Wess-Zumino Model & free chiral supermultiplet

1) Lagrangian

$$\mathcal{L} = \mathcal{L}_{scalar} + \mathcal{L}_{fermion}$$

where

$$\mathcal{L}_{scalar} = \partial_\mu \phi^\dagger \partial^\mu \phi$$

$$\mathcal{L}_{fermion} = i \, \bar{\chi} \, \bar{\sigma}^\mu \partial_\mu \chi$$

2) SUSY transformations

$$\begin{cases} \delta_\varepsilon \phi = \sqrt{2} \, \varepsilon \chi \quad ; \quad \delta_\varepsilon \phi^\dagger = \sqrt{2} \, \bar{\varepsilon} \bar{\chi} \\ \delta_\varepsilon \chi_\alpha = -i \sqrt{2} \, (\sigma^\mu \bar{\varepsilon})_\alpha \, \partial_\mu \phi \\ \delta_\varepsilon \bar{\chi}^{\dot{\alpha}} = -i \sqrt{2} \, (\bar{\sigma}^\mu \varepsilon)^{\dot{\alpha}} \, \partial_\mu \phi^\dagger \end{cases}$$

$$\delta_\varepsilon \mathcal{L}_{scalar} = \sqrt{2} \, \bar{\varepsilon} \, \partial_\mu \bar{\chi} \, \partial^\mu \phi + \sqrt{2} \, \partial_\mu \phi^\dagger \, \varepsilon \partial^\mu \chi$$

$$\delta_\varepsilon \mathcal{L}_{fermion} = i \, \delta_\varepsilon \bar{\chi}_{\dot{\alpha}} \, (\bar{\sigma}^\mu \partial_\mu \chi)^{\dot{\alpha}} + i \, (\bar{\chi} \bar{\sigma}^\mu)^\beta \, \partial_\mu \delta_\varepsilon \chi_\beta$$

$$= i \, \varepsilon_{\dot{\alpha}\dot{\beta}} \, \delta_\varepsilon \bar{\chi}^{\dot{\beta}} (\bar{\sigma}^\mu \partial_\mu \chi)^{\dot{\alpha}} + i \, (\bar{\chi} \bar{\sigma}^\mu)^\beta \left(-i\sqrt{2} \, (\sigma^\nu \bar{\varepsilon})_\beta \, \partial_\mu \partial_\nu \phi\right.$$

$$= i \, \varepsilon_{\dot{\alpha}\dot{\beta}} \left(-i\sqrt{2} \, (\bar{\sigma}^\nu \varepsilon)^{\dot{\beta}} \, \partial_\nu \phi^\dagger \, (\bar{\sigma}^\mu \partial_\mu \chi)^{\dot{\alpha}}\right.$$

$$+ \sqrt{2} \, \bar{\chi} \, \bar{\varepsilon} \, \partial_\mu \partial^\mu \phi$$

where we used.

$$\sigma^\mu \bar{\sigma}^\nu \equiv \eta^{\mu\nu} + \sigma^{\mu\nu} \quad ; \quad \sigma^{\mu\nu} \equiv \tfrac{1}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$$

$$\bar{\sigma}^\mu \sigma^\nu = \eta^{\mu\nu} + \bar{\sigma}^{\mu\nu} \quad ; \quad \bar{\sigma}^{\mu\nu} = \tfrac{1}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$$

Now using $\varepsilon_{\dot\alpha\dot\beta} = (-i\sigma_2)_{\alpha\dot\beta}$ and

$$\sigma_2 \bar{\sigma}^\nu \sigma_2 = \sigma^{\nu T} \Rightarrow (\sigma_2 \bar{\sigma}^\nu \sigma_2)^T = \sigma^\nu$$

or

$$(\sigma_2 \bar{\sigma}^\nu \sigma_2)_{\dot\alpha\beta} = (\sigma^\nu)_{\beta\dot\alpha}$$

we get

$$\delta_\varepsilon \mathcal{L}_{\text{fermions}} = -\sqrt{2}\,(\varepsilon\sigma^\nu\bar{\sigma}^\mu\partial_\mu\chi)\,\partial_\nu\phi^\dagger + \sqrt{2}\,\bar{\chi}\,\bar{\varepsilon}\,\partial_\mu\partial^\mu\phi$$

$$= -\delta_\varepsilon \mathcal{L}_{\text{scalar}} + \partial_\mu X^\mu$$

where

$$X^\mu = \sqrt{2}\,\bar{\varepsilon}\bar{\chi}\,\partial^\mu\phi - \sqrt{2}\,(\varepsilon\sigma^{\nu\mu}\chi)\,\partial_\nu\phi^\dagger$$

therefore

$$\delta_\varepsilon \mathcal{L} = \partial_\mu X^\mu$$

and, as the total derivative does not affect the equations of motion $\mathcal{L}$ is invariant.

3) Closure of the Algebra

For the Scalars:

$$(\delta_{\varepsilon_2}\delta_{\varepsilon_1} - \delta_{\varepsilon_1}\delta_{\varepsilon_2})\phi = \delta_{\varepsilon_2}(\sqrt{2}\varepsilon_1\chi) - \delta\varepsilon_1(\sqrt{2}\varepsilon_2\chi)$$

$$= -2i\ \varepsilon_1\sigma^\mu\bar{\varepsilon}_2\partial_\mu\phi + 2i\ \varepsilon_2\sigma^\mu\bar{\varepsilon}_1\partial_\mu\phi$$

$$= 2\left(-\varepsilon_1\sigma^\mu\bar{\varepsilon}_2 + \varepsilon_2\sigma^\mu\bar{\varepsilon}_1\right)i\,\partial_\mu\phi$$

ok, (remember $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = i(\sigma^\mu)_{\alpha\dot\beta}P_\mu$, see below)

For the fermion fields:

$$(\delta_{\varepsilon_2}\delta_{\varepsilon_1} - \delta_{\varepsilon_1}\delta_{\varepsilon_2})\chi_\alpha = \delta_{\varepsilon_2}\left(-i\sqrt{2}\,(\sigma^\mu\bar{\varepsilon}_1)_\alpha\partial_\mu\phi\right) - \varepsilon_1\Leftrightarrow\varepsilon_2$$

$$= -i\sqrt{2}\,(\sigma^\mu\bar{\varepsilon}_1)_\alpha\sqrt{2}\,\varepsilon_2\partial_\mu\chi - \varepsilon_1\Leftrightarrow\varepsilon_2$$

$$= -i2\,(\sigma^\mu\bar{\varepsilon}_1)_\alpha\,\varepsilon_2\partial_\mu\chi - (\varepsilon_1\Leftrightarrow\varepsilon_2)$$

we now use

$$\theta_\alpha(\xi\eta) = -\xi_\alpha(\eta\theta) - \eta_\alpha(\theta\xi)$$

[Hint: $\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta} = \varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta} - \varepsilon^{\alpha\delta}\varepsilon^{\gamma\beta}$]

with $\theta_\alpha = (\sigma^\mu\bar{\varepsilon}_1)_\alpha$; $\xi = \varepsilon_2$; $\eta = \partial_\mu\chi$

to get

$$(\delta_{\varepsilon_2}\delta_{\varepsilon_1} - \delta_{\varepsilon_2}\delta_{\varepsilon_1})\chi = 2i\ \varepsilon_{2\alpha}(\partial_\mu\chi\,\sigma^\mu\bar{\varepsilon}_1) + 2i\,\partial_\mu\chi_\alpha(\varepsilon_2\sigma^\mu\bar{\varepsilon}_1)$$
$$- (\varepsilon_1\Leftrightarrow\varepsilon_2)$$

$$= 2\left(-\epsilon_1 \sigma^\mu \bar{\epsilon}_2 + \epsilon_2 \sigma^\mu \bar{\epsilon}_1\right) i\partial_\mu \chi_\alpha$$

$$- 2i\, \epsilon_{2\alpha}\, \bar{\epsilon}_1 \bar{\sigma}^\mu \partial_\mu \chi + 2i\, \epsilon_{1\alpha} \bar{\epsilon}_2 \bar{\sigma}^\mu \partial_\mu \chi$$

So the algebra closes on-shell !

$$\boxed{\bar{\sigma}^\mu \partial_\mu \chi = 0}$$

2) <u>Off-shell Closure of the Algebra</u>

Counting de degrees of freedom we have

	ϕ	χ	F
on-shell	2	2	0
off-shell	2	4	2

we need an extra complex scalar f. Assume

$\mathcal{L}_F = F^+ F$ so $[f] = 2$ in terms of mass.

we write (on dimensional grounds)

$$\delta_\varepsilon F = b\, \bar{\varepsilon} \bar{\sigma}^\mu \partial_\mu \chi \quad ; \quad \delta_\varepsilon f^+ = b^* \partial_\mu \bar{\chi} \bar{\sigma}^\mu \varepsilon$$

and modify the transformation laws for χ

$$\delta_\varepsilon \chi_\alpha = -i\sqrt{2}\,(\sigma^\mu \bar{\varepsilon})_\alpha \partial_\mu \phi + c\, \varepsilon_\alpha F$$

As there was closure on ϕ we can't change the transformations laws for ϕ. With this we get for χ:

$$(\delta_{\varepsilon_2}\delta_{\varepsilon_1} - \delta_{\varepsilon_2}\delta_{\varepsilon_1})\chi_\alpha = 2(-\varepsilon_1\sigma^\mu\bar{\varepsilon}_2 + \varepsilon_2\sigma^\mu\bar{\varepsilon}_1)i\partial_\mu\chi_\alpha$$

$$-2i\,\varepsilon_{2\alpha}\,\bar{\varepsilon}_1\bar{\sigma}^\mu\partial_\mu\chi + 2i\,\varepsilon_{1\alpha}\bar{\varepsilon}_2\bar{\sigma}^\mu\partial_\mu\chi$$

$$+\,c\,\delta_{\varepsilon_2}(\varepsilon_{1\alpha}F) - c\,\delta_{\varepsilon_1}(\varepsilon_{2\alpha}f)$$

$$= 2(-\varepsilon_1\sigma^\mu\bar{\varepsilon}_2 + \varepsilon_2\sigma^\mu\bar{\varepsilon}_1)i\partial_\mu\chi_\alpha$$

$$-2i\,\varepsilon_{2\alpha}\bar{\varepsilon}_1\bar{\sigma}^\mu\partial_\mu\chi + 2i\,\varepsilon_{1\alpha}\bar{\varepsilon}_2\bar{\sigma}^\mu\partial_\mu\chi$$

$$+bc\,\varepsilon_{1\alpha}\bar{\varepsilon}_2\bar{\sigma}^\mu\partial_\mu\chi - bc\,\varepsilon_{2\alpha}\bar{\varepsilon}_1\bar{\sigma}^\mu\partial_\mu\chi$$

So the algebra closes if

$$\boxed{bc = -2i}$$

Now we verify for ϕ:

$$(\delta_{\varepsilon_2}\delta_{\varepsilon_1} - \delta_{\varepsilon_2}\delta_{\varepsilon_1})\phi = 2(-\varepsilon_1\sigma^\mu\bar{\varepsilon}_2 + \varepsilon_2\sigma^\mu\bar{\varepsilon}_1)i\partial_\mu\phi$$

$$+\sqrt{2}(\varepsilon_1\varepsilon_2)f - \sqrt{2}(\varepsilon_2\varepsilon_1)F \qquad \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1$$

$$= 2(-\varepsilon_1\sigma^\mu\bar{\varepsilon}_2 + \varepsilon_2\sigma^\mu\bar{\varepsilon}_1)i\partial_\mu\phi$$

finally we verify for F:

(6)

$$(\delta_{\varepsilon_2}\delta_{\varepsilon_1} - \delta_{\varepsilon_1}\delta_{\varepsilon_2}) f = \delta_{\varepsilon_2}\left(b\,\bar\varepsilon_1 \bar\sigma^\mu \partial_\mu \chi\right) - (\varepsilon_1 \Leftrightarrow \varepsilon_2)$$

$$= b\;\bar\varepsilon_1 \bar\sigma^\mu \left(-i\sqrt{2}\,\sigma^\nu \bar\varepsilon_2\right)\partial_\nu\partial_\mu\phi + bc\;\bar\varepsilon_1 \bar\sigma^\mu \varepsilon_2 \partial_\mu F$$

$$-(\varepsilon_1 \Leftrightarrow \varepsilon_2)$$

$$= -i\sqrt{2}\,b\left(\bar\varepsilon_1 \bar\varepsilon_2\, \partial_\mu \partial^\mu \phi - \bar\varepsilon_2 \bar\varepsilon_1\, \partial_\mu \partial^\nu \phi\right)$$

$$+ i bc\left(-\bar\varepsilon_1 \bar\sigma^\mu \varepsilon_2 + \bar\varepsilon_2 \bar\sigma^\mu \varepsilon_1\right) i \partial_\mu f$$

so

$$\boxed{(\delta_{\varepsilon_2}\delta_{\varepsilon_1} - \delta_{\varepsilon_1}\delta_{\varepsilon_2}) f = 2\left(-\bar\varepsilon_1 \bar\sigma^\mu \varepsilon_2 + \bar\varepsilon_2 \bar\sigma^\mu \varepsilon_1\right) i\partial_\mu F}$$

if

$$i bc = 2 \quad \Rightarrow \quad bc = -2i$$

We chose $c = \sqrt{2}$ $b = -\sqrt{2}\,i$

so, in summary:

$$\begin{cases} \delta_\varepsilon \phi = \varepsilon\chi \quad ; \quad \delta\phi^\dagger = \bar\varepsilon \bar\chi \\ \delta_\varepsilon \chi_\alpha = -i\sqrt{2}\,(\sigma^\mu \bar\varepsilon)_\alpha \partial_\mu \phi + \sqrt{2}\,\varepsilon_\alpha f \\ \delta_\varepsilon \bar\chi^{\dot\alpha} = -i\sqrt{2}\,(\bar\sigma^\mu \varepsilon)^{\dot\alpha} \partial_\mu \phi + \sqrt{2}\,\bar\varepsilon^{\dot\alpha} F^\dagger \\ \delta_\varepsilon f = -i\sqrt{2}\;\bar\varepsilon \bar\sigma^\mu \partial_\mu \chi ; \quad \delta_\varepsilon f^\dagger = i\sqrt{2}\,\partial_\mu \bar\chi \bar\sigma^\mu \varepsilon \end{cases}$$