Advanced Quantum Field Theory
1º Semester 2018/2019

I give here a brief description of the proposed topics. You should write a small text on the chosen topic and make an oral presentation of 20 minutes. As we discussed you send to me the small text a week before the presentation. These will take place on February 7th so that I can deliver the grades until Friday the 8th.

IST, 20 of December, 2018
Jorge C. Romão

1 Renormalization Group and Unified Theories

Objective: Evaluate the evolution of the coupling constants using the renormalization group, for the Standard Model and for the MSSM. The student should do the explicit calculations for the coefficients of the β functions that are needed.

Bibliography:

- *Advanced Quantum Field Theory* , Jorge C. Romão, Chapter 7.

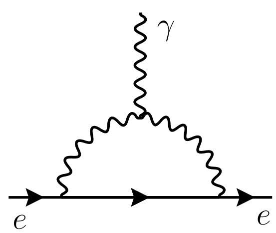
Student:

2 QED in a non-linear gauge

Consider QED with a non-linear gauge condition,

$$F = \partial_\mu A^\mu + \frac{\lambda}{2} \, A_\mu A^\mu \ .$$

1. Write $\mathcal{L}_{eff}$ and show that $s\mathcal{L}_{eff} = 0$, where s is the Slavnov operator.
2. Write the Feynman rules for the new vertices and propagators. Then evaluate at tree level $\gamma + \gamma \to \gamma + \gamma$. Compare with the results in the usual linear gauge.
3. Evaluate the vacuum polarization at one-loop.
4. Show that the diagram of the figure, that would be potentially dangerous for the anomalous magnetic moment of the electron (would be proportional to λ) vanishes.



Student:

3 Vacuum Polarization in QCD

Consider the theory that describes the interactions of quarks with gluons (QCD) given by the Lagrangian:

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \sum_{\alpha=1}^{n} \overline{\psi}_i^\alpha (i\not{D} - m_\alpha)_{ij} \psi_j^\alpha$$

where

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \\ (D_\mu)_{ij} &= \delta_{ij}\partial_\mu - ig \left(\frac{\lambda^a}{2}\right)_{ij} A^a_\mu \ . \end{aligned}$$

The index $\alpha = 1, 2, \ldots, n$ denotes the different quark flavours (*up*,*down*,...,*top*). In order to quantize the theory use the linear gauge condition,

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} \ (\partial_\mu A^{\mu a})^2 \ ,$$

that gives the following Lagrangian for the ghosts,

$$\mathcal{L}_G = \partial_\mu \overline{\omega}^a \partial^\mu \omega^a + g f^{abc} \partial^\mu \overline{\omega}^a A^b_\mu \omega^c$$

To renormalize the theory one needs the following counter-term Lagrangian,

$$\begin{aligned} \Delta\mathcal{L} \ = \ & -\frac{1}{4}(Z_3 - 1)\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - (Z_4 - 1) g f^{abc} \partial_\mu A^a_\nu A^{\mu b} A^{\nu c} \\ & -\frac{1}{4} g^2 (Z_5 - 1) f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e} + \sum_\alpha (Z_2 - 1) i \overline{\psi}_i^\alpha \gamma^\mu \partial_\mu \psi_i^\alpha \\ & - \sum_\alpha m_\alpha (Z_{m_\alpha} - 1) \overline{\psi}_i^\alpha \psi_i^\alpha + (Z_1 - 1) g \sum_\alpha \overline{\psi}_i^\alpha \gamma^\mu \left(\frac{\lambda^a}{2}\right)_{ij} \psi_j^\alpha A^a_\mu \\ & + (Z_6 - 1) \partial_\mu \overline{\omega}^a \partial^\mu \omega^a + (Z_7 - 1) g f^{abc} \partial^\mu \overline{\omega}^a A^b_\mu \omega^c \ . \end{aligned}$$

1. Verify the expression for $\mathcal{L}_G$.
2. Verify the Feynman rules given in the text.
3. Evaluate the vacuum polarization.

Student: Lorenzo Annulli

4 Renormalization if QCD

Consider the theory that describes the interactions of quarks with gluons (QCD) given by the Lagrangian:

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \sum_{\alpha=1}^{n} \overline{\psi}^{\alpha}_i (i\not{D} - m_\alpha)_{ij} \psi^{\alpha}_j$$

where

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \\ (D_\mu)_{ij} &= \delta_{ij}\partial_\mu - ig \left(\frac{\lambda^a}{2}\right)_{ij} A^a_\mu \ . \end{aligned}$$

The index $\alpha = 1, 2, \ldots, n$ denotes the different quark flavours (*up*,*down*,...,*top*). In order to quantize the theory use the linear gauge condition,

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1. Show that the following relations must be true

$$\frac{Z_1}{Z_2} = \frac{Z_4}{Z_3} = \frac{Z_7}{Z_6} = \frac{\sqrt{Z_5}}{\sqrt{Z_3}}$$

2. Evaluate Z_1, Z_2, Z_3, Z_4, Z_6 e Z_7 using Minimal Subtraction (MS). Show explicitly that $Z_1 Z_6 = Z_2 Z_7$.

3. Evaluate the contribution of the fermions to Z_4 and Z_5. Show that they are in agreement with the previous relations.

Student:

5 Renormalization of Scalar Electrodynamics

Consider Scalar Electrodynamics, that is the gauge theory of interactions of photons with charged scalar particles.

1. Write the Lagrangian for this theory.
2. Derive the Feynman rules.
3. Identify the divergent diagrams.
4. Do the on-shell renormalization for the self-energies of the photon and charged scalar particle.

Student:

6 Unitarity in Non-Abelian Gauge Theories

Consider the theory that describes the interactions of quarks with gluons (QCD) given by the Lagrangian:

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \sum_{\alpha=1}^{n} \overline{\psi}^\alpha_i (i\not{D} - m_\alpha)_{ij} \psi^\alpha_j$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$
$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - ig \left(\frac{\lambda^a}{2}\right)_{ij} A^a_\mu \ .$$

The index $\alpha = 1, 2, \ldots, n$ denotes the different quark flavours (*up*,*down*,...,*top*). In order to quantize the theory use the linear gauge condition,

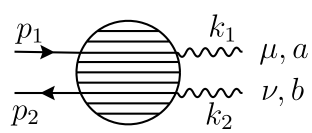
$$\mathcal{L}_{GF} = -\frac{1}{2\xi} \ (\partial_\mu A^{\mu a})^2 \ ,$$

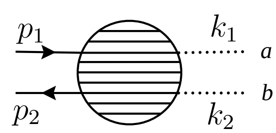
that gives the following Lagrangian for the ghosts,

$$\mathcal{L}_G = \partial_\mu \overline{\omega}^a \partial^\mu \omega^a + g f^{abc} \partial^\mu \overline{\omega}^a A^b_\mu \omega^c$$

1. Verify the expression for $\mathcal{L}_G$.
2. Show explicitly that the action is BRS invariant.

   Consider now the amplitudes

$iT^{ab}_{\mu\nu} \equiv$ 

$iT^{ab} \equiv$ 

3. Evaluate $T^{ab}_{\mu\nu}$ at tree level. Verify that, for off-shell gluons, we have $k_1^\mu T^{ab}_{\mu\nu} \neq 0$. What happens for on-shell gluons?
4. Verify, at tree level, the Ward identities

$$k_1^\mu \ T^{ab}_{\mu\nu} = k_2^\nu \ T^{ab}$$

5. Use the above results to explicitly prove unitarity at one-loop level, showing that the optical theorem holds in this case.

Student: Francisco Faro

7 Feynman Rules for QED using the Path Integral

The generating functional for the Green functions in QED is given by,

$$Z(J_\mu, \eta, \overline{\eta}) = \int \mathcal{D}(A_\mu, \psi, \overline{\psi})\; e^{i\int d^4x(\mathcal{L}_{QED}+\mathcal{L}_{GF}+J^\mu A_\mu+\overline{\eta}\psi+\overline{\psi}\eta)}\ . \tag{1}$$

where

$$\begin{aligned}
\mathcal{L}_{QED} &= -\frac{1}{4}\; F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi \\
\mathcal{L}_{GF} &= -\frac{1}{2\xi}(\partial \cdot A)^2 \\
D_\mu &= \partial_\mu + ieA_\mu\ .
\end{aligned}$$

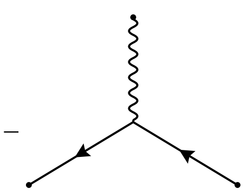
a) Determine $Z_0[J^\mu, \eta, \overline{\eta}]$
b) Show that

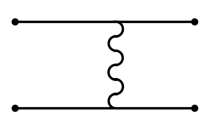
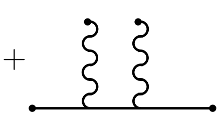
$$Z[J^\mu, \eta, \overline{\eta}] = \exp\left\{(-ie)\int d^4x \frac{\delta}{\delta\eta_\alpha(x)}\,(\gamma^\mu)_{\alpha\beta}\,\frac{\delta}{\delta\overline{\eta}_\beta(x)}\frac{\delta}{\delta J_\mu(x)}\right\} Z_0[J^\mu, \eta, \overline{\eta}]\ . \tag{2}$$

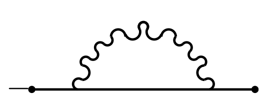
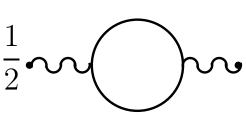
c) Expand

$$Z = Z_0\left[1 + (-ie)Z_1 + (-ie)^2 Z_2 + \cdots\right] \tag{3}$$

where we have subtracted the vacuum-vacuum amplitudes in Z_i, that is, $Z_i[0] = 0 \rightarrow Z[0] = 1$. Show that

$Z_1 = -$  (4)

$Z_2 = \frac{1}{2}Z_1^2$ $+$ $\frac{1}{2}$  $+$ 

 $+$ $\frac{1}{2}$  (5)

d) Discuss the numerical factors and signs of the previous diagrams.
e) Evaluate in lowest order

$$\langle 0|\, T A^\mu(x)\psi_\beta(y)\overline{\psi}_\alpha(z)\,|0\rangle = \frac{\delta^3 Z}{i\delta\eta_\alpha(z) i\delta\overline{\eta}_\beta(y) i\delta J_\mu(x)} \tag{6}$$

and verify that it coincides with the Feynman rules for the vertex

f) Determine the amplitude for the Compton scattering in lowest order, that is,

$$\langle 0| TA^{\mu}(x)A^{\nu}(y)\psi_{\beta}(z)\overline{\psi}_{\alpha}(w)|0\rangle = \frac{\delta^4 Z}{i\delta\eta_{\alpha}(w)i\delta\overline{\eta}_{\beta}(z)i\delta J_{\nu}(y)i\delta J_{\mu}} \tag{7}$$

and verify that it reproduces the result obtained from the usual Feynman rules.

Student: Miguel Oliveira

8 β Function in a general SU(N) gauge theory: AAA

8.1 Definitions

We define here the theory to have all conventions consistent. Notice that some of these conventions differ from the textbook.

Classical Theory

Consider the non-abelian **SU(N)** gauge theory defined by its classical Lagrangian,

$$\mathcal{L}_{\rm SU(n)} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \overline{\psi}_i(i\not{D} - m_F)_{ij}\psi_j + (D_\mu\phi)^\dagger_i D^\mu\phi_i - m_S^2\phi^\dagger_i\phi_i \tag{8}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc}A^b_\mu A^c_\nu \tag{9}$$

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu + ig\,(T^a)_{ij}\,A^a_\mu\ . \tag{10}$$

and T^a_{ij} are the generators in the representation to which the fermion and scalar belong (possibly different ones). To quantify the theory we have to introduce the gauge fixing term, that we choose to be of the form,

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}\ (\partial_\mu A^{\mu a})^2\ , \tag{11}$$

for which we have the following ghost Lagrangian,

$$\mathcal{L}_G = \partial_\mu\overline{\omega}^a\partial^\mu\omega^a - g f^{abc}\partial^\mu\overline{\omega}^a A^b_\mu\omega^c\ . \tag{12}$$

Counterterm Lagrangian

To renormalize the theory we need the following counterterm Lagrangian

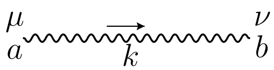
$$\begin{aligned}\Delta\mathcal{L} \ =\ & -\frac{1}{4}(Z_{\rm A}-1)\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + (Z_{\rm AAA}-1)g f^{abc}\partial_\mu A^a_\nu A^{\mu b}A^{\nu c}\\ & -\frac{1}{4}g^2(Z_{\rm AAAA}-1)f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e} + (Z_{\rm F}-1)i\overline{\psi}_i\gamma^\mu\partial_\mu\psi_i - m_F(Z_{m_\psi}-1)\overline{\psi}_i\psi_i\\ & -(Z_{\psi\psi\rm A}-1)g\,\overline{\psi}_i\gamma^\mu\,(T^a)_{ij}\,\psi_j A^a_\mu + (Z_{\rm G}-1)\partial_\mu\overline{\omega}^a\partial^\mu\omega^a - (Z_{\rm GGA}-1)g f^{abc}\partial^\mu\overline{\omega}^a A^b_\mu\omega^c\\ & +(Z_S-1)\partial_\mu\phi^*_i\partial^\mu\phi_i - m_S^2(Z_S-1)\phi^*_i\phi_i - (Z_{SSA}-1)\left[igA^a_\mu\phi^\dagger_i T^a_{ij}\partial^\mu\phi_j - igA^a_\mu\partial^\mu\phi^\dagger_i T^a_{ij}\phi_j\right]\\ & +(Z_{SSAA}-1)g^2\phi^\dagger_i T^a_{ij}\phi_j A^a_\mu\phi^\dagger_k T^b_{km}\phi_m A^{bmu}\ .\end{aligned} \tag{13}$$

Feynman Rules

For completeness we give here the Feynman rules for this theory.

- **Propagators:**

i) Gauge bosons

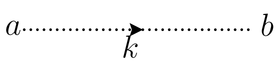


$$-i\delta_{ab}\left[\frac{g^{\mu\nu}}{k^2+i\epsilon}-(1-\xi)\,\frac{k^\mu k^\nu}{(k^2+i\epsilon)^2}\right]\equiv i\delta_{ab}\frac{N^{\mu\nu}(k,\xi)}{k^2+i\epsilon} \tag{14}$$

where, for future use, we have defined the numerator of the propagator in an arbitrary R_ξ gauge as,

$$N^{\mu\nu}(k,\xi)\equiv-\left[g^{\mu\nu}-(1-\xi)\,\frac{k^\mu k^\nu}{k^2}\right] \tag{15}$$

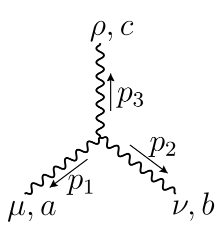
ii) Ghosts



$$\frac{i}{k^2+i\epsilon}\;\delta_{ab} \tag{16}$$

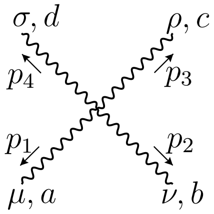
- **Vertices:**

i)Triple gauge boson vertex



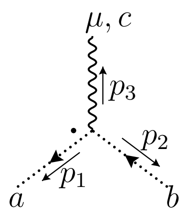
$$\begin{aligned} gf^{abc}[\quad & g^{\mu\nu}(p_1-p_2)^\rho+g^{\nu\rho}(p_2-p_3)^\mu \\ & +g^{\rho\mu}(p_3-p_1)^\nu]\equiv\Gamma^{\mu\nu\rho}_{abc}(p_1,p_2,p_3) \\ & p_1+p_2+p_3=0 \end{aligned} \tag{17}$$

ii) Quartic gauge boson vertex



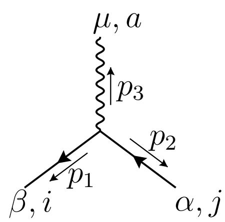
$$\begin{aligned} -ig^2\Big[\quad & f_{eab}f_{ecd}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) \\ & +f_{eac}f_{edb}(g^{\mu\sigma}g^{\rho\nu}-g^{\mu\nu}g^{\rho\sigma}) \\ & +f_{ead}f_{ebc}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\Big]\equiv\Gamma^{\mu\nu\rho\sigma}_{abcd} \\ & p_1+p_2+p_3+p_4=0 \end{aligned} \tag{18}$$

iii) Ghost-Gauge boson interaction



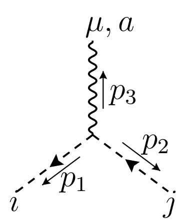
$$\begin{aligned} & -g\ f^{abc} p_1^{\mu} \equiv \Gamma^{\mu}_{abc}(p_1) \\ p_1 & + p_2 + p_3 = 0 \end{aligned} \tag{19}$$

iv) Fermion-Gauge boson interaction



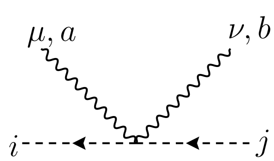
$$-ig(\gamma^{\mu})_{\beta\alpha} T^a_{ij} \tag{20}$$

v) Scalar-Gauge boson interaction: Cubic term



$$-ig(p_1 - p_2)^{\mu} T^a_{ij} \equiv V^{\mu}_S(p_1, p_2) T^a_{ij} \tag{21}$$

vi) Scalar-Gauge boson interaction: Quartic term



$$i g^2 g_{\mu\nu} \{T^a, T^b\}_{ij} \tag{22}$$

Notice that in the definition of $\Gamma^{\mu\nu\rho}_{abc}(p_1, p_2, p_3)$ in Eq. (95) all the momenta are outgoing. This explains the different sign when comparing with Ref.[1].

Group factors

We summarize here some useful formulas for dealing with some group theory factors. Our generators obey the defining commutation relations,

$$\left[T^a, T^b\right] = i\, f^{abc} T^c \tag{23}$$

where the structure constants of the Lie group G are completely antisymmetric, and the generators in a representation R of G are normalized as follows,

$$\mathrm{Tr}[T^a T^b] = T_R \delta^{ab} \tag{24}$$

The structure constants obey the Jacobi identity

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0 \tag{25}$$

and we define the two Casimir invariants

$$f^{abd} f^{dbc} = C_A \delta^{ad}, \quad T^a T^a = C_R \mathbf{1} \ . \tag{26}$$

Useful relations are,

$$T_R \, r = d_R \, C_A \tag{27}$$

$$\mathrm{Tr}[T^a T^b T^c] - \mathrm{Tr}[T^a T^c T^b] = i \, T_R \, f^{abc} \tag{28}$$

$$T^{abcd} + T^{abdc} + T^{acdb} + T^{adcb} - 2T^{acbd} - 2T^{adbc} = T_R \left(f^{ade} f^{bce} + f^{ace} f^{bde} \right) \ , \tag{29}$$

where r is the dimension of G, d_R the dimension of the representation R of G and $T^{abcd} \equiv \mathrm{Tr}[T^a T^b T^c T^d]$. For $SU(N)$ we have,

$$r = N^2 - 1 \qquad\qquad d_N = N \tag{30}$$

$$T_N = \frac{1}{2} \qquad\qquad C_N = \frac{N^2-1}{2N} \tag{31}$$

$$C_A = T_{Adj} = N \qquad\qquad . \tag{32}$$

8.2 Calculate the β function

The β function can be obtained in many ways. Here use as starting point

$$Z_g = Z_{AAA} \, Z_A^{-3/2} \tag{33}$$

In all calculations consider the gauge with $\xi = 1$ and evaluate the counter-terms using the MS scheme (just consider the coefficient of the pole).

1. Calculate the pure gauge contribution to Z_A at one loop.
2. Calculate the pure gauge contribution to Z_{AAA} at one loop.
3. Calculate the fermion contribution to Z_A and Z_{AAA} at one loop.
4. Calculate the scalar contribution to Z_A and Z_{AAA} at one loop.
5. Finally evaluate the β function for this theory. Check that you recover Eq. 7.181 of the textbook.

Student:

9 β Function in a general SU(N) gauge theory: FFA

9.1 Definitions

We define here the theory to have all conventions consistent. Notice that some of these conventions differ from the textbook.

Classical Theory

Consider the non-abelian **SU**(**N**) gauge theory defined by its classical Lagrangian,

$$\mathcal{L}_{\rm SU(n)} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \overline{\psi}_i(i\not{D} - m_F)_{ij}\psi_j + (D_\mu\phi)^\dagger_i D^\mu\phi_i - m_S^2\phi_i^\dagger\phi_i \tag{34}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc}A^b_\mu A^c_\nu \tag{35}$$

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu + ig\,(T^a)_{ij}\,A^a_\mu\ . \tag{36}$$

and T^a_{ij} are the generators in the representation to which the fermion and scalar belong (possibly different ones). To quantify the theory we have to introduce the gauge fixing term, that we choose to be of the form,

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}\ (\partial_\mu A^{\mu a})^2\ , \tag{37}$$

for which we have the following ghost Lagrangian,

$$\mathcal{L}_G = \partial_\mu\overline{\omega}^a\partial^\mu\omega^a - g f^{abc}\partial^\mu\overline{\omega}^a A^b_\mu\omega^c\ . \tag{38}$$

Counterterm Lagrangian

To renormalize the theory we need the following counterterm Lagrangian

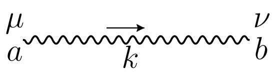
$$\begin{aligned}\Delta\mathcal{L} \ =\ & -\frac{1}{4}(Z_{\rm A}-1)\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + (Z_{\rm AAA}-1)g f^{abc}\partial_\mu A^a_\nu A^{\mu b}A^{\nu c} \\ & -\frac{1}{4}g^2(Z_{\rm AAAA}-1)f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e} + (Z_{\rm F}-1)i\overline{\psi}_i\gamma^\mu\partial_\mu\psi_i - m_F(Z_{m_\psi}-1)\overline{\psi}_i\psi_i \\ & -(Z_{\psi\psi\rm A}-1)g\,\overline{\psi}_i\gamma^\mu\,(T^a)_{ij}\,\psi_j A^a_\mu + (Z_{\rm G}-1)\partial_\mu\overline{\omega}^a\partial^\mu\omega^a - (Z_{\rm GGA}-1)g f^{abc}\partial^\mu\overline{\omega}^a A^b_\mu\omega^c \\ & +(Z_S-1)\partial_\mu\phi_i^*\partial^\mu\phi_i - m_S^2(Z_S-1)\phi_i^*\phi_i - (Z_{SSA}-1)\left[igA^a_\mu\phi_i^\dagger T^a_{ij}\partial^\mu\phi_j - igA^a_\mu\partial^\mu\phi_i^\dagger T^a_{ij}\phi_j\right] \\ & +(Z_{SSAA}-1)g^2\phi_i^\dagger T^a_{ij}\phi_j A^a_\mu\phi_k^\dagger T^b_{km}\phi_m A^{bmu}\ .\end{aligned} \tag{39}$$

Feynman Rules

For completeness we give here the Feynman rules for this theory.

- **Propagators:**

i) Gauge bosons



$$-i\delta_{ab}\left[\frac{g^{\mu\nu}}{k^2+i\epsilon}-(1-\xi)\,\frac{k^\mu k^\nu}{(k^2+i\epsilon)^2}\right]\equiv i\delta_{ab}\frac{N^{\mu\nu}(k,\xi)}{k^2+i\epsilon} \tag{40}$$

where, for future use, we have defined the numerator of the propagator in an arbitrary R_ξ gauge as,

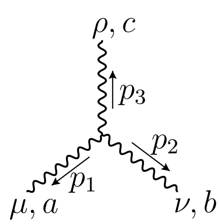
$$N^{\mu\nu}(k,\xi)\equiv-\left[g^{\mu\nu}-(1-\xi)\,\frac{k^\mu k^\nu}{k^2}\right] \tag{41}$$

ii) Ghosts

$$\frac{i}{k^2+i\epsilon}\;\delta_{ab} \tag{42}$$

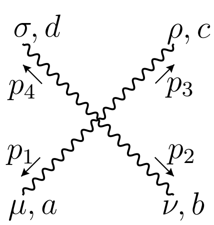
- **Vertices:**

i)Triple gauge boson vertex



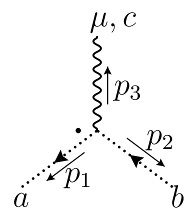
$$\begin{aligned} g f^{abc}[\quad & g^{\mu\nu}(p_1-p_2)^\rho+g^{\nu\rho}(p_2-p_3)^\mu \\ & +g^{\rho\mu}(p_3-p_1)^\nu]\equiv\Gamma^{\mu\nu\rho}_{abc}(p_1,p_2,p_3) \\ & p_1+p_2+p_3=0 \end{aligned} \tag{43}$$

ii) Quartic gauge boson vertex



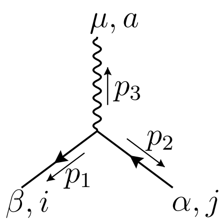
$$\begin{aligned} -ig^2\Big[\quad & f_{eab}f_{ecd}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) \\ & +f_{eac}f_{edb}(g^{\mu\sigma}g^{\rho\nu}-g^{\mu\nu}g^{\rho\sigma}) \\ & +f_{ead}f_{ebc}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\Big]\equiv\Gamma^{\mu\nu\rho\sigma}_{abcd} \\ & p_1+p_2+p_3+p_4=0 \end{aligned} \tag{44}$$

iii) Ghost-Gauge boson interaction



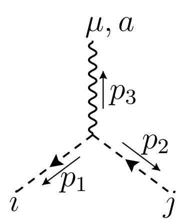
$$-g\ f^{abc} p_1^\mu \equiv \Gamma^\mu_{abc}(p_1)$$
$$p_1 \quad + p_2 + p_3 = 0 \tag{45}$$

iv) Fermion-Gauge boson interaction



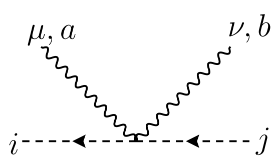
$$-ig(\gamma^\mu)_{\beta\alpha} T^a_{ij} \tag{46}$$

v) Scalar-Gauge boson interaction: Cubic term



$$-ig(p_1 - p_2)^\mu T^a_{ij} \equiv V^\mu_S(p_1, p_2) T^a_{ij} \tag{47}$$

vi) Scalar-Gauge boson interaction: Quartic term



$$ig^2 g_{\mu\nu} \{T^a, T^b\}_{ij} \tag{48}$$

Notice that in the definition of $\Gamma^{\mu\nu\rho}_{abc}(p_1, p_2, p_3)$ in Eq. (95) all the momenta are outgoing. This explains the different sign when comparing with Ref.[1].

Group factors

We summarize here some useful formulas for dealing with some group theory factors. Our generators obey the defining commutation relations,

$$\left[T^a, T^b\right] = i\, f^{abc} T^c \tag{49}$$

where the structure constants of the Lie group G are completely antisymmetric, and the generators in a representation R of G are normalized as follows,

$$\mathrm{Tr}[T^aT^b] = T_R\delta^{ab} \tag{50}$$

The structure constants obey the Jacobi identity

$$f^{abd}f^{dce} + f^{bcd}f^{dae} + f^{cad}f^{dbe} = 0 \tag{51}$$

and we define the two Casimir invariants

$$f^{abd}f^{dbc} = C_A\delta^{ad}, \quad T^aT^a = C_R\,\mathbf{1}\ . \tag{52}$$

Useful relations are,

$$T_R\ r = d_R\ C_A \tag{53}$$

$$\mathrm{Tr}[T^aT^bT^c] - \mathrm{Tr}[T^aT^cT^b] = i\,T_R\,f^{abc} \tag{54}$$

$$T^{abcd} + T^{abdc} + T^{acdb} + T^{adcb} - 2T^{acbd} - 2T^{adbc} = T_R\left(f^{ade}f^{bce} + f^{ace}f^{bde}\right)\ , \tag{55}$$

where r is the dimension of G, d_R the dimension of the representation R of G and $T^{abcd} \equiv \mathrm{Tr}[T^aT^bT^cT^d]$. For $SU(N)$ we have,

$$r = N^2 - 1 \qquad\qquad d_N = N \tag{56}$$

$$T_N = \frac{1}{2} \qquad\qquad C_N = \frac{N^2-1}{2N} \tag{57}$$

$$C_A = T_{Adj} = N \qquad\qquad . \tag{58}$$

9.2 Calculate the β function

The β function can be obtained in many ways. Here use as starting point

$$Z_g = Z_{FFA}\,Z_A^{-1/2}\,Z_F^{-1} \tag{59}$$

In all calculations consider the gauge with $\xi = 1$ and evaluate the counter-terms using the MS scheme (just consider the coefficient of the pole).

1. Calculate the pure gauge contribution to Z_A at one loop.
2. Calculate Z_{FFA} and Z_F at one loop.
3. Calculate the fermion contribution to Z_A at one loop.
4. Calculate the scalar contribution to Z_A at one loop.
5. Finally evaluate the β function for this theory. Check that you recover Eq. 7.181 of the textbook.

Student:

10 β Function in a general SU(N) gauge theory: GGA

10.1 Definitions

We define here the theory to have all conventions consistent. Notice that some of these conventions differ from the textbook.

Classical Theory

Consider the non-abelian **SU(N)** gauge theory defined by its classical Lagrangian,

$$\mathcal{L}_{\mathrm{SU(n)}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \overline{\psi}_i (i\not{D} - m_F)_{ij} \psi_j + (D_\mu \phi)^\dagger_i D^\mu \phi_i - m_S^2 \phi^\dagger_i \phi_i \tag{60}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \tag{61}$$

$$(D_\mu)_{ij} = \delta_{ij} \partial_\mu + ig \left(T^a\right)_{ij} A^a_\mu \ . \tag{62}$$

and T^a_{ij} are the generators in the representation to which the fermion and scalar belong (possibly different ones). To quantify the theory we have to introduce the gauge fixing term, that we choose to be of the form,

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} \ (\partial_\mu A^{\mu a})^2 \ , \tag{63}$$

for which we have the following ghost Lagrangian,

$$\mathcal{L}_G = \partial_\mu \overline{\omega}^a \partial^\mu \omega^a - g f^{abc} \partial^\mu \overline{\omega}^a A^b_\mu \omega^c \ . \tag{64}$$

Counterterm Lagrangian

To renormalize the theory we need the following counterterm Lagrangian

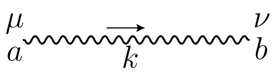
$$\begin{aligned}
\Delta\mathcal{L} \ = \ & -\frac{1}{4}(Z_{\mathrm{A}} - 1) \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + (Z_{\mathrm{AAA}} - 1) g f^{abc} \partial_\mu A^a_\nu A^{\mu b} A^{\nu c} \\
& -\frac{1}{4} g^2 (Z_{\mathrm{AAAA}} - 1) f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e} + (Z_{\mathrm{F}} - 1) i \overline{\psi}_i \gamma^\mu \partial_\mu \psi_i - m_F (Z_{m_\psi} - 1) \overline{\psi}_i \psi_i \\
& -(Z_{\psi\psi\mathrm{A}} - 1) g \, \overline{\psi}_i \gamma^\mu \left(T^a\right)_{ij} \psi_j A^a_\mu + (Z_{\mathrm{G}} - 1) \partial_\mu \overline{\omega}^a \partial^\mu \omega^a - (Z_{\mathrm{GGA}} - 1) g f^{abc} \partial^\mu \overline{\omega}^a A^b_\mu \omega^c \\
& +(Z_S - 1) \partial_\mu \phi^*_i \partial^\mu \phi_i - m_S^2 (Z_S - 1) \phi^*_i \phi_i - (Z_{SSA} - 1) \left[ig A^a_\mu \phi^\dagger_i T^a_{ij} \partial^\mu \phi_j - ig A^a_\mu \partial^\mu \phi^\dagger_i T^a_{ij} \phi_j \right] \\
& +(Z_{SSAA} - 1) g^2 \phi^\dagger_i T^a_{ij} \phi_j A^a_\mu \phi^\dagger_k T^b_{km} \phi_m A^{bmu} \ .
\end{aligned} \tag{65}$$

Feynman Rules

For completeness we give here the Feynman rules for this theory.

- **Propagators:**

i) Gauge bosons



$$-i\delta_{ab}\left[\frac{g^{\mu\nu}}{k^2+i\epsilon}-(1-\xi)\,\frac{k^\mu k^\nu}{(k^2+i\epsilon)^2}\right]\equiv i\delta_{ab}\frac{N^{\mu\nu}(k,\xi)}{k^2+i\epsilon} \tag{66}$$

where, for future use, we have defined the numerator of the propagator in an arbitrary R_ξ gauge as,

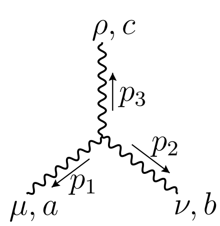
$$N^{\mu\nu}(k,\xi)\equiv-\left[g^{\mu\nu}-(1-\xi)\,\frac{k^\mu k^\nu}{k^2}\right] \tag{67}$$

ii) Ghosts

$$\frac{i}{k^2+i\epsilon}\;\delta_{ab} \tag{68}$$

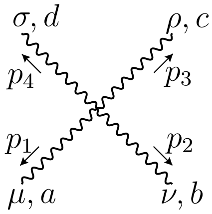
- **Vertices:**

i)Triple gauge boson vertex



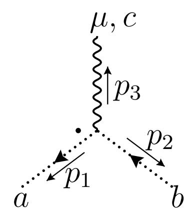
$$\begin{aligned} g f^{abc}[\;\; & g^{\mu\nu}(p_1-p_2)^\rho+g^{\nu\rho}(p_2-p_3)^\mu \\ & +g^{\rho\mu}(p_3-p_1)^\nu]\equiv\Gamma^{\mu\nu\rho}_{abc}(p_1,p_2,p_3) \\ & p_1+p_2+p_3=0 \end{aligned} \tag{69}$$

ii) Quartic gauge boson vertex



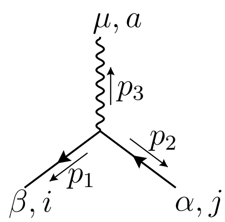
$$\begin{aligned} -ig^2\Big[\;\; & f_{eab}f_{ecd}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) \\ & +f_{eac}f_{edb}(g^{\mu\sigma}g^{\rho\nu}-g^{\mu\nu}g^{\rho\sigma}) \\ & +f_{ead}f_{ebc}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\Big]\equiv\Gamma^{\mu\nu\rho\sigma}_{abcd} \\ & p_1+p_2+p_3+p_4=0 \end{aligned} \tag{70}$$

iii) Ghost-Gauge boson interaction



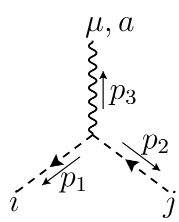
$$-g\ f^{abc}p_1^{\mu} \equiv \Gamma^{\mu}_{abc}(p_1)$$
$$p_1 \quad +p_2 + p_3 = 0 \tag{71}$$

iv) Fermion-Gauge boson interaction



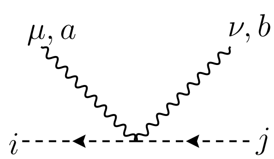
$$-ig(\gamma^{\mu})_{\beta\alpha}T^a_{ij} \tag{72}$$

v) Scalar-Gauge boson interaction: Cubic term



$$-ig(p_1 - p_2)^{\mu}T^a_{ij} \equiv V^{\mu}_S(p_1, p_2)T^a_{ij} \tag{73}$$

vi) Scalar-Gauge boson interaction: Quartic term



$$ig^2 g_{\mu\nu}\{T^a, T^b\}_{ij} \tag{74}$$

Notice that in the definition of $\Gamma^{\mu\nu\rho}_{abc}(p_1, p_2, p_3)$ in Eq. (95) all the momenta are outgoing. This explains the different sign when comparing with Ref.[1].

Group factors

We summarize here some useful formulas for dealing with some group theory factors. Our generators obey the defining commutation relations,

$$\left[T^a, T^b\right] = i\, f^{abc}T^c \tag{75}$$

where the structure constants of the Lie group G are completely antisymmetric, and the generators in a representation R of G are normalized as follows,

$$\mathrm{Tr}[T^a T^b] = T_R \delta^{ab} \tag{76}$$

The structure constants obey the Jacobi identity

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0 \tag{77}$$

and we define the two Casimir invariants

$$f^{abd} f^{dbc} = C_A \delta^{ad}, \quad T^a T^a = C_R \, \mathbf{1} \ . \tag{78}$$

Useful relations are,

$$T_R \, r = d_R \, C_A \tag{79}$$

$$\mathrm{Tr}[T^a T^b T^c] - \mathrm{Tr}[T^a T^c T^b] = i \, T_R \, f^{abc} \tag{80}$$

$$T^{abcd} + T^{abdc} + T^{acdb} + T^{adcb} - 2T^{acbd} - 2T^{adbc} = T_R \left(f^{ade} f^{bce} + f^{ace} f^{bde} \right) \ , \tag{81}$$

where r is the dimension of G, d_R the dimension of the representation R of G and $T^{abcd} \equiv \mathrm{Tr}[T^a T^b T^c T^d]$. For $SU(N)$ we have,

$$r = N^2 - 1 \qquad\qquad d_N = N \tag{82}$$

$$T_N = \frac{1}{2} \qquad\qquad C_N = \frac{N^2-1}{2N} \tag{83}$$

$$C_A = T_{Adj} = N \qquad\qquad . \tag{84}$$

10.2 Calculate the β function

The β function can be obtained in many ways. Here use as starting point

$$Z_g = Z_{GGA} \, Z_A^{-1/2} \, Z_G^{-1} \tag{85}$$

In all calculations consider the gauge with $\xi = 1$ and evaluate the counter-terms using the MS scheme (just consider the coefficient of the pole).

1. Calculate the pure gauge contribution to Z_A at one loop.
2. Calculate Z_{GGA} and Z_G at one loop.
3. Calculate the fermion contribution to Z_A at one loop.
4. Calculate the scalar contribution to Z_A at one loop.
5. Finally evaluate the β function for this theory. Check that you recover Eq. 7.181 of the textbook.

Student:

11 β Function in a general SU(N) gauge theory: SSA

11.1 Definitions

We define here the theory to have all conventions consistent. Notice that some of these conventions differ from the textbook.

Classical Theory

Consider the non-abelian **SU(N)** gauge theory defined by its classical Lagrangian,

$$\mathcal{L}_{\mathrm{SU(n)}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \overline{\psi}_i(i\not{D} - m_F)_{ij}\psi_j + (D_\mu\phi)^\dagger_i D^\mu\phi_i - m_S^2\phi^\dagger_i\phi_i \tag{86}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \tag{87}$$

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu + ig\,(T^a)_{ij}\,A^a_\mu\ . \tag{88}$$

and T^a_{ij} are the generators in the representation to which the fermion and scalar belong (possibly different ones). To quantify the theory we have to introduce the gauge fixing term, that we choose to be of the form,

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}\ (\partial_\mu A^{\mu a})^2\ \ , \tag{89}$$

for which we have the following ghost Lagrangian,

$$\mathcal{L}_G = \partial_\mu\overline{\omega}^a\partial^\mu\omega^a - g f^{abc}\partial^\mu\overline{\omega}^a A^b_\mu\omega^c\ . \tag{90}$$

Counterterm Lagrangian

To renormalize the theory we need the following counterterm Lagrangian

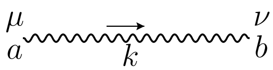
$$\begin{aligned}\Delta\mathcal{L} \;=\; & -\frac{1}{4}(Z_{\mathrm{A}}-1)\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + (Z_{\mathrm{AAA}}-1) g f^{abc}\partial_\mu A^a_\nu A^{\mu b}A^{\nu c} \\ & -\frac{1}{4}g^2(Z_{\mathrm{AAAA}}-1) f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e} + (Z_{\mathrm{F}}-1) i\overline{\psi}_i\gamma^\mu\partial_\mu\psi_i - m_F(Z_{m_\psi}-1)\overline{\psi}_i\psi_i \\ & -(Z_{\psi\psi\mathrm{A}}-1) g\,\overline{\psi}_i\gamma^\mu\,(T^a)_{ij}\,\psi_j A^a_\mu + (Z_{\mathrm{G}}-1)\partial_\mu\overline{\omega}^a\partial^\mu\omega^a - (Z_{\mathrm{GGA}}-1) g f^{abc}\partial^\mu\overline{\omega}^a A^b_\mu\omega^c \\ & +(Z_S-1)\partial_\mu\phi^*_i\partial^\mu\phi_i - m_S^2(Z_S-1)\phi^*_i\phi_i - (Z_{SSA}-1)\left[igA^a_\mu\phi^\dagger_i T^a_{ij}\partial^\mu\phi_j - igA^a_\mu\partial^\mu\phi^\dagger_i T^a_{ij}\phi_j\right] \\ & +(Z_{SSAA}-1) g^2\phi^\dagger_i T^a_{ij}\phi_j A^a_\mu\phi^\dagger_k T^b_{km}\phi_m A^{bmu}\ .\end{aligned} \tag{91}$$

Feynman Rules

For completeness we give here the Feynman rules for this theory.

- **Propagators:**

i) Gauge bosons



$$-i\delta_{ab}\left[\frac{g^{\mu\nu}}{k^2+i\epsilon}-(1-\xi)\,\frac{k^\mu k^\nu}{(k^2+i\epsilon)^2}\right]\equiv i\delta_{ab}\frac{N^{\mu\nu}(k,\xi)}{k^2+i\epsilon} \tag{92}$$

where, for future use, we have defined the numerator of the propagator in an arbitrary R_ξ gauge as,

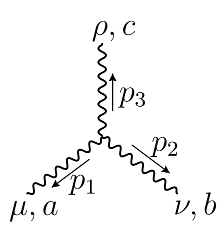
$$N^{\mu\nu}(k,\xi)\equiv-\left[g^{\mu\nu}-(1-\xi)\,\frac{k^\mu k^\nu}{k^2}\right] \tag{93}$$

ii) Ghosts

$$\frac{i}{k^2+i\epsilon}\;\delta_{ab} \tag{94}$$

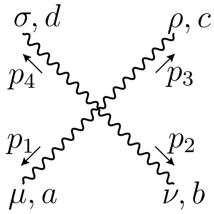
- **Vertices:**

i)Triple gauge boson vertex



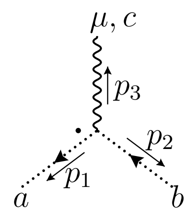
$$\begin{aligned} g f^{abc}[\;& g^{\mu\nu}(p_1-p_2)^\rho+g^{\nu\rho}(p_2-p_3)^\mu \\ & +g^{\rho\mu}(p_3-p_1)^\nu]\equiv\Gamma^{\mu\nu\rho}_{abc}(p_1,p_2,p_3) \\ & p_1+p_2+p_3=0 \end{aligned} \tag{95}$$

ii) Quartic gauge boson vertex



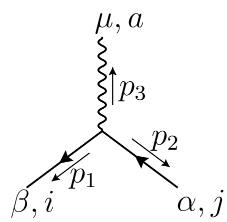
$$\begin{aligned} -ig^2\Big[\;& f_{eab}f_{ecd}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) \\ & +f_{eac}f_{edb}(g^{\mu\sigma}g^{\rho\nu}-g^{\mu\nu}g^{\rho\sigma}) \\ & +f_{ead}f_{ebc}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\Big]\equiv\Gamma^{\mu\nu\rho\sigma}_{abcd} \\ & p_1+p_2+p_3+p_4=0 \end{aligned} \tag{96}$$

iii) Ghost-Gauge boson interaction



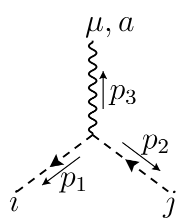
$$\begin{aligned} &-g\ f^{abc} p_1^{\mu} \equiv \Gamma^{\mu}_{abc}(p_1) \\ p_1 \quad &+ p_2 + p_3 = 0 \end{aligned} \tag{97}$$

iv) Fermion-Gauge boson interaction



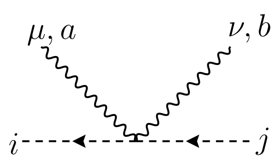
$$-ig(\gamma^{\mu})_{\beta\alpha} T^a_{ij} \tag{98}$$

v) Scalar-Gauge boson interaction: Cubic term



$$-ig(p_1 - p_2)^{\mu} T^a_{ij} \equiv V^{\mu}_S(p_1, p_2) T^a_{ij} \tag{99}$$

vi) Scalar-Gauge boson interaction: Quartic term



$$ig^2 g_{\mu\nu} \{T^a, T^b\}_{ij} \tag{100}$$

Notice that in the definition of $\Gamma^{\mu\nu\rho}_{abc}(p_1, p_2, p_3)$ in Eq. (95) all the momenta are outgoing. This explains the different sign when comparing with Ref.[1].

Group factors

We summarize here some useful formulas for dealing with some group theory factors. Our generators obey the defining commutation relations,

$$\left[T^a, T^b\right] = i\, f^{abc} T^c \tag{101}$$

where the structure constants of the Lie group G are completely antisymmetric, and the generators in a representation R of G are normalized as follows,

$$\mathrm{Tr}[T^a T^b] = T_R \delta^{ab} \tag{102}$$

The structure constants obey the Jacobi identity

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0 \tag{103}$$

and we define the two Casimir invariants

$$f^{abd} f^{dbc} = C_A \delta^{ad}, \quad T^a T^a = C_R \mathbf{1} \ . \tag{104}$$

Useful relations are,

$$T_R \, r = d_R \, C_A \tag{105}$$

$$\mathrm{Tr}[T^a T^b T^c] - \mathrm{Tr}[T^a T^c T^b] = i \, T_R \, f^{abc} \tag{106}$$

$$T^{abcd} + T^{abdc} + T^{acdb} + T^{adcb} - 2T^{acbd} - 2T^{adbc} = T_R \left(f^{ade} f^{bce} + f^{ace} f^{bde} \right) \ , \tag{107}$$

where r is the dimension of G, d_R the dimension of the representation R of G and $T^{abcd} \equiv \mathrm{Tr}[T^a T^b T^c T^d]$. For $SU(N)$ we have,

$$r = N^2 - 1 \qquad\qquad d_N = N \tag{108}$$

$$T_N = \frac{1}{2} \qquad\qquad C_N = \frac{N^2 - 1}{2N} \tag{109}$$

$$C_A = T_{Adj} = N \qquad\qquad . \tag{110}$$

11.2 Calculate the β function

The β function can be obtained in many ways. Here use as starting point

$$Z_g = Z_{SSA} \, Z_A^{-1/2} \, Z_S^{-1} \tag{111}$$

In all calculations consider the gauge with $\xi = 1$ and evaluate the counter-terms using the MS scheme (just consider the coefficient of the pole).

1. Calculate the pure gauge contribution to Z_A at one loop.
2. Calculate Z_{SSA} and Z_S at one loop.
3. Calculate the fermion contribution to Z_A at one loop.
4. Calculate the scalar contribution to Z_A at one loop.
5. Finally evaluate the β function for this theory. Check that you recover Eq. 7.181 of the textbook.

Student:

References

[1] J. C. Romao and J. P. Silva, Int. J. Mod. Phys. **A27**, 1230025 (2012), [1209.6213].