\frac{1}{g'}\sin\theta_W - \frac{1}{g}\cos\theta_W\right)\partial_\mu\epsilon(x) = 0$$

$$\delta A_\mu = \left[\frac{\cos\theta_W}{g'} + \frac{\sin\theta_W}{g}\right]\partial_\mu\epsilon(x)$$

$$= \frac{1}{gg'}[g\cos\theta_W + g'\sin\theta_W]\,\partial_\mu\epsilon(x) = \frac{\sqrt{g^2+g'^2}}{gg'}\partial_\mu\epsilon(x) = \frac{1}{e}\partial_\mu\epsilon(x). \tag{7.32}$$

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8. Phenomenology of the model. Incorporation of hadrons

Since both $g/[g^2+g'^2]^{1/2}$ and $g'/[g^2+g'^2]^{1/2}$ are less than 1, we can conclude from (7.31) that

$$g\sin\theta = e, \qquad g'\cos\theta = e, \tag{8.1}$$

so both g and g' are greater than e.

From (7.22), the mass of the W is given by $M_W^2 = \frac{1}{4}g^2v^2$. From (7.26), $v^2 = 1/G\sqrt{2}$. Therefore

$$M_W^2 = \frac{g^2v^2}{4} = \frac{e^2}{\sin^2\theta_W}\,\frac{1}{4\sqrt{2}\,G}\,. \tag{8.2}$$

The W mass must be quite large,

$$M_W = \left[\frac{\pi\alpha}{\sqrt{2}G}\right]^{1/2}\frac{1}{\sin\theta_W} \approx \frac{38}{\sin\theta_W}\ \text{GeV}. \tag{8.3}$$

Evidently, in this model, the minimum value of M_W is too large to be produced in present-day accelerators; nevertheless, it is not nearly as large as the unitarity bound, which is of the order of hundreds of GeV.

The Z meson is even heavier. From eq. (7.24)

$$M_Z = \frac{vg}{2\cos\theta_W} = \frac{M_W}{\cos\theta_W} = \frac{38\ \text{GeV}}{\frac{1}{2}\sin 2\theta_W}. \tag{8.4}$$

Since $g' = 0$ is not allowed, $\cos\theta_W < 1$, and

$$M_Z > M_W, \qquad M_Z > 76\ \text{GeV}. \tag{8.5}$$

The value of the dimensionless e–e–η coupling constant G_e can be obtained from (7.18) and (7.26)

$$G_e = \sqrt{2}\, m_e/v = \sqrt{2}\, m_e \cdot \sqrt[4]{2}\sqrt{G} \sim 2 \times 10^{-6} \tag{8.6}$$

which is small, indicating that graphs with ηee vertices can often be ignored compared to graphs with photon or Z vertices.

What is the effect of the W on the spectrum for $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$? The (μ^-, ν_μ) doublet is easily incorporated into the model in exact analogy to the (e^-, ν_e) doublet, and the μ-mass generated by the μ–ν_μ–ϕ coupling. The coupling constant G_μ must have the value

$$G_\mu = (m_\mu/m_e)G_e$$

which is larger than (8.6) but still very small. The amplitude for μ^--decay is

$$\frac{-ig^2}{16\pi^2}\,\bar{u}(\nu_\mu)\gamma^\mu(1-\gamma_5)u(\mu)\bar{u}(e)\gamma^\nu(1-\gamma_5)v(\bar{\nu}_e)\,\frac{[g_{\mu\nu} - k_\mu k_\nu/M_W^2]}{k^2 - M_W^2} \tag{8.7}$$

where $k = p(\mu) - p(\nu_\mu) = p(e) + p(\bar{\nu}_e)$. In (8.7), the $g_{\mu\nu}$ term reproduces the point interaction spectrum up to terms of the order k^2/M_W^2. The second term is of the order $m_e m_\mu/M_W^2$. So the effect on the spectrum is very small.

The most accessible test of the model seems to be $\bar{\nu} - e^-$ elastic scattering. The W contribution comes from fig. 8.1(a).

At low energies, the contribution of fig. 8.1(a) is indistinguishable from the Fermi theory:

$$T^{(a)} = \frac{iG}{2\sqrt{2}\pi^2}\,v(\bar{\nu}')\gamma^\mu(1-\gamma_5)v(\bar{\nu})\bar{u}(e'^-)\gamma_\mu(1-\gamma_5)u(e) \tag{8.8}$$

where we have applied a Fierz transformation to the $(V-A)(V-A)$ coupling. The sign in (8.8) is the product of a minus sign from Fermi statistics and a minus sign from the Fierz transformation. The Z-exchange contribution can be obtained from (7.30). At low energies it is

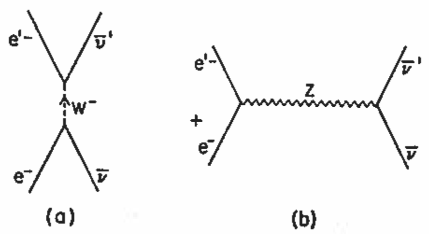

Fig. 8.1. Graphs for νe^- elastic scattering in the Weinberg–Salam model.

$$\frac{iG}{2\sqrt{2}\pi^2} v(\nu')\gamma^\mu(1-\gamma_5)v(\bar\nu)\bar u(e)\gamma_\mu[2\sin^2\theta_W - \tfrac{1}{2} + \tfrac{1}{2}\gamma_5]u(e). \tag{8.9}$$

In general, we may write the amplitude for $\nu + e \to \nu + e^-$ as

$$\frac{iG}{2\sqrt{2}\pi^2} v(\bar\nu')\gamma^\mu(1-\gamma_5)v(\nu)\bar u(e^{-\prime})\gamma_\mu[C_V - \gamma_5 C_A]u(e^-). \tag{8.10}$$

Then W exchange alone (or Fermi coupling) predicts

$$C_V = C_A = 1 \tag{8.11}$$

while the present model predicts

$$C_V = 2\sin^2\theta_W + \tfrac{1}{2}, \qquad C_A = \tfrac{1}{2}. \tag{8.12}$$

From (8.10), the spin-averaged differential cross section can be calculated. The cross section into solid angle $d\Omega$ (in the center of mass frame) is

$$\frac{d\sigma}{d\Omega} = \frac{G^2}{4\pi^2 s}[(C_V - C_A)^2(p\cdot q)^2 + (C_V + C_A)^2(p\cdot q')^2 - m_e^2(C_V^2 - C_A^2)(p\cdot p')] \tag{8.13}$$

where p and p' are the initial and final neutrino momenta, q and q' the initial and final electron momenta, and $s = (p+q)^2$. In terms of the lab-frame electron recoil energy, T, we obtain from (8.13)

$$\frac{d\sigma}{dT} = \frac{G^2}{2\pi} m_e \left[(C_V - C_A)^2 + (C_V + C_A)^2\left[1 - \frac{T}{\omega}\right]^2 - (C_V^2 - C_A^2)\frac{m_e T}{\omega^2}\right] \tag{8.14}$$

where ω is the neutrino energy in the initial electron's rest frame (the lab frame). The last term is small for $\omega \gg m_e$. In the V − A model ($C_V = C_A = 1$), $d\sigma/dT$ decreases, for fixed T, like $1/\omega$. Otherwise, there is a constant term. If $C_V = -C_A$ (not possible in the W.-S. model), $d\sigma/dT$ would be entirely independent of ω.

Gurr, Reines and Sobel have looked for νe events from anti-neutrinos produced by a Savannah River Plant reactor. What they measure is the rate given by (8.14) integrated from a minimum to a maximum value of T, folded into the neutrino spectrum. T_{max} is just the neutrino energy ω,

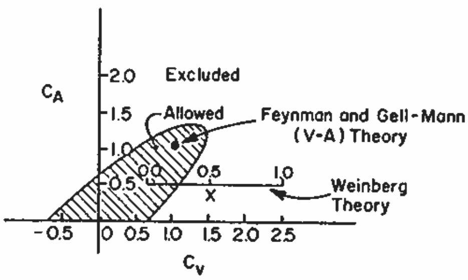

Fig. 8.2. Region of values of C_V and C_A in agreement with the experiment of Gurr, Reines and Sobel.

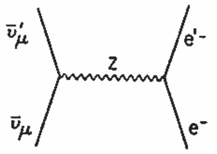

Fig. 8.3. Z-exchange graph for $\nu_\mu e^-$ scattering.

and $T_{\min}$ is determined by the experimental conditions. They have established that the cross section is less than twice that predicted by $C_V = C_A = 1$. Fig. 8.2 is from their paper. It is a map of the $C_V - C_A$ space, the shaded region being the value allowed by their experiment. The V – A theory is not excluded, and the W.-S. model is acceptable for $\sin^2\theta_W \leqslant 0.35$, corresponding to a W mass greater than 60 GeV.

The amplitude for $\nu_\mu + e \to \nu_\mu + e$ can be parametrized in a similar way. It is a particularly interesting process because it is forbidden if only charged currents exist, since $\nu_\mu - e$ does not couple to W. If there is a neutral Z, elastic $\nu_\mu e$ scattering will be mediated by Z exchange, as in fig. 8.3.

The effective interaction is

$$-i\frac{G}{\sqrt{2}}\bar{\nu}_{(\mu)}\gamma^\mu(1-\gamma_5)\nu_\mu[\bar{e}\gamma_\mu(C'_V - C'_A\gamma_5)e]. \tag{8.17}$$

In the W.-S. model,

$$C'_A = \tfrac{1}{2}, \qquad C'_V = \tfrac{1}{2} - 2\sin^2\theta_W. \tag{8.18}$$

In pure V – A theory, $C'_A = C'_V = 0$.

Recent experiments at CERN have put bounds on both the $\nu_\mu e$ and $\bar{\nu}_\mu e$ elastic cross sections. Like $\nu_e + e$, both grow linearly with the (anti) neutrino energy ω for $\omega \ll M_Z$. If ω is measured in GeV, the cross sections are less than $0.7 \times 10^{-41}\ \omega\ \mathrm{cm}^2$ and $1.1 \times 10^{-41}\ \omega\ \mathrm{cm}^2$, respectively. A formula like (8.15) describes these cross sections also in terms of θ_W. The experimental bounds restrict $\sin^2\theta_W$ to be less than about 0.6, which so far is less restrictive than the bounds obtained from elastic $e^-\nu_e$ scattering.

There have been many attempts to include hadrons in a W.-S. type model. One of the principal difficulties is that a realistic theory must have $\Delta S = 1$, $\Delta Q = 1$ currents, but no neutral strangeness-changing currents. The question of hadronic neutral non-strange currents is still open experimentally.

A straightforward way to add hadrons to the model without changing its basic structure is to add three fundamental "quark" fields, which we shall call p, n, λ. We will not worry about approximate SU(3) symmetry here, but assume that the Lagrangian contains some very strong, symmetric term, like a vector gluon interaction, which does not affect the rest of the discussion.

Next we group the left-handed quarks into an $SU(2)_L$ doublet

$$N_L = \begin{pmatrix} p_L \\ n_L\cos\theta + \lambda_L\sin\theta \end{pmatrix} \equiv \begin{pmatrix} p_L \\ n_{c_L} \end{pmatrix} \tag{8.19}$$

where θ is the Cabibbo angle introduced in the eq. (5.22). The remaining singlets are the right-handed quarks, n_R, p_R and λ_R, and the combination orthogonal to the bottom line in (8.19), namely

$$\lambda_c = (\lambda_L \cos\theta - n_L \sin\theta). \tag{8.20}$$

Since we are not interested here in SU(3) transformations, we assign $Y = 1$ to n_L, $Y = 2$ to p_R, and $Y = 0$ to the rest. Then p has unit positive charge, and λ and n are neutral. (The conventional quark charges can be obtained by shifting the Y assignments.)

The Lagrangian $\mathcal{L}$ must be symmetric under SU(2) × U(1). The lepton and gauge field pieces already are, and so can be the very strong vector gluon coupling term. A quark mass term, of the form

$$m_p[\bar{p}_R p_L + \bar{p}_L p_R]$$

is forbidden by the symmetry, so cannot appear in $\mathcal{L}$. The quark masses arise from interaction with the scalar doublet ϕ.

To write the most general interaction, we need

$$\tilde{\phi} = \mathrm{i}\,\sigma_2\,\phi^* = \begin{pmatrix}\phi^0\\ -\phi^-\end{pmatrix} \tag{8.21}$$

which also transforms like a doublet under SU(2), but has $Y = -1$. The general quark-scalar interaction has the form

$$G_1[\bar{N}_L\tilde{\phi}p_R + \text{h.c.}] + G_2[\bar{N}_L\phi n_R + \text{h.c.}] + G_3[\bar{N}_L\phi\lambda_R + \text{h.c.}] + G_4[n_R\lambda_c] + G_5[\lambda_R\lambda_c]. \tag{8.22}$$

The quark mass matrix is obtained by replacing ϕ by its vacuum expectation value:

$$\langle\phi^+\rangle = 0, \qquad \langle\phi^0\rangle = v/\sqrt{2}.$$

The term quadratic in the fermion fields in eq. (8.22) becomes

$$\frac{v}{\sqrt{2}}[G_1\bar{p}p + G_2(\bar{n}\cos\theta + \bar{\lambda}\sin\theta)n + G_3(\bar{n}\cos\theta + \bar{\lambda}\sin\theta)\lambda$$
$$+ G_4\bar{n}(\lambda\cos\theta - n\sin\theta) + G_5\bar{\lambda}(\lambda\cos\theta - n\sin\theta)]. \tag{8.23}$$

Evidently, $G_1, \ldots G_5$ must be adjusted so that the mass of the p-quark is m_p, etc., and the physical n- and λ-quarks are mass eigenstates. This determines the couplings $G_1, \ldots G_5$ completely.

In terms of the quark masses and the Cabibbo angle, (8.22) may be written

$$\frac{\sqrt{2}}{v}\left\{m_p(N_L\tilde{\phi}p_R + \text{h.c.}) + m_n\left[\bar{n}_R\left(\phi^+N_L\cos\theta - \frac{v}{\sqrt{2}}\lambda_c\sin\theta\right) + \text{h.c.}\right]\right.$$
$$\left.+ m_\lambda\left[\bar{\lambda}_R\left(\phi^+N_L\sin\theta + \frac{v}{\sqrt{2}}\lambda_c\cos\theta\right) + \text{h.c.}\right]\right\}. \tag{8.24}$$

Since v is determined by eq. (7.26) in terms of the Fermi constant G, we conclude that the coupling constants G_i in (8.23) are quite small, of the order 1%, and therefore the Higgs scalar couples to the quarks weakly.

Let us examine the neutral quark currents. The coupling to B_μ and A_μ^3 is

$$g'B_\mu[\bar{p}_R\gamma^\mu p_R + \tfrac{1}{2}(\bar{n}_L\gamma^\mu n_L\cos^2\theta + \bar{\lambda}_L\gamma^\mu\lambda_L\sin^2\theta + \cos\theta\,\sin\theta(\bar{n}_L\gamma^\mu\lambda_L + \bar{\lambda}_L\gamma^\mu n_L))] + gA_\mu^3 j^{3\mu} \quad (8.25)$$

where

$$j^{(3)\mu} = \tfrac{1}{2}[\bar{p}_L\gamma^\mu p_L - \bar{n}_L\gamma^\mu n_L\cos^2\theta - \bar{\lambda}_L\gamma^\mu\lambda_L\sin^2\theta - \cos\theta\,\sin\theta(\bar{n}_L\gamma^\mu\lambda_L + \bar{\lambda}_L\gamma^\mu n_L)]\,. \quad (8.26)$$

The terms proportional to $\cos\theta\,\sin\theta$ are the strangeness-changing neutral currents. In terms of A_μ and Z_μ, the neutral vector eigenstates given by (7.23) in terms of A_μ^3 and B_μ, the interaction (8.25) can be written

$$\frac{gg'}{\sqrt{g^2+g'^2}}A_\mu j^{\mu(\mathrm{em})} + \sqrt{g^2+g'^2}\,Z_\mu[j^{\mu(3)} - \sin^2\theta_W\, j^{\mu(\mathrm{em})}] \quad (8.27)$$

where $j_\mu^{(3)}$ is given by (8.26) and

$$j_\mu^{(\mathrm{em})} = \bar{p}\gamma^\mu p.$$

For other charge assignments eq. (8.27) still holds, with appropriate $j^{(\mathrm{em})}$. Since $j_\mu^{(\mathrm{em})}$ contains no terms with λ or n, Z_μ does couple to a strangeness-changing neutral current.

This result is impossible to avoid with only those quarks, whatever their charge assignments. Because of the limits placed experimentally on such currents by the absence of $K^+ \to \pi^+ + e^+ + e^-$ or $K_L^0 \to \mu^+ + \mu^-$, it is desirable to eliminate them. A model which does this has been suggested by Glashow, Iliopoulos and Maiani. They add a fourth quark, q^+, and group the quarks into two $SU(2)_L$ doublets:

$$\begin{pmatrix} p \\ n_c \end{pmatrix}_L \quad \text{and} \quad \begin{pmatrix} q^+ \\ \lambda_c \end{pmatrix}_L \,. \quad (8.28)$$

If the mass of the q^+ is very high, no unwanted effects will appear. Instead of (8.26), the neutral current is now

$$j_\mu^{(3)} = \tfrac{1}{2}[\bar{p}_L\gamma^\mu p_L - \bar{n}_{cL}\gamma_\mu n_{cL} + \bar{q}_L\gamma_\mu q_L - \bar{\lambda}_c\gamma_\mu\lambda_{cL}]\,. \quad (8.29)$$

Because (n_c, λ_L) is obtained from (n, λ) by making a unitary transformation (5.22), the combination $\bar{n}_{cL}\gamma^\mu n_{cL} + \bar{\lambda}_{cL}\gamma^\mu\lambda_{cL}$ in (8.29) is just $\bar{n}\gamma^\mu u + \bar{\lambda}\gamma^\mu\lambda$. The cross terms proportional to $\cos\theta \times \sin\theta$ cancel, and the unwanted currents are eliminated.

In this model, the Z-hadron coupling is still given by (8.27), with $j_\mu^{(3)}$ given by (8.29), and the Z-lepton coupling is unchanged [see eq. (7.30) or (8.9)]. Specifically, the Z coupling to hadrons and neutrinos takes the form

$$\sqrt{g^2+g'^2}\,Z^\mu[j_\mu^{(3)} - \sin^2\theta_W j_\mu^{(\mathrm{em})} + \tfrac{1}{2}\bar{\nu}\gamma_\mu\tfrac{1}{2}(1-\gamma_5)\nu]\,.$$

For low energies, the amplitude for $\nu + a \to \nu' + b$, where a and b are hadron states, is proportional to

$$\frac{(g^2+g'^2)}{4M_Z^2}\langle b|j_\mu^{(3)} - \sin^2\theta_W\, j_\mu^{(\mathrm{em})}|a\rangle\bar{\nu}'\gamma^\mu(1-\gamma_5)\nu. \quad (8.30)$$

Using (7.24) for M_Z, we obtain

$$(g^2 + g'^2)/4M_Z = 1/v^2 = \sqrt{2}\, G. \qquad (8.31)$$

The rates are therefore independent of the Z mass at low energies, when the Z-propagator can be approximated by $-g_{\mu\nu}/M_Z^2$.

For example, the amplitude for elastic νp scattering has been measured to be (0.12 ± 0.06) times the rate for $\nu + \mathrm{n} \to \ell^- + \mathrm{p}$. To make a theoretical prediction, the matrix element from (8.30) can be obtained as follows: The matrix elements of $j^{(\mathrm{em})}$ are well known from electromagnetic form factors. The current $j_\mu^{(3)}$ is the neutral component of a triplet whose charged member is just what is measured in $\nu + \mathrm{n} \to \mu^- + \mathrm{p}$. Thus the amplitude from (8.30) is known experimentally. Pais and Treiman predict the branching ratio to be

$$0.15 \leqslant \sigma(\nu + \mathrm{p} \to \nu + \mathrm{p})/\sigma(\nu + \mathrm{n} \to \mu^- + \mathrm{p}) \leqslant 0.25 \qquad (8.32)$$

provided $\theta_W < 0.35$, as required by the $\mathrm{e}^-\nu$ elastic scattering experiments.

Even more stringent bounds can be obtained from experiments looking for weak pion production. We will say only a few words and refer you to the literature for details. Consider the process $\nu + \mathrm{p} \to \nu + \mathrm{p} + \pi^0$. We need the matrix element

$$\langle \mathrm{p}\pi^0 | j_\mu^{(3)} - \sin^2\theta_W\, j_\mu^{(\mathrm{em})} | \mathrm{p} \rangle . \qquad (8.33)$$

The electromagnetic current can be measured in π^0 electroproduction. The charged version of $j_\mu^{(3)}$, $\langle \mathrm{p}\pi^0 | j_\mu^+ | \mathrm{n}\rangle$ can be measured in $\nu + \mathrm{n} \to \mathrm{p} + \pi^0 + \mu^-$ experiments. Actually this matrix element is not simply related to $\langle \mathrm{p}\pi^0 | j_\mu^{(3)} | \mathrm{p}\rangle$ by isospin, because $\mathrm{p}\pi^0$ can have either $I = \frac{1}{2}$ or $I = \frac{3}{2}$. However, inequalities can be deduced, π^+ and π^- amplitudes may be averaged, isospin zero nuclei may be used for targets, or events may be selected where the 3–3 resonance is known to dominate.

There are experimental bounds on many branching ratios for neutral-to-charged neutrino-induced pion production processes. One of the most stringent is

$$R = \frac{\sigma(\nu + \mathrm{p} \to \nu + \mathrm{p} + \pi^0) + \sigma(\nu + \mathrm{n} \to \nu + \mathrm{n} + \pi^0)}{2\sigma(\nu + \mathrm{n} \to \mu^- + \mathrm{p} + \pi^0)} \leqslant 0.14.$$

Theoretical arguments, with inputs from other experiments, predict $R \geqslant 0.2$. Although these numbers are subject to considerable theoretical and experimental uncertainties, it is beginning to look as if there may not be any neutral hadron currents which couple to neutrinos. However, only more detailed measurements can settle this point.

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9. Models with heavy leptons

In this section we shall describe models without neutral vector mesons coupling to neutrinos. In these models, the rates for all neutrino processes described in the last section vanish to order G^2.

All these sections involve heavy leptons. The reason is simply that the graph in fig. 9.1 for $\nu + \bar{\nu} \to W^+ + W^-$ exists in all models.

The amplitude calculated from this graph grows linearly with s, and therefore violates the unitarity bound. This behavior leads to a non-renormalizable theory, because the box graph occurring in the fourth order $\nu + \bar{\nu}$ elastic amplitude is quadratically divergent. In the Weinberg–Salam theory, the leading asymptotic behavior of the graph in fig. 9.1 is cancelled by the graph in fig. 9.2. The skeptical reader should calculate the ZWW vertex and verify this cancellation.

If $\bar{\nu}\nu$Z vertices are to be banned, the linear growth of the graph in fig. 9.1 must be cancelled somehow. The only other alternative is more leptons, as in fig. 9.3.

The linear term in fig. 9.3 has the opposite sign to the linear term in fig. 9.1, and therefore they can cancel with appropriate coupling constants, leading to a theory which may be renormalizable. The hypothetical E^+ is a "heavy" lepton, because if it were lighter than the K^+ meson, it would already have been seen in $K^+ \to E^+ + \bar{\nu}$.

Heavy leptons can be introduced in the context of an SU(2) × U(1) model, where one of their functions is to eliminate the $Z\bar{\nu}\nu$ coupling. For example, we may introduce a left-handed triplet

$$\begin{pmatrix} E^+ \\ \nu \\ e^- \end{pmatrix}_L . \tag{9.1}$$

In addition the model contains right-handed $SU(2)_L$ singlets, e^-_R and E^+_R. The triplet can be assigned $Y = 0$. The electron and E^+ have $Y = -2$ and $Y = +2$ respectively. Then the neutral current is

$$j^{(3)}_\mu = \bar{E}^+_L \gamma_\mu E^+_L - \bar{e}^-_L \gamma_\mu e_L \tag{9.2}$$

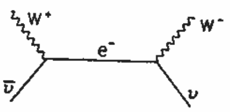

Fig. 9.1. Electron exchange graph for $\nu + \nu \to W^+ + W^-$.

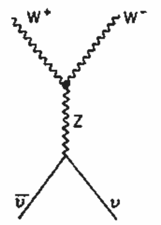

Fig. 9.2. Z annihilation graph for $\nu + \nu \to W^+ + W^-$.

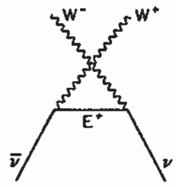

Fig. 9.3. Heavy lepton exchange graph for $\nu + \nu \to W^+ + W^-$.

which contains no $\bar{\nu}\gamma^\mu\nu$ term. Neither $A_\mu^{(3)}$ nor B_μ couple to the neutrinos, so neither do the linear combinations A_μ or Z_μ.

Another possibility is to add a neutral E° to the scheme just described, and group the leptons into two doublets

$$\begin{pmatrix} (\nu + E^\circ)/\sqrt{2} \\ e^- \end{pmatrix}_L \qquad \begin{pmatrix} E^+ \\ (\nu - E^\circ)/\sqrt{2} \end{pmatrix}_L \tag{9.3}$$

with $Y = -1$ and $Y = +1$ respectively. E_R° has $Y = 0$. The hypercharge current

$$\tfrac{1}{2}(\bar{\nu}_L + \bar{E}_L^\circ)\gamma^\mu(\nu_L + E_L^\circ) - (\bar{\nu}_L - \bar{E}_L^\circ)\gamma^\mu(\nu_L - E_L^\circ)$$

contains no term in $\bar{\nu}\gamma^\mu\nu$ and neither does $j_\mu^{(3)}$. In such a model one would expect $\nu + e^- \to E^\circ + e^-$ at sufficiently high energy, but no elastic $\nu + e$ scattering. The former model is known as the LPZ model; the latter as the PZ II model.

A rather different idea has been suggested by Georgi and Glashow. Instead of SU(2) × U(1), let the basic gauge group be O(3). Then there will be only one neutral current, and it must be just $j^{(em)}$. In this model there is no other neutral current at all, so that there is no parity violation predicted in electromagnetic processes like $e^- + e^- \to e^- + e^-$ or $e^- + p \to e^- + p$.

The simplest way to realize this idea is to add a neutral lepton E° and group it together with E^+, ν, and e^- into a triplet:

$$L = \begin{pmatrix} E^+ \\ \nu \sin\beta + E^\circ \cos\beta \\ e^- \end{pmatrix}_L . \tag{9.4}$$

E° must have a mass, so we can form a right-handed triplet also

$$R = \begin{pmatrix} E^+ \\ E^\circ \\ e^- \end{pmatrix}_R . \tag{9.5}$$

There remains a left-handed singlet:

$$(E^\circ \sin\beta - \nu\cos\beta)_L .$$

The interaction of the leptons with the gauge fields A_μ^i is, according to the general prescription

$$g A^i_\mu j^{\mu i} = A^{(\mathrm{o})}_\mu j^{\mu(\mathrm{o})} + A^+_\mu j^{\mu +} + A^-_\mu j^{\mu -} \tag{9.6}$$

where

$$A^\pm_\mu = \frac{A^{(1)}_\mu \mp \mathrm{i} A^{(2)}_\mu}{\sqrt{2}}$$

$$j^\pm_\mu = \frac{j^{(1)}_\mu \pm \mathrm{i} j^{(2)}_\mu}{\sqrt{2}} \tag{9.7}$$

and therefore

$$j^\pm_\mu = \bar{L}_\alpha \gamma_\mu T^\pm_{\alpha\beta} L_\beta + \bar{R}_\alpha \gamma^\mu T^\pm_{\alpha\beta} R_\beta$$

$$j_\mu = \bar{L}_\alpha \gamma^\mu T^\mathrm{o}_{\alpha\beta} L_\beta + \bar{R}_\alpha \gamma^\mu T^\mathrm{o}_{\alpha\beta} R_\beta . \tag{9.8}$$

In the spherical representation

$$T^+ = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^- = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^\mathrm{o} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

[The phase is chosen so that for a neutral triplet $(\phi^+, \phi^\mathrm{o}, \phi^-)$ we have $\phi^+ = (\phi^-)^\dagger$.]

We identify $A^\pm_\mu$ with $W^\pm_\mu$, and $A^{(\mathrm{o})}$ with the photon, A_μ.

The neutral term in eq. (9.6) is

$$g A_\mu [\bar{e}^- \gamma_\mu e^- - \bar{E}^+ \gamma^\mu E^+] \tag{9.10}$$

and does not violate parity. Therefore we can identify

$$g = e. \tag{9.11}$$

The charged term is

$$eW_\mu[(\bar{\nu}_\mathrm{L} \sin\beta + \bar{E}^\mathrm{o}_\mathrm{L} \cos\beta)\gamma^\mu e_\mathrm{L} - \bar{E}^+_\mathrm{L} \gamma^\mu(\nu_\mathrm{L} \sin\beta + E^\mathrm{o}_\mathrm{L} \cos\beta) + \bar{E}^\mathrm{o}_\mathrm{R} \gamma^\mu e_\mathrm{R} - \bar{E}^+_\mathrm{R} \gamma^\mu E^\mathrm{o}_\mathrm{R}] + \mathrm{H.C.} \tag{9.12}$$

The term in (9.12) which couples electrons to neutrinos is

$$e \sin\beta\, W^+ \bar{\nu}_\mathrm{L} \gamma^\mu e^- + \mathrm{H.C.} = \tfrac{1}{2} e \sin\beta\, W^+ \bar{\nu} \gamma^\mu (1 - \gamma_5) e^- + \mathrm{H.C.} \tag{9.13}$$

Therefore

$$G/\sqrt{2} = e^2 \sin^2\beta / 4M^2_\mathrm{W}. \tag{9.14}$$

In the W.-S. model – compare eq. (8.2) – we had

$$G/\sqrt{2} = e^2/(8M^2_\mathrm{W} \sin^2\theta_\mathrm{W}). \tag{9.15}$$

The Georgi–Glashow model therefore has an *upper* bound for the W mass

$$M_\mathrm{W} \leqslant \sqrt{2}\,(38\ \mathrm{GeV}) = 53\ \mathrm{GeV}.$$

Obviously, the muon and its neutrino can be introduced analogously, at the cost of two new heavy muons.

The Higgs mechanism for this model can be constructed from a triplet of scalar fields ϕ^i. The gauge invariant kinetic energy term for the ϕ^i is given by (3.10). and we saw in the third section that if

$$\langle\phi_1\rangle = \langle\phi_2\rangle = 0, \qquad \langle\phi_3\rangle = v, \tag{9.16}$$

then the W meson acquires a mass ev.

It will be useful later to write the Lagrangian in terms of the charge eigenstates $\phi_o = \phi_3$ and $s^\pm = (\phi_1 \mp i\phi_2)/\sqrt{2}$. Then (s^+, ϕ_o, s^-) form a basis for the representation (9.9) of O(3). The Lagrangian term for the Higgs scalars is

$$\mathcal{L}_{\text{scalars}} = \tfrac{1}{2}(\partial_\mu\phi_i - \mathrm{i}\, e\, A_\mu\cdot(T)_{ik}\phi_k)(\partial_\mu\phi_i - \mathrm{i}\, e\, A^\mu\cdot(T)_{ik}\phi_k) \tag{9.17}$$

which becomes, in terms of $s^\pm$ and ϕ^o,

$$\mathcal{L}_{\text{scalars}} = \tfrac{1}{2}\,[\partial_\mu\phi^o + \mathrm{i}\, e(W_\mu^- s^+ - W_\mu^+ s^-)]\,[\partial^\mu\phi^o - \mathrm{i}\, e(W^{\mu+}s^- - W^{\mu-}s^+)]$$
$$+ (\partial_\mu s^+ - \mathrm{i}\, e\, A_\mu s^+ + \mathrm{i}\, e\,\phi^o W^+)(\partial_\mu s^- + \mathrm{i}\, e\, A^\mu s^- - \mathrm{i}\, e\,\phi^o W^{-\mu}). \tag{9.18}$$

Write $\phi^o = v + \psi$. Then it is evident from (9.18) that the photon remains massless, while $W^\pm$ acquires a mass μ:

$$\mu = ev. \tag{9.19}$$

We see that there is a direct $s^- W_\mu^+ A_\nu$ coupling term

$$-e\,\mu\, g^{\mu\nu} \tag{9.20}$$

in addition to those explicit in (9.18). Of course, it is possible to eliminate the $s^\pm$-fields by writing

$$\begin{pmatrix} s^+ \\ \phi^o \\ s^- \end{pmatrix} = \exp\{\mathrm{i}(T_+\xi_+ + T_-\xi_-)/v\}\begin{pmatrix} 0 \\ v+\psi \\ 0 \end{pmatrix}$$

and performing the gauge transformation $U = \exp(-\mathrm{i})[T_+\xi_+ + T_-\xi_-]$ on the scalar, vector and fermion fields. This is the U-gauge discussed previously. However in a later section we will need the Feynman rules in other gauges, and for this reason we have written the Lagrangian in terms of the (s^+, ϕ^o, s^-) fields, without eliminating the fictitious components.

Finally there may be fermion mass terms and fermion-scalar couplings. An invariant fermion mass term has the form

$$\mathcal{L}_{\text{mass}} = -m_o[\bar{L}R + \bar{R}L]$$
$$= -m_o[\overline{E^+}E^+ + \cos\beta\,\overline{E^o}E^o + \tfrac{1}{2}\sin\beta\,(\bar{\nu}(1+\gamma_5)E^o + \overline{E^o}(1-\gamma_5)\nu) + \bar{e}^-e^-]. \tag{9.21}$$

There are two possible invariant coupling terms

$$\mathcal{L}_{\text{coupling}} = G_1[\bar{L}(\mathrm{T}\cdot\phi)R + \text{H.C.}] + G_2[\overline{E^{\mathrm{o}}_{\mathrm{L}}}\sin\beta - \bar{\nu}_{\mathrm{L}}\cos\beta]\phi\cdot\mathrm{R} + \text{H.C.} \tag{9.22}$$

where $\phi\cdot\mathrm{R}$ means $s^+e^-_{\mathrm{R}} + s^-E^+_{\mathrm{R}} + \phi^{\mathrm{o}}E^{\mathrm{o}}_{\mathrm{R}}$. Replacing ϕ^{o} with v, we see that (9.22) contributes another fermion mass term to the Lagrangian. It is

$$G_1v[\overline{E^+}E^+ - \overline{R^-}e^-] + \sin\beta\, G_2v\,\overline{E^{\mathrm{o}}}E^{\mathrm{o}} - \tfrac{1}{2}\cos\beta\, G_2v[\bar{\nu}(1+\gamma_5)E^{\mathrm{o}} + \overline{E^{\mathrm{o}}}(1-\gamma_5)\nu]. \tag{9.23}$$

From (9.21) and (9.23), we obtain a fermion mass matrix, which we should diagonalize, and then impose the condition that the field we denoted by ν is indeed massless. (Since there is one more neutral left-handed fermion than right-handed fermion, there is bound to be a massless left-handed field.) This condition gives, from eqs. (9.21) and (9.23)

$$m_{\mathrm{o}}\sin\beta + G_2v\cos\beta = 0. \tag{9.24}$$

The heavy leptons which occur in the models we have discussed may actually be reasonably light, and, if they exist, may be discovered long before the heavy vector mesons. All we really know is that they are all heavier than the K meson. They can probably be produced most easily in colliding e^+e^- beam, which can set lower limits on their masses close to the beam energy. Reactions like $\nu + \mathrm{p} \to \mathrm{E}^+ +$ hadrons have also been studied, and appear to be feasible experiments at NAL energies. Decay modes like $\mathrm{E}^+ \to \mathrm{e}^+ + \nu_{\mathrm{e}} + \nu_{\mathrm{e}}$, $\mathrm{E}^+ \to \nu_{\mathrm{e}} + \mu^+ + \nu_\mu$, $\mathrm{E}^+ \to \mathrm{E}^{\mathrm{o}} +$ hadrons, or $\mathrm{E}^+ \to \nu_{\mathrm{e}} +$ hadrons, should all be easy to identify because of the apparent violation of momentum conservation. We have listed some recent references in the bibliography.

The masses of the fermions can be expressed in terms of m_{o}, G_1, G_2 and v:

$$m_{\mathrm{E}^+} = m_{\mathrm{o}} - G_1v$$

$$m_{\mathrm{E}^{\mathrm{o}}} = \cos\beta\, m_{\mathrm{o}} - \sin\beta\, G_2v$$

$$m_{\mathrm{e}^-} = m_{\mathrm{o}} + G_1v. \tag{9.25}$$

From the first and third equations in (9.25), we obtain

$$m_{\mathrm{o}} = \tfrac{1}{2}(m_{\mathrm{E}^+} + m_{\mathrm{e}^-}) \tag{9.26}$$

and from the remaining relation and (9.24), we obtain

$$m_{\mathrm{E}^+} + m_{\mathrm{e}^-} = 2\cos\beta\, M_{\mathrm{E}^{\mathrm{o}}} \tag{9.27}$$

which is a general constraint on the masses of the leptons in this model. Then from (9.19), (9.25), (9.26) and (9.27),

$$G_2 = -\frac{e}{\mu}\sin\beta\, M_{\mathrm{E}^{\mathrm{o}}} \tag{9.28}$$

and

$$G_1 = \frac{e}{\mu}\frac{M_{\mathrm{e}^-} - M_{\mathrm{E}^+}}{2} = \frac{e}{\mu}(m_{\mathrm{e}^-} - \cos\beta\, M_{\mathrm{E}^{\mathrm{o}}}). \tag{9.29}$$

Thus all the scalar-fermion couplings are fixed in terms of β and the e^- and E^{o} masses. Alternatively, β can be expressed through (9.27) in terms of the three masses. We shall use these results in Part II to calculate the anomalous magnetic moment of the muon of this model.

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10. More on model building

> A copy of the universe is not what is required of art;
> one of the damned thing is ample.
>
> Rebecca West

In this section we shall try to describe various ramifications of gauge models of weak and electromagnetic interactions based on O(3) or U(2), their defects, and possible other avenues in model building. We will not dwell upon any one idea in detail, but rather try to present a panoramic overview on these developments. Instead of presenting a long list of recent articles and preprints exhaustively, we will cite representative works that have been at least partly digested by us.

We have seen a few examples of models based on SU(2) or U(2) gauge symmetries. The basic strategy of model building may be stated as follows:

A. Choose a gauge group.
B. Choose the representation of the Higgs scalar fields and their charge assignments.
C. Choose the representations of the spin $\frac{1}{2}$ chiral fermions.
D. Couple the gauge fields invariantly to the Higgs scalars and the fermions.
E. Couple the Higgs fields to themselves invariantly and renormalizably, so that the potential of the Higgs fields attains the minimum when neutral Higgs fields acquire non-vanishing vacuum-expected values.
F. Couple the Higgs fields invariantly to the fermions.

When these steps are taken,

a. Some gauge bosons acquire masses:

$$\tfrac{1}{2}(\partial_\mu \phi + gW_\mu \phi)^2 \to \tfrac{1}{2} g^2 \langle\phi\rangle^2 W_\mu^2 .$$

b. Some fermions acquire masses:

$$f(\bar{\psi}_R \psi_L \phi + \text{h.c.}) \to f\langle\phi\rangle \bar{\psi}\psi .$$

c. At least one vector boson remains massless, because electric charge conservation is unbroken

d. Some of the Higgs fields undergo a transmutation: they turn into the longitudinal components of the massive vector bosons.

in this strategy, the left-handed lepton (e_L or μ_L) and its neutrino are placed in a multiplet of SU(2), the right-handed component to another multiplet, by inventing heavy leptons as they are needed. If the multiplets chosen are such that $Q = T_3$, a neutral massive vector boson is not needed, and the unification can be achieved in an O(3) framework. Otherwise we need an SU(2) × U(1) scheme. Bjorken and Llewellyn-Smith have considered many schemes of this type:

1. J.D. Bjorken and C.H. Llewellyn-Smith, Phys. Rev. D7 (1973) 887, Appendix A. So far, we have closed our eyes to the *CP*-violation in weak interactions.

There are a few attempts to incorporate it in a unified gauge model. See

2. R.N. Mohapatra, Phys. Rev. D6 (1972) 2023.
3. A. Pais, Phys. Rev. Letters 29 (1973) 1712.

The latter scheme is based on the O(4) gauge group, which deserves attention on its own right.

Quite apart from this line of development, the Higgs mechanism provides us with a means of constructing renormalizable models of strong interactions based on the notion of "field algebra":

4. T.D. Lee, S. Weinberg and B. Zumino, Phys. Rev. Letters 18 (1967) 1029.

The field algebra is the field theoretic expression for vector dominance, by equating the hadronic currents with massive gauge bosons. In the past, the mass term for the gauge bosons was put in "by hand" – such a procedure breaks the renormalizability of the theory. The Higgs mechanism allows endowing the gauge bosons with masses. This was first noticed by 't Hooft;

5. G. 't Hooft, Nucl. Phys. B35 (1971) 167.

and has since been generalized and elaborated on:

6. B.W. Lee and J. Zinn-Justin, Phys. Rev. D5 (1972) 3137, Appendix.
7. K. Bardakci and M.B. Halpern, Phys. Rev. D6 (1972) 696.

These are a number of applications of this idea to hadron physics. For example 't Hooft discussed the $\pi^\pm - \pi^0$ mass difference from this point of view. For other applications, see

8. K. Bardakci, to be published.
9. H. Georgi and T. Goldman, Phys. Rev. Letters 30 (1973) 514.
10. D.Z. Freedman and W. Kummer, Phys. Rev. D7 (1973) 1829.
11. A. Duncan and P. Schattner, Phys. Rev. D7 (1973) 1861.

There have been many attempts to incorporate three triplets of hadronic building blocks (such as the Han-Nambu, or three-color-quark schemes) which seem better suited to correlate various facets of hadron physics. See

12. H. Lipkin, Phys. Rev. Letters 28 (1972) 63.
13. H. Georgi and S.L. Glashow, Phys. Rev. DI (1973) 561.
14. M. Tonin, preprint.
15. Y. Achiman, Weinberg's Gauge Model for Weak and Electromagnetic Interactions with Han-Nambu Quarks, Heidelberg preprint.
16. M.A.B. Beg and A. Zee, Phys. Rev. Letters 30 (1973) 675.

The defect of the models discussed in previous sections is their inability to accomodate hadrons in a realistic, and "natural" manner. Let us illustrate this remark in terms of the scheme discussed in section 8, in which the quartet of spin $\frac{1}{2}$ fundamental hadronic building blocks is incorporated in the Weinberg-Salam model. The necessity of including four, rather than three, such objects arose from the absence of the $\Delta S = \pm 1$ neutral current, and this fact should not be considered as a defect. Rather, it must be considered as heralding, possibly, a new dimension in hadron spectroscopy, with a new quantum number associated with the "fourth quark". The defect lies in that the approximate hadronic symmetries such as SU(2), SU(3) or chiral SU(2) × SU(2) are purely accidental in this scheme. For example, the hadronic isospin symmetry SU(2) has to be explained in this scheme as a consequence of an approximate equality of m_p and m_n, which is not demanded by the gauge or other symmetries of the Lagrangian. It has long been the conviction (prejudice?) of particle physicists that the proton-neutron mass difference is due to electromagnetism and possibly also due to weak interaction, so that in an ultimate theory the mass difference should be computable. In the model under discussion, this mass difference is not zero even in lowest order, but is a free parameter.

The following papers discuss various conditions and circumstances under which intramultiplet mass differences are computable, as well as the definition of computability:

17. S. Weinberg, Phys. Rev. Letters 29 (1972) 388.
18. H. Georgi and S.L. Glashow, Phys. Rev. D6 (1972) 2977.
19. T. Hagiwara and B.W. Lee, Phys. Rev. D7 (1973) 459.
20. H. Georgi and S.L. Glashow, Phys. Rev. D8 (1973) 2457.

The central idea underlying these discussions is that any relationship which is true in lowest order in the presence of all gauge invariant, renormalization counterterms is also true in high orders with a finite computable correction.

Thus, if the mass difference within a hadronic multiplet is to be computable, the underlying hadron symmetry must not be broken by any renormalization counterterms in the Lagrangian. Future developments in model building ought to lie in the construction of models in which hadronic symmetries are accounted for naturally. There have been two important developments in this direction.

The first is the works of Bars, Halpern and Yoshimura and of de Wit:

21. I. Bars, M.B. Halpern and M. Yoshimura, Phys. Rev. Letters 29 (1972) 969.
22. B. de Wit, Nucl. Phys. B51 (1973) 237.

The models proposed by these authors treat the hadronic and leptonic worlds as separate up to a point, each having its own set of gauge bosons; the two worlds communicate to one another through the intermediary of a new kind of Higgs mesons which carry both leptonic and hadronic quantum numbers and whose vacuum expectation values are responsible for the coupling of the two kinds of gauge bosons, in much the same way as in the field algebra. The following work is very similar to the above two in this respect:

23. J.C. Pati and A. Salam, Phys. Rev. D8 (1973) 1240.

The second is perhaps more profound in its concept. Weinberg notes that under certain circumstances the potential of the Higgs scalar fields cannot help but having a symmetry $\overline{G}$ larger than the gauge symmetry of weak and electromagnetic interaction). If the symmetry $\overline{G}$ is spontaneously broken so that the vacuum expectation value of the scalar fields, determined by minimizing the potential, leaves the subgroup S, $S \subset \overline{G}$ unbroken, then, in *lowest order,* there are Goldstone bosons corresponding to the generators of the cosets $\overline{G}/S$. Presumably in a realistic theory, the intersection $G \cap S$ is just the U(1) corresponding to the electric charge conservation. The Goldstone bosons corresponding to the remaining generators of the gauge group G are the unphysical Higgs scalars which become the longitudinal components of the massive vector bosons. The remaining Goldstone bosons which do not correspond to any generators of the group G of

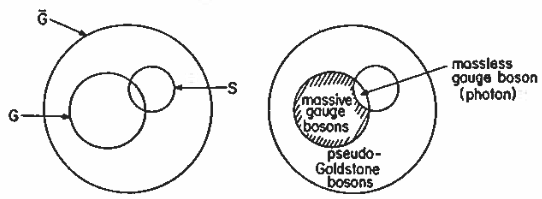

Fig. 10.1. Diagrammatic representation of Lie algebras G, $\bar{G}$ and S, and their correspondence to massive and massless gauge bosons and pseudo-Goldstone bosons.

the entire Lagrangian then acquire computable masses in higher order due to the fact that the pseudosymmetry $\bar{G}$ is broken down by weak and electromagnetic interactions, and are called pseudo-Goldstone bosons. See fig. 10.1.

24. S. Weinberg, Phys. Rev. Letters 28 (1972) 1698.

25. S. Weinberg, Phys. Rev. D7 (1973) 2887.

The idea here is that $\bar{G}$ includes some approximate hadronic symmetry, and the pseudo-Goldstone bosons discussed here are the would-be Goldstone bosons (such as pions) seen in nature. This view has many very profound implications on the nature of hadronic symmetries and their breaking. So far no realistic model has been written down which realizes this view.