Advanced Quantum Field Theory
Chapter 1
Canonical Quantization of Free Fields

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Lecture 1

Canonical quantization for particles

- Let us start with a system that consists of one particle with just one degree of freedom, like a particle moving in one space dimension. The classical equations of motion are obtained from the action,

$$S = \int_{t_1}^{t_2} dt L(q, \dot{q}) \ .$$

- The condition for the minimization of the action, $\delta S = 0$, gives the Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

which are the equations of motion.

- Before proceeding to the quantization, it is convenient to change to the Hamiltonian formulation. We start by defining the conjugate momentum p, to the coordinate q, by

$$p = \frac{\partial L}{\partial \dot{q}}$$

Quantization:The Hamiltonian and Poisson Brackets

- Then we introduce the Hamiltonian using the Legendre transform

$$H(p,q) = p\dot{q} - L(q,\dot{q})$$

- In terms of H the equations of motion are,

$$\{H,q\}_{\text{PB}} = \frac{\partial H}{\partial p} = \dot{q}$$

$$\{H,p\}_{\text{PB}} = -\frac{\partial H}{\partial q} = \dot{p}$$

- The Poisson Bracket (PB) is defined by

$$\{f(p,q), g(p,q)\}_{\text{PB}} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}$$

obviously satisfying

$$\{p,q\}_{\text{PB}} = 1\ .$$

Quantization: The Schrödinger picture

- The quantization is done by promoting p and q to hermitian operators that will satisfy the commutation relation $(\hbar = 1)$,

$$[p, q] = -i$$

which is trivially satisfied in the coordinate representation where $p = -i\dfrac{\partial}{\partial q}$.

- The dynamics is given by the Schrödinger equation

$$H(p, q) \, |\Psi_S(t)\rangle = i \frac{\partial}{\partial t} \, |\Psi_S(t)\rangle$$

- If we know the state of the system in $t = 0$, $|\Psi_S(0)\rangle$, then the previous equation completely determines the state $|\Psi_s(t)\rangle$ and therefore the value of any physical observable.

- This description, where the states are time dependent and the operators, on the contrary, do not depend on time, is known as the Schrödinger representation.

Quantization: The Heisenberg picture

- There exists and alternative description, where the time dependence goes to the operators and the states are time independent. This is called the Heisenberg representation.

- To define this representation, we formally integrate to obtain

$$|\Psi_S(t)\rangle = e^{-iHt}|\Psi_S(0)\rangle = e^{-iHt}|\Psi_H\rangle \ .$$

- The state in the Heisenberg representation, $|\Psi_H\rangle$, is defined as the state in the Schrödinger representation for $t=0$. The unitary operator e^{-iHt} allows us to go from one representation to the other.

- If we define the operators in the Heisenberg representation as,

$$O_H(t) = e^{iHt} O_S e^{-iHt}$$

then the matrix elements are representation independent. In fact,

$$\begin{aligned}\langle\Psi_S(t)|O_S|\Psi_S(t)\rangle &= \left\langle\Psi_S(0)|e^{iHt}O_S e^{-iHt}|\Psi_S(0)\right\rangle \\ &= \langle\Psi_H|O_H(t)|\Psi_H\rangle \ .\end{aligned}$$

Quantization: The Heisenberg picture

- The time evolution of the operator $O_H(t)$ is then given by the equation

$$\frac{dO_H(t)}{dt} = i[H, O_H(t)] + \frac{\partial O_H}{\partial t}$$

The last term is only present if O_S explicitly depends on time.

- In the non-relativistic theory the difference between the two representations is very small if we work with energy eigenfunctions. For the relativistic theory, the Heisenberg representation is more convenient, because Lorentz covariance is more easily handled in the Heisenberg representation, because time and spatial coordinates are together in the field operators.

- In the Heisenberg representation the fundamental commutation relation is now

$$[p(t), q(t)] = -i$$

- The dynamics is now given by

$$\frac{dp(t)}{dt} = i[H, p(t)] \quad ; \quad \frac{dq(t)}{dt} = i[H, q(t)]$$

Quantization: The Heisenberg picture

- Notice that in this representation the fundamental equations are similar to the classical equations with the substitution,

$$\{,\}_{PB} \Longrightarrow -i[,]$$

- In the case of a system with n degrees of freedom the generalization is

$$[p_i(t), q_j(t)] = -i\delta_{ij}$$
$$[p_i(t), p_j(t)] = 0$$
$$[q_i(t), q_j(t)] = 0$$

and

$$\dot{p}_i(t) = i[H, p_i(t)] \ ; \ \dot{q}_i(t) = i[H, q_i(t)]$$

- Because it is an important example let us look at the harmonic oscillator. The Hamiltonian is

$$H = \frac{1}{2}(p^2 + \omega_0^2 q^2)$$

Harmonic Oscillator

- The equations of motion are

$$\dot{p} = i[H,p] = -\omega_0^2 q\,, \quad \dot{q} = i[H,q] = p \Longrightarrow \ddot{q} + \omega_0^2 q = 0\ .$$

- It is convenient to introduce the operators

$$a = \frac{1}{\sqrt{2\omega_0}}(\omega_0 q + ip)\ ;\ a^\dagger = \frac{1}{\sqrt{2\omega_0}}(\omega_0 q - ip)$$

- The equations of motion for a and $a^\dagger$ are very simple:

$$\dot{a}(t) = -i\omega_0 a(t) \text{ e } \dot{a}^\dagger(t) = i\omega_0 a^\dagger(t)\ .$$

They have the solution

$$a(t) = a_0 e^{-i\omega_0 t}\ ;\ a^\dagger(t) = a_0^\dagger e^{i\omega_0 t}$$

- They obey the commutation relations

$$[a, a^\dagger] = [a_0, a_0^\dagger] = 1$$

$$[a, a] = [a_0, a_0] = 0\,, \quad [a^\dagger, a^\dagger] = [a_0^\dagger, a_0^\dagger] = 0$$

Harmonic Oscillator

- In terms of $a, a^\dagger$ the Hamiltonian reads

$$\begin{aligned} H =& \frac{1}{2}\,\omega_0(a^\dagger a + a a^\dagger) = \frac{1}{2}\,\omega_0(a_0^\dagger a_0 + a_0 a_0^\dagger) \\ =& \omega_0 a_0^\dagger a_0 + \frac{1}{2}\,\omega_0 \end{aligned}$$

where we have used

$$[H, a_0] = -\omega_0 a_0, \quad [H, a_0^\dagger] = \omega_0 a_0^\dagger$$

- We see that a_0 decreases the energy of a state by the quantity ω_0 while $a_0^\dagger$ increases the energy by the same amount.

- As the Hamiltonian is a sum of squares the eigenvalues must be positive. Then it should exist a ground state (state with the lowest energy), $|0\rangle$, defined by the condition

$$a_0\,|0\rangle = 0$$

Harmonic Oscillator

- The state $|n\rangle$ is obtained by the application of $\left(a_0^\dagger\right)^n$. If we define

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a_0^\dagger\right)^n |0\rangle$$

then

$$\langle m|n\rangle = \delta_{mn}$$

and

$$H\,|n\rangle = \left(n+\frac{1}{2}\right)\omega_0\,|n\rangle$$

- We will see that, in the quantum field theory, the equivalent of a_0 and $a_0^\dagger$ are the creation and annihilation operators.

Canonical quantization for fields

- Let us move now to field theory, that is, systems with an infinite number of degrees of freedom. To specify the state of the system, we must give for all space-time points one number (more if we are not dealing with a scalar field)

- The equivalent of the coordinates $q_i(t)$ and velocities, $\dot{q}_i$, are here the fields $\varphi(\vec{x},t)$ and their derivatives, $\partial^\mu\varphi(\vec{x},t)$. The action is now

$$S = \int d^4x \mathcal{L}(\varphi, \partial^\mu \varphi)$$

where the Lagrangian density $\mathcal{L}$, is a functional of the fields φ and their derivatives $\partial^\mu\varphi$.

- Let us consider closed systems for which $\mathcal{L}$ does not depend explicitly on the coordinates x^μ (energy and linear momentum are therefore conserved).

- For simplicity let us consider systems described by n scalar fields $\varphi_r(x), r = 1, 2, \cdots n$. The stationarity of the action, $\delta S = 0$, implies the equations of motion, the so-called Euler-Lagrange equations,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} - \frac{\partial \mathcal{L}}{\partial \varphi_r} = 0 \qquad r = 1, \cdots n$$

Canonical quantization for fields

- For the case of real scalar fields with no interactions that we are considering, we can easily see that the Lagrangian density should be,

$$\mathcal{L} = \sum_{r=1}^{n} \left[\frac{1}{2} \partial^\mu \varphi_r \partial_\mu \varphi_r - \frac{1}{2} m^2 \varphi_r \varphi_r \right]$$

  in order to obtain the Klein-Gordon equations as the equations of motion,

$$(\Box + m^2)\varphi_r = 0 \quad ; \quad r = 1, \cdots n$$

- To define the canonical quantization rules we have to change to the Hamiltonian formalism, in particular we need to define the conjugate momentum $\pi(x)$ for the field $\varphi(x)$.

- To make an analogy with systems with n degrees of freedom, we divide the 3-dimensional space in cells with elementary volume ΔV_i. Then we introduce the coordinate $\varphi_i(t)$ as the average of $\varphi(\vec{x}, t)$ in the volume element ΔV_i, that is,

$$\varphi_i(t) \equiv \frac{1}{\Delta V_i} \int_{(\Delta V_i)} d^3x \, \varphi(\vec{x}, t)$$

Canonical quantization for fields

- Also

$$\dot{\varphi}_i(t) \equiv \frac{1}{\Delta V_i} \int_{(\Delta V_i)} d^3x \dot{\varphi}(\vec{x}, t) \ .$$

- Then

$$L = \int d^3x \mathcal{L} \to \sum_i \Delta V_i \overline{\mathcal{L}}_i \ .$$

- Therefore the canonical momentum is now

$$p_i(t) = \frac{\partial L}{\partial \dot{\varphi}_i(t)} = \Delta V_i \frac{\partial \overline{\mathcal{L}}_i}{\partial \dot{\varphi}_i(t)} \equiv \Delta V_i \pi_i(t)$$

- And the Hamiltonian

$$H = \sum_i p_i \dot{\varphi}_i - L = \sum_i \Delta V_i (\pi_i \dot{\varphi}_i - \overline{\mathcal{L}}_i)$$

- In the limit of the continuum, we define the conjugate momentum,

$$\pi(\vec{x}, t) \equiv \frac{\partial \mathcal{L}(\varphi, \dot{\varphi})}{\partial \dot{\varphi}(\vec{x}, t)}$$

in such a way that its average value in ΔV_i is $\pi_i(t)$

Canonical quantization for fields

- This suggests the introduction of an Hamiltonian density such that

$$
\begin{aligned}
H &= \int d^3x \mathcal{H} \\
\mathcal{H} &= \pi\dot{\varphi} - \mathcal{L} \ .
\end{aligned}
$$

- To define the rules of the canonical quantization we start with the coordinates $\varphi_i(t)$ and conjugate momenta $p_i(t)$. We have

$$
\begin{aligned}
&[p_i(t), \varphi_j(t)] = -i\delta_{ij} \\
&[\varphi_i(t), \varphi_j(t)] = 0 \\
&[p_i(t), p_j(t)] = 0
\end{aligned}
$$

- In terms of momentum $\pi_i(t)$ we have

$$
[\pi_i(t), \varphi_j(t)] = -i\frac{\delta_{ij}}{\Delta V_i} \ .
$$

Canonical quantization for fields

- Going into the continuum limit, $\Delta V_i \to 0$, we obtain

$$[\varphi(\vec{x},t), \varphi(\vec{x}\,',t)] = 0\,, \qquad [\pi(\vec{x},t), \pi(\vec{x}\,',t)] = 0$$
$$[\pi(\vec{x},t), \varphi(\vec{x}\,',t)] = -i\delta(\vec{x}-\vec{x}\,')$$

- These relations are the basis of the canonical quantization. For the case of n scalar fields, the generalization is:

$$[\varphi_r(\vec{x},t), \varphi_s(\vec{x}\,',t)] = 0 \qquad [\pi_r(\vec{x},t), \pi_s(\vec{x}\,',t)] = 0$$
$$[\pi_r(\vec{x},t), \varphi_s(\vec{x}\,',t)] = -i\delta_{rs}\delta(\vec{x}-\vec{x}\,')$$

where

$$\pi_r(\vec{x},t) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_r(\vec{x},t)}$$

- The Hamiltonian is

$$H = \int d^3x \mathcal{H} \quad \text{with} \quad \mathcal{H} = \sum_{r=1}^{n} \pi_r \dot{\varphi}_r - \mathcal{L}\ .$$

Symmetries and conservation laws: Noether's Theorem

- The Lagrangian formalism gives us a powerful method to relate symmetries and conservation laws. At the classical level the fundamental result is
- **Noether's Theorem**

  To each continuous symmetry transformation that leaves $\mathcal{L}$ and the equations of motion invariant, corresponds one conservation law.
- Instead of making the proof for all cases, we will consider three very important particular cases:
- *Translations*

  Let us consider an infinitesimal translation

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}$$

Then

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = \varepsilon^{\mu}\frac{\partial \mathcal{L}}{\partial x^{\mu}}$$

and $\mathcal{L}'$ leads to the same equations of motion as $\mathcal{L}$, as they differ only by a 4-divergence. If $\mathcal{L}$ is invariant for translations, then it can not depend explicitly on the coordinates x^{μ}.

Symmetries and conservation laws: Translations

❐ Therefore

$$\delta\mathcal{L} = \sum_r \left[\frac{\partial\mathcal{L}}{\partial\varphi_r}\delta\varphi_r + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}\delta(\partial_\mu\varphi_r) \right] = \partial_\mu \left[\sum_r \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}\varepsilon^\nu\partial_\nu\varphi_r \right]$$

where we have used the equations of motion and $\delta\varphi_r = \varepsilon^\nu\partial_\nu\varphi_r$.

❐ From the above and using the fact that ε^μ is arbitrary we get

$$\partial_\mu T^{\mu\nu} = 0$$

where $T^{\mu\nu}$ is the *energy-momentum tensor* defined by

$$T^{\mu\nu} = -g^{\mu\nu}\mathcal{L} + \sum_r \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}\partial^\nu\ \varphi_r$$

❐ Using these relations we can define the conserved quantities

$$P^\mu \equiv \int d^3x T^{0\mu} \quad \Rightarrow \quad \frac{dP^\mu}{dt} = 0$$

Noticing that $T^{00} = \mathcal{H}$, it is easy to realize that P^μ should be the 4-momentum vector. Therefore we conclude that invariance for translations leads to the conservation of energy and momentum.

Symmetries and conservation laws: Lorentz transformations

- Consider the infinitesimal *Lorentz transformations*

$$x'^{\mu} = x^{\mu} + \omega^{\mu}{}_{\nu}\, x^{\nu}$$

- The transformation for the coordinates will induce the following transformation for the fields,

$$\varphi'_r(x') = S_{rs}(\omega)\varphi_s(x)$$

- For the case of scalar fields $S_{rs} = \delta_{rs}$ and for spinors we know that $S_{rs} = \delta_{rs} + \frac{1}{8}[\gamma_\mu, \gamma_\nu]_{rs}\omega^{\mu\nu}$. In general the variation of φ_r comes from two different effects
- We have

$$\begin{aligned}\delta\varphi_r(x) \equiv& \varphi'_r(x) - \varphi_r(x) = S^{-1}_{rs}(\omega)\varphi_s(x') - \varphi_r(x)\\ =& -\frac{1}{2}\omega_{\alpha\beta}\left[(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\delta_{rs} + \Sigma^{\alpha\beta}_{rs}\right]\varphi_s\end{aligned}$$

where we have defined

$$S_{rs}(\omega) = \delta_{rs} + \frac{1}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta}_{rs} \; .$$

Symmetries and conservation laws: Lorentz transformations

- Then

$$\delta\mathcal{L} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \delta\varphi_r \right]$$

which gives

$$\partial_\mu M^{\mu\alpha\beta} = 0 \quad \text{with} \quad M^{\mu\alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \Sigma_{rs}^{\alpha\beta} \varphi_s$$

- The conserved angular momentum is then

$$M^{\alpha\beta} = \int d^3x M^{0\alpha\beta} = \int d^3x \left[x^\alpha T^{0\beta} - x^\beta T^{0\alpha} + \sum_{r,s} \pi_r \Sigma_{rs}^{\alpha\beta} \varphi_s \right]$$

with

$$\frac{dM^{\alpha\beta}}{dt} = 0$$

Symmetries and conservation laws: Internal symmetries

- Let us consider that the Lagrangian is invariant for an infinitesimal internal symmetry transformation

$$\delta\varphi_r(x) = -i\varepsilon\lambda_{rs}\varphi_s(x)$$

- Then we can easily show that

$$\partial_\mu J^\mu = 0 \quad \text{where} \quad J^\mu = -i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}\lambda_{rs}\varphi_s$$

- This leads to the conserved charge

$$Q(\lambda) = -i\int d^3x\pi_r\lambda_{rs}\varphi_s \quad ; \quad \frac{dQ}{dt} = 0$$

Symmetries and conservation laws: Quantum Theory

- These relations between symmetries and conservation laws were derived for the classical theory
- In the quantum theory the fields $\varphi_r(x)$ become operators acting on the Hilbert space of the states. The physical observables are related with the matrix elements of these operators. We have therefore to require Lorentz covariance for those matrix elements
- This means that the classical fields relation

$$\varphi'_r(x') = S_{rs}(a)\varphi_s(x)$$

should be in the quantum theory

$$\left\langle \Phi'_\alpha | \varphi_r(x') | \Phi'_\beta \right\rangle = S_{rs}(a) \left\langle \Phi_\alpha | \varphi_s(x) | \Phi_\beta \right\rangle$$

- There should exist an unitary transformation $U(a,b)$ that should relate the two inertial frames

$$|\Phi'\rangle = U(a,b)\,|\Phi\rangle$$

where $a^\mu{}_\nu$ e b^μ are defined by

$$x'^\mu = a^\mu{}_\nu x^\nu + b^\mu$$

Symmetries and conservation laws: Quantum Theory

- Therefore we get that the field operators should transform as

$$U(a,b)\varphi_r(x)U^{-1}(a,b) = S_{rs}^{(-1)}(a)\varphi_s(ax+b)$$

- Let us look at the consequences of this relation for translations and Lorentz transformations. We consider first the translations. We get

$$U(b)\varphi_r(x)U^{-1}(b) = \varphi_r(x+b)$$

- For infinitesimal translations we can write

$$U(\varepsilon) \equiv e^{i\varepsilon_\mu \mathcal{P}^\mu} \simeq 1 + i\varepsilon_\mu \mathcal{P}^\mu$$

where $\mathcal{P}^\mu$ is an hermitian operator.

- This gives

$$i[\mathcal{P}^\mu, \varphi_r(x)] = \partial^\mu \varphi_r(x)$$

- The correspondence with classical mechanics and non relativistic quantum theory suggests that we identify $\mathcal{P}^\mu$ with the 4-momentum, that is, $\mathcal{P}^\mu \equiv P^\mu$ where P^μ has been defined before

Symmetries and conservation laws: Quantum Theory

- For Lorentz transformations $x'^{\mu} = a^{\mu}{}_{\nu}\, x^{\nu}$, we write for an infinitesimal transformation

$$a^{\mu}{}_{\nu} = g^{\mu}_{\nu} + \omega^{\mu}{}_{\nu} + O(\omega^2)$$

and therefore

$$U(\omega) \equiv 1 - \frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}$$

- We then obtain the requirement

$$i[\mathcal{M}^{\mu\nu}, \varphi_r(x)] = x^{\mu}\partial^{\nu}\varphi_r - x^{\nu}\partial^{\mu}\varphi_r + \Sigma^{\mu\nu}_{rs}\varphi_s(x)$$

- As we have an explicit expression for P^{μ} and $M^{\mu\nu}$ and we know the commutation relations of the quantum theory, these equations become an additional requirement that the theory has to verify in order to be invariant under translations. We will see explicitly that this is indeed the case for the theories in which we are interested.

Real scalar field

- The real scalar field described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi\varphi$$

to which corresponds the Klein-Gordon equation

$$(\Box + m^2)\varphi = 0$$

is the simplest example, and in fact was already used to introduce the general formalism.

- As we have seen the conjugate momentum is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}$$

- The commutation relations are

$$[\varphi(\vec{x},t), \varphi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$$

$$[\pi(\vec{x},t), \varphi(\vec{x}',t)] = -i\delta^3(\vec{x} - \vec{x}')$$

Real scalar field

- The Hamiltonian is given by,

$$H = P^0 = \int d^3x \mathcal{H} = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}|\vec{\nabla}\varphi|^2 + \frac{1}{2}m^2\varphi^2 \right]$$

  and the linear momentum is

$$\vec{P} = -\int d^3x \pi \vec{\nabla}\varphi$$

- Using these equations it is easy to verify that

$$i[P^\mu, \varphi] = \partial^\mu \varphi$$

  showing the invariance of the theory for the translations. In the same way we can verify the invariance under Lorentz transformations, with $\Sigma^{\mu\nu}_{rs} = 0$ (spin zero).

- In order to define the states of the theory it is convenient to have eigenstates of energy and momentum.

Real scalar field

- To build these states we start by making a spectral Fourier decomposition of $\varphi(\vec{x},t)$ in plane waves:

$$\varphi(\vec{x},t)=\int \widetilde{dk}\;\left[a(k)e^{-ik\cdot x}+a^\dagger(k)e^{ik\cdot x}\right]$$

where

$$\widetilde{dk}\equiv\frac{d^3k}{(2\pi)^3 2\omega_k}\;;\;\omega_k=+\sqrt{|\vec{k}|^2+m^2}$$

is the Lorentz invariant integration measure.

- As in the quantum theory φ is an operator, also $a(k)$ e $a^\dagger(k)$ should be operators. As φ is real, then $a^\dagger(k)$ should be the hermitian conjugate to $a(k)$. In order to determine their commutation relations we start by solving in order to $a(k)$ and $a^\dagger(k)$. Using the properties of the delta function, we get

$$a(k)=i\int d^3x e^{ik\cdot x}\overleftrightarrow{\partial}_0\varphi(x)\,,\quad a^\dagger(k)=-i\int d^3x e^{-ik\cdot x}\overleftrightarrow{\partial}_0\varphi(x)$$

where we have introduced the notation

$$a\overleftrightarrow{\partial}_0 b=a\frac{\partial b}{\partial t}-\frac{\partial a}{\partial t}b$$

Real scalar field

- $a(k)$ and $a^\dagger(k)$ time independent as can be checked explicitly (see the Problems). This observation is important in order to be able to choose equal times in the commutation relations. We get

$$[a(k), a^\dagger(k')] = \int d^3x \int d^3y \, \left[e^{ik\cdot x} \overleftrightarrow{\partial}_0 \varphi(\vec{x}, t), e^{-ik'\cdot y} \overleftrightarrow{\partial}_0 \varphi(\vec{y}, t) \right]$$

$$= (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}')$$

and

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0$$

- We then see that, except for a small difference in the normalization, $a(k)$ e $a^\dagger(k)$ should be interpreted as annihilation and creation operators of states with momentum k^μ. To show this, we observe that

$$H = \frac{1}{2} \int \widetilde{dk} \; \omega_k \left[a^\dagger(k) a(k) + a(k) a^\dagger(k) \right]$$

$$\vec{P} = \frac{1}{2} \int \widetilde{dk} \; \vec{k} \left[a^\dagger(k) a(k) + a(k) a^\dagger(k) \right]$$

Real scalar field

- Using these explicit forms we can then obtain

$$[P^\mu, a^\dagger(k)] = k^\mu a^\dagger(k) \quad [P^\mu, a(k)] = -k^\mu a(k)$$

showing that $a^\dagger(k)$ adds momentum k^μ and that $a(k)$ destroys momentum k^μ. That the quantization procedure has produced an infinity number of oscillators should come as no surprise. In fact $a(k), a^\dagger(k)$ correspond to the quantization of the normal modes of the classical Klein-Gordon field.

- By analogy with the harmonic oscillator, we are now in position of finding the eigenstates of H. We start by defining the base state, that in quantum field theory is called the vacuum. We have

$$a(k)\left|0\right\rangle_k = 0 \quad ; \quad \forall_k$$

- Then the vacuum, that we will denote by $|0\rangle$, will be formally given by

$$|0\rangle = \Pi_k \left|0\right\rangle_k$$

and we will assume that it is normalized, that is $\langle 0|0\rangle = 1$. If now we calculate the vacuum energy, we find immediately the first problem with infinities in Quantum Field Theory (QFT).

Real scalar field

- In fact

$$
\begin{aligned}
\langle 0|H|0\rangle =& \frac{1}{2}\int \widetilde{dk}\; \omega_k \left\langle 0\right| \left[a^\dagger(k)a(k)+a(k)a^\dagger(k)\right] \left|0\right\rangle \\
=& \frac{1}{2}\int \widetilde{dk}\; \omega_k \left\langle 0\right| \left[a(k),a^\dagger(k)\right] \left|0\right\rangle \\
=& \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\; \omega_k (2\pi)^3 2\omega_k \delta^3(0) \\
=& \frac{1}{2}\int d^3k\; \omega_k \delta^3(0) = \infty
\end{aligned}
$$

- This infinity can be understood as the the (infinite) sum of the zero point energy of all quantum oscillators. In the discrete case we would have, $\sum_k \frac{1}{2}\omega_k = \infty$. This infinity can be easily removed. We start by noticing that we only measure energies as differences with respect to the vacuum energy, and those will be finite. We will then define the energy of the vacuum as being zero. Technically this is done as follows.

Real scalar field

- We define a new operator $P^{\mu}_{N.O.}$ as

$$\begin{aligned} P^{\mu}_{N.O.} \equiv & \frac{1}{2}\int \widetilde{dk}\ k^{\mu}\left[a^{\dagger}(k)a(k)+a(k)a^{\dagger}(k)\right] \\ & -\frac{1}{2}\int \widetilde{dk}\ k^{\mu}\left\langle 0\right|\left[a^{\dagger}(k)a(k)+a(k)a^{\dagger}(k)\right]\left|0\right\rangle \\ = & \int \widetilde{dk}\ k^{\mu}a^{\dagger}(k)a(k) \end{aligned}$$

- Now $\langle 0|P^{\mu}_{N.O.}|0\rangle = 0.$ The ordering of operators where the annihilation operators appear on the right of the creation operators is called *normal ordering* and the usual notation is

$$:\frac{1}{2}(a^{\dagger}(k)a(k)+a(k)a^{\dagger}(k)):\equiv a^{\dagger}(k)a(k)$$

- To remove the infinity of the energy and momentum corresponds to choose the normal ordering. We will adopt this convention in the following dropping the subscript "$N.O.$" to simplify the notation. This should not appear as an *ad hoc* procedure. In fact, in going from the classical theory where we have products of fields into the quantum theory where the fields are operators, we should have a prescription for the correct ordering of such products.

Real scalar field

- Once we have the vacuum we can build the states by applying the creation operators $a^\dagger(k)$. As in the case of the harmonic oscillator, we can define the *number* operator,

$$N = \int \widetilde{dk}\; a^\dagger(k) a(k)$$

- It is easy to see that N commutes with H and therefore the eigenstates of H are also eigenstates of N.

- The state with one particle of momentum k^μ is obtained as $a^\dagger(k)\left|0\right\rangle$. In fact we have

$$\begin{aligned} P^\mu a^\dagger(k)\left|0\right\rangle &= \int \widetilde{dk}' k'^\mu a^\dagger(k') a(k') a^\dagger(k)\left|0\right\rangle \\ &= \int d^3k' k'^\mu \delta^3(\vec{k}-\vec{k}') a^\dagger(k)\left|0\right\rangle \\ &= k^\mu a^\dagger(k)\left|0\right\rangle \end{aligned}$$

and

$$N a^\dagger(k)\left|0\right\rangle = a^\dagger(k)\left|0\right\rangle$$

Real scalar field

- In a similar way, the state $a^\dagger(k_1)...a^\dagger(k_n)\left|0\right\rangle$ would be a state with n particles. However, the sates that we have just defined have a problem. They are not normalizable and therefore they can not form a basis for the Hilbert space of the quantum field theory, the so-called Fock space.

- The origin of the problem is related to the use of plane waves and states with exact momentum. This can be solved forming states that are superpositions of plane waves

$$\left|1\right\rangle = \lambda \int \widetilde{dk}\, C(k) a^\dagger(k) \left|0\right\rangle$$

- Then

$$\begin{aligned}\left\langle 1|1\right\rangle =& \lambda^2 \int \widetilde{dk_1}\widetilde{dk_2}\, C^*(k_1)C(k_2) \left\langle 0|a(k_1)a^\dagger(k_2)|0\right\rangle\\ =& \lambda^2 \int \widetilde{dk} |C(k)|^2 = 1\end{aligned}$$

and therefore

$$\lambda = \left(\int \widetilde{dk}\ |C(k)|^2\right)^{-1/2} \quad \text{if} \quad \int \widetilde{dk}\ |C(k)|^2 < \infty$$

Real scalar field

- If k is only different from zero in a neighborhood of a given 4-momentum k^μ, then the state will have a well defined momentum (within some experimental error).
- A basis for the Fock space can then be constructed from the n–particle normalized states

$$|n\rangle = \left(n! \int \widetilde{dk}_1 \cdots \widetilde{dk}_n |C(k_1, \cdots k_n)|^2 \right)^{-1/2} \int \widetilde{dk}_1 \cdots \widetilde{dk}_n C(k_1, \cdots k_n) a^\dagger(k_1) \cdots a^\dagger(k_n) \, |0\rangle$$

that satisfy $\langle n|n\rangle = 1 \,, \quad N\,|n\rangle = n\,|n\rangle$

- Due to the commutation relations of the operators $a^\dagger(k)$, the functions $C(k_1 \cdots k_n)$ are symmetric (obey the Bose–Einstein statistics),

$$C(\cdots k_i, \cdots k_j, \cdots) = C(\cdots k_j \cdots k_i \cdots)$$

- This interpretation in terms of particles, with creation and annihilation operators, that results from the canonical quantization, is usually called *second quantization*, as opposed to the description in terms of wave functions (the *first quantization*)

Microscopic causality

- Classically, the fields can be measured with an arbitrary precision. In a relativistic quantum theory we have several problems. The first, results from the fact that the fields are now operators. This means that the observables should be connected with the matrix elements of the operators and not with the operators.

- Besides this question, we can only speak of measuring φ in two space-time points x and y if $[\varphi(x), \varphi(y)]$ vanishes. Let us look at the conditions needed for this to occur.

$$
\begin{aligned}
[\varphi(x), \varphi(y)] &= \int \widetilde{dk}_1 \widetilde{dk}_2 \left\{ \left[a(k_1), a^\dagger(k_2)\right] e^{-ik_1\cdot x + ik_2 \cdot y} + \left[a^\dagger(k_1), a(k_2)\right] \right. \\
&= \int \widetilde{dk}_1 \left(e^{-ik_1\cdot(x-y)} - e^{ik_1\cdot(x-y)} \right) \\
&\equiv i\Delta(x-y)
\end{aligned}
$$

- The function $\Delta(x-y)$ is Lorentz invariant and satisfies the relations

$$
(\Box_x + m^2)\Delta(x-y) = 0
$$

$$
\Delta(x-y) = -\Delta(y-x) \qquad \Delta(\vec{x}-\vec{y}, 0) = 0
$$

Microscopic causality

- The last relation ensures that the equal time commutator of two fields vanishes. Lorentz invariance implies then,

$$\Delta(x-y) = 0 \quad ; \quad \forall\ (x-y)^2 < 0$$

- This means that for two points that can not be physically connected, that is for which $(x-y)^2 < 0$, the fields interpreted as physical observables, can then be independently measured. This result is known as *Microscopic Causality*.

- We also note that

$$\partial^0 \Delta(x-y)|_{x^0=y^0} = -\delta^3(\vec{x}-\vec{y})$$

which ensures the canonical commutation relation

Vacuum fluctuations

- It is well known from Quantum Mechanics that, in an harmonic oscillator, the coordinate is not well defined for the energy eigenstates, that is

$$\left\langle n|q^2|n\right\rangle \; > \; (\langle n|q|n\rangle)^2 = 0$$

- In Quantum Field Theory, we deal with an infinite set of oscillators, and therefore we will have the same behavior, that is,

$$\langle 0|\varphi(x)\varphi(y)|0\rangle \neq 0$$

although

$$\langle 0|\varphi(x)|0\rangle = 0$$

- We can calculate the above expression. We have

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \int \widetilde{dk}_1\widetilde{dk}_2 e^{-ik_1\cdot x}e^{ik_2\cdot y}\left\langle 0|a(k_1)a^\dagger(k_2)|0\right\rangle$$

$$= \int \widetilde{dk}_1 e^{-ik\cdot(x-y)} \equiv \Delta_+(x-y)$$

Vacuum fluctuations

- The function $\Delta_+(x-y)$ corresponds to the positive frequency part of $\Delta(x-y)$. When $y \to x$ this expression diverges quadratically,

$$\langle 0|\varphi^2(x)|0\rangle = \Delta_+(0) = \int \widetilde{dk}_1 = \int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}}$$

- This divergence can not be eliminated in the way we did with the energy of the vacuum. In fact these vacuum fluctuations, as they are known, do have observable consequences like, for instance, the *Lamb shift*.

- We will be less worried with this result, if we notice that for measuring the square of the operator φ at x we need frequencies arbitrarily large, that is, an infinite amount of energy. Physically only *averages* over a finite space-time region have meaning.

Charged scalar field

- The description in terms of real fields does not allow the distinction between particles and anti-particles. It applies only the those cases were the particle and anti-particle are identical, like the π^0. For the more usual case where particles and anti-particles are distinct, it is necessary to have some charge (electric or other) that allows us to distinguish them. For this we need complex fields.

- The theory for the scalar complex field can be easily obtained from two real scalar fields φ_1 and φ_2 with the same mass. If we denote the complex field φ by,

$$\varphi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}$$

then

$$\mathcal{L} = \mathcal{L}(\varphi_1) + \mathcal{L}(\varphi_2) =: \partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi :$$

which leads to the equations of motion

$$(\Box + m^2)\varphi = 0 \; ; \; (\Box + m^2)\varphi^\dagger = 0$$

Charged scalar field

- The classical theory has, at the classical level, a conserved current, $\partial_\mu J^\mu = 0$,

$$J^\mu = i\varphi^\dagger \overleftrightarrow{\partial}^\mu \varphi$$

- Therefore we expect, at the quantum level, the charge Q

$$Q = \int d^3x : i(\varphi^\dagger \dot{\varphi} - \dot{\varphi}^\dagger \varphi) :$$

  to be conserved, that is, $[H, Q] = 0$.

- To show this we need to know the commutation relations for the field φ. The definition and the commutation relations for φ_1 and φ_2 allow us to obtain the following relations for φ and $\varphi^\dagger$:

$$[\varphi(x), \varphi(y)] = [\varphi^\dagger(x), \varphi^\dagger(y)] = 0$$

$$[\varphi(x), \varphi^\dagger(y)] = i\Delta(x - y)$$

Charged scalar field

- For equal times we can get

$$[\pi(\vec{x},t),\varphi(\vec{y},t)] = [\pi^\dagger(\vec{x},t),\varphi^\dagger(\vec{y},t)] = -i\delta^3(\vec{x}-\vec{y})$$

where

$$\pi = \dot{\varphi}^\dagger \quad ; \quad \pi^\dagger = \dot{\varphi}$$

- The plane waves expansion is then

$$\varphi(x) = \int \widetilde{dk}\ \left[a_+(k)e^{-ik\cdot x} + a_-^\dagger(k)e^{ik\cdot x}\right]$$

$$\varphi^\dagger(x) = \int \widetilde{dk}\ \left[a_-(k)e^{-ik\cdot x} + a_+^\dagger(k)e^{ik\cdot x}\right]$$

where the definition of $a_\pm(k)$ is

$$a_\pm(k) = \frac{a_1(k) \pm i a_2(k)}{\sqrt{2}}\ ;\ a_\pm^\dagger = \frac{a_1^\dagger(k) \mp i a_2^\dagger(k)}{\sqrt{2}}$$

Charged scalar field

- The algebra of the operators $a_\pm$ it is easily obtained from the algebra of the operators $a_{i'}s$. We get the following non-vanishing commutators:

$$[a_+(k), a_+^\dagger(k')] = [a_-(k), a_-^\dagger(k')] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}')$$

therefore allowing us to interpret a_+ and $a_+^\dagger$ as annihilation and creation operators of quanta of type $+$, and similarly for the quanta of type $-$.

- We can construct the number operators for those quanta:

$$N_\pm = \int \widetilde{dk}\; a_\pm^\dagger(k) a_\pm(k)$$

- One can easily verify that

$$N_+ + N_- = N_1 + N_2$$

where

$$N_i = \int \widetilde{dk}\; a_i^\dagger(k) a_i(k)$$

Charged scalar field

- The energy-momentum operator can be written in terms of the $+$ and $-$ operators,

$$P^\mu = \int \widetilde{dk}\; k^\mu \left[a_+^\dagger(k) a_+(k) + a_-^\dagger(k) a_-(k) \right]$$

where we have already considered the normal ordering.

- We also obtain for the charge Q:

$$\begin{aligned} Q =& \int d^3x : i(\varphi^\dagger \dot{\varphi} - \dot{\varphi}^\dagger \dot{\varphi}) : \\ =& \int \widetilde{dk}\; \left[a_+^\dagger(k) a_+(k) - a_-^\dagger(k) a_-(k) \right] \\ =& N_+ - N_- \end{aligned}$$

- Using the commutation relations one can easily verify that

$$[H, Q] = 0$$

showing that the charge Q is conserved. The previous equation allows us to interpret the $\pm$ quanta as having charge ± 1.

Charged scalar field

- However, before introducing interactions, the theory is symmetric, and we can not distinguish between the two types of quanta.
- From the commutation relations we obtain,

$$[P^\mu, a_+^\dagger(k)] = k^\mu a_+^\dagger(k)$$
$$[Q, a_+^\dagger(k)] = +a_+^\dagger(k)$$

- This shows that $a_+^\dagger(k)$ creates a quanta with 4-momentum k^μ and charge $+1$. In a similar way we can show that $a_-^\dagger$ creates a quanta with charge -1 and that $a_\pm(k)$ annihilate quanta of charge ± 1, respectively.

Time ordered product and the Feynman propagator

- The operator $\varphi^\dagger$ creates a particle with charge $+1$ or annihilates a particle with charge -1. In both cases it adds a total charge $+1$. In a similar way φ annihilates one unit of charge. Let us construct a state of one particle (not normalized) with charge $+1$ by application of $\varphi^\dagger$ in the vacuum:

$$|\Psi_+(\vec{x},t)\rangle \equiv \varphi^\dagger(\vec{x},t)\,|0\rangle$$

- The amplitude to propagate the state $|\Psi_+\rangle$ into the future to the point $(\vec{x}',t')$ with $t'>t$ is given by

$$\theta(t'-t)\,\langle\Psi_+(\vec{x}',t')|\Psi_+(\vec{x},t)\rangle = \theta(t'-t)\,\left\langle 0|\varphi(\vec{x}',t')\varphi^\dagger(\vec{x},t)|0\right\rangle$$

- In $\varphi^\dagger(\vec{x},t)\,|0\rangle$ only the operator $a_+^\dagger(k)$ is active, while in $\langle 0|\,\varphi(\vec{x}',t')$ the same happens to $a_+(k)$. Therefore this is the matrix element that creates a quanta of charge $+1$ in $(\vec{x},t)$ and annihilates it in $(\vec{x}',t')$ with $t'>t$.

- There exists another way of increasing the charge by $+1$ unit in $(\vec{x},t)$ and decreasing it by -1 in $(\vec{x}',t')$. This is achieved if we create a quanta of charge -1 in $\vec{x}'$ at time t' and let it propagate to $\vec{x}$ where it is absorbed at time $t>t'$. The amplitude is then,

$$\theta(t-t')\,\langle\Psi_-(\vec{x},t)|\Psi_-(\vec{x}',t')\rangle = \left\langle 0|\varphi^\dagger(\vec{x},t)\varphi(\vec{x}',t')|0\right\rangle\theta(t-t')$$

Time ordered product and the Feynman propagator

- Since we can not distinguish the two paths we must sum of the two amplitudes. This is the so-called *Feynman propagator*. It can be written in a more compact way if we introduce the time ordered product. Given two operators $a(x)$ and $b(x')$ we define the time ordered product T by,

$$T a(x) b(x') \equiv \theta(t-t') a(x) b(x') + \theta(t'-t) b(x') a(x)$$

- In this prescription the older times are always to the right of the more recent times. It can be applied to an arbitrary number of operators. With this definition, the Feynman propagator reads,

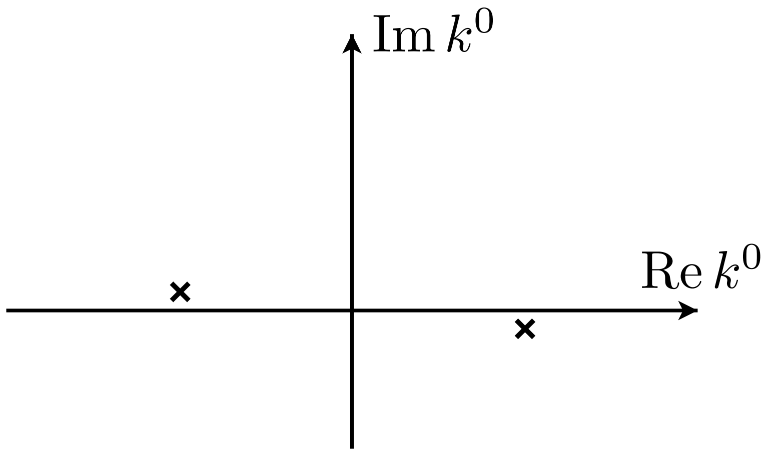
$$\Delta_F(x'-x) = \langle 0 | T \varphi(x') \varphi^\dagger(x) | 0 \rangle$$

- Using the φ and $\varphi^\dagger$ decomposition we can calculate Δ_F (for free fields)

$$\begin{aligned}
\Delta_F(x'-x) &= \int \widetilde{dk} \left[\theta(t'-t) e^{-ik\cdot(x'-x)} + \theta(t-t') e^{ik\cdot(x'-x)} \right] \\
&= \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik\cdot(x'-x)} \\
&\equiv \int \frac{d^4k}{(2\pi)^4} \Delta_F(k) e^{-ik\cdot(x'-x)}
\end{aligned}$$

Time ordered product and the Feynman propagator

- Where $\Delta_F(k) \equiv \dfrac{i}{k^2 - m^2 + i\varepsilon}$
- $\Delta_F(k)$ is the propagator in momenta space (Fourier transform). The equivalence between the previous equations is done using integration in the complex plane of the time component k^0, with the help of the residue theorem. The contour is defined by the $i\varepsilon$ prescription, as indicated in in the figure



- Applying the operator $(\Box'_x + m^2)$ to $\Delta_F(x' - x)$ one can show that

$$(\Box'_x + m^2)\Delta_F(x' - x) = -i\delta^4(x' - x)$$

that is, $\Delta_F(x' - x)$ is the Green's function for the Klein-Gordon equation with Feynman boundary conditions.

Lecture 2

Canonical formalism for the Dirac field

- Let us now apply the formalism of second quantization to the Dirac field. Something has to be changed, otherwise we would be led to a theory obeying Bose statistics, while we know that electrons obey Fermi statistics.

- The Lagrangian density that leads to the Dirac equation is

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi$$

The conjugate momentum to ψ_α is

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\alpha} = i\psi^\dagger_\alpha$$

while the conjugate momentum to $\psi^\dagger_\alpha$ vanishes. The Hamiltonian density is

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi$$

- The requirement of translational and Lorentz invariance for $\mathcal{L}$ leads to the tensors $T^{\mu\nu}$ and $M^{\mu\nu\lambda}$. We get for energy-momentum tensor

$$T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\mathcal{L}$$

Canonical formalism for the Dirac field

- And for the angular-momentum tensor density

$$M^{\mu\nu\lambda} = i\overline{\psi}\gamma^\mu(x^\nu\partial^\lambda - x^\lambda\partial^\nu + \sigma^{\nu\lambda})\psi - \left(x^\nu g^{\mu\lambda} - x^\mu g^{\nu\lambda}\right)\mathcal{L}$$

  where

$$\sigma^{\nu\lambda} = \frac{1}{4}[\gamma^\nu, \gamma^\lambda]$$

- The 4-momentum P^μ and the angular momentum tensor $M^{\nu\lambda}$ are then given by,

$$P^\mu \equiv \int d^3x T^{0\mu}, \quad M^{\mu\lambda} \equiv \int d^3x M^{0\nu\lambda}$$

- This gives for the energy and linear momentum

$$H \equiv \int d^3x \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi$$

$$\vec{P} \equiv \int d^3x \psi^\dagger(-i\vec{\nabla})\psi$$

Canonical formalism for the Dirac field

- If we define the angular momentum vector $\vec{J} \equiv (M^{23}, M^{31}, M^{12})$ we get

$$\vec{J} = \int d^3x \psi^\dagger \left(\vec{r} \times \frac{1}{i} \vec{\nabla} + \frac{1}{2} \vec{\Sigma} \right) \psi$$

which has the familiar aspect $\vec{J} = \vec{L} + \vec{S}$.

- We can also identify a conserved current, $\partial_\mu j^\mu = 0$, with $j^\mu = \overline{\psi} \gamma^\mu \psi$, which will give the conserved charge

$$Q = \int d^3x \psi^\dagger \psi$$

- All that we have done so far is at the classical level. To apply the canonical formalism we have to enforce commutation relations and verify the Lorentz invariance of the theory. This will lead us into problems. To see what these are and how to solve them, we will introduce the plane wave expansions,

$$\psi(x) = \int \widetilde{dp} \sum_s \left[b(p,s) u(p,s) e^{-ip\cdot x} + d^\dagger(p,s) v(p,s) e^{ip\cdot x} \right]$$

$$\psi^\dagger(x) = \int \widetilde{dp} \sum_s \left[b^\dagger(p,s) u^\dagger(p,s) e^{+ip\cdot x} + d(p,s) v^\dagger(p,s) e^{-ip\cdot x} \right]$$

Canonical formalism for the Dirac field

- $u(p,s)$ and $v(p,s)$ are the spinors for positive and negative energy, respectively, introduced in the study of the Dirac equation and $b, b^\dagger, d$ and $d^\dagger$ are operators.

- To see what are the problems with the canonical quantization of fermions, let us calculate P^μ. We get

$$P^\mu = \int \widetilde{dk}\ k^\mu \sum_s \left[b^\dagger(k,s) b(k,s) - d(k,s) d^\dagger(k,s) \right]$$

  where we have used the orthogonality and closure relations for the spinors.

- We realize that if we define the vacuum as $b(k,s)\left|0\right\rangle = d(k,s)\left|0\right\rangle = 0$ and if we quantize with commutators then particles b and particles d will contribute with opposite signs to the energy and the theory will not have a stable ground state.

- In fact, this was the problem already encountered in the study of the negative energy solutions of the Dirac equation, and this is the reason for the negative sign. Dirac's hole theory required Fermi statistics for the electrons and we will see how spin and statistics are related.

Canonical formalism for the Dirac field

- To discover what are the relations that $b, b^\dagger, d$ and $d^\dagger$ should obey, we recall that at the quantum level it is always necessary to verify Lorentz invariance.

$$i[P_\mu, \psi(x)] = \partial_\mu \psi \ ; \ i[P_\mu, \overline{\psi}(x)] = \partial_\mu \overline{\psi}$$

- We start with the above equations and we will discover the appropriate relations for the operators. Using the plane wave expansions we can show that the above equations lead to

$$[P_\mu, b(k,s)] = -k_\mu b(k,s) \ ; \ [P_\mu, b^\dagger(k,s)] = k_\mu b^\dagger(k,s)$$

$$[P_\mu, d(k,s)] = -k_\mu d(k,s) \ ; \ [P_\mu, d^\dagger(k,s)] = k_\mu d^\dagger(k,s)$$

- From the definition of P_μ we get

$$\sum_{s'} \left[\left(b^\dagger(p,s') b(p,s') - d(p,s') d^\dagger(p,s') \right), b(k,s) \right] = -(2\pi)^3 2k^0 \delta^3(\vec{k}-\vec{p}) b(k,s)$$

and three other similar relations.

Canonical formalism for the Dirac field

- If we assume that

$$[d^\dagger(p,s')d(p,s'),b(k,s)]=0$$

the previous condition reads

$$\sum_{s'}\left[b^\dagger(p,s')\{b(p,s'),b(k,s)\}-\{b^\dagger(p,s'),b(k,s)\}b(p,s')\right]=$$

$$=-(2\pi)^3 2k^0\delta^3(\vec{p}-\vec{k})b(k,s)$$

where the parenthesis $\{,\}$ denote anti-commutators.

- It is easy to see that this relation is verified if we impose the canonical anti-commutation relations. We should have

$$\{b^\dagger(p,s),b(k,s)\}=(2\pi)^3 2k^0\delta^3(\vec{p}-\vec{k})\delta_{ss'}$$

$$\{d^\dagger(p,s'),d(k,s)\}=(2\pi)^3 2k^0\delta^3(\vec{p}-\vec{k})\delta_{ss'}$$

and all the other anti-commutators vanish. Note that as b anti-commutes with d and $d^\dagger$, then it commutes with $d^\dagger d$ as we have assumed.

Canonical formalism for the Dirac field

- With the anti-commutator relations both contributions to P^μ are positive. As in boson case we have to subtract the zero point energy. This is done, as usual, by taking all quantities normal ordered. Therefore we have for P^μ,

$$
\begin{aligned}
P^\mu &= \int \widetilde{dk}\; k^\mu \sum_s : \left(b^\dagger(k,s) b(k,s) - d(k,s) d^\dagger(k,s) \right) : \\
&= \int \widetilde{dk}\; k^\mu \sum_s : \left(b^\dagger(k,s) b(k,s) + d^\dagger(k,s) d(k,s) \right) :
\end{aligned}
$$

- For the charge

$$
Q = \int d^3x : \psi^\dagger(x)\psi(x) := \int \widetilde{dk} \sum_s \left[b^\dagger(k,s) b(k,s) - d^\dagger(k,s) d(k,s) \right]
$$

which means that the quanta of b type have charge $+1$ while those of d type have charge -1. It is interesting to note that was the second quantization of the Dirac field that introduced the $-$ sign in the charge, making the charge operator without a definite sign, while in Dirac theory was the probability density that was positive defined. The reverse is true for bosons.

Canonical formalism for the Dirac field

- We can easily show that

$$\begin{aligned} &[Q, b^\dagger(k,s)] = b^\dagger(k,s)\,, \quad [Q, d(k,s)] = d(k,s), \\ &[Q, b(k,s)] = -b(k,s)\,, \quad [Q, d^\dagger(k,s)] = -d^\dagger(k,s) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} &[Q,\psi] = -\psi \\ &[Q,\overline{\psi}] = \overline{\psi} \end{aligned}$$

- In `QED` the charge is given by eQ $(e<0)$. Therefore we see that ψ creates positrons and annihilates electrons and the opposite happens with $\overline{\psi}$.
- We can introduce the number operators

$$N^+(p,s) = b^\dagger(p,s)b(p,s) \quad ; \quad N^-(p,s) = d^\dagger(p,s)d(p,s)$$

and we can rewrite

$$P^\mu = \int \widetilde{dk}\; k^\mu \sum_s (N^+(k,s) + N^-(k,s))$$

$$Q = \int \widetilde{dk}\; \sum_s (N^+(k,s) - N^-(k,s))$$

- Using the anti-commutator relations it is now easy to verify that the theory is Lorentz invariant, that is (see Problems)

$$i[M^{\mu\nu},\psi] = (x^\mu\partial^\nu - x^\nu\partial^\mu)\psi + \Sigma^{\mu\nu}\psi\ .$$

Microscopic causality

- The anti-commutation relations can be used to find the anti-commutation relations at equal times for the fields. We get

$$\begin{aligned}\{\psi_\alpha(\vec{x},t),\psi^\dagger_\beta(\vec{y},t)\} =&\delta^3(\vec{x}-\vec{y})\delta_{\alpha\beta}\\ \{\psi_\alpha(\vec{x},t),\psi_\beta(\vec{y},t)\} =&\{\psi^\dagger_\alpha(\vec{x},t),\psi^\dagger_\beta(\vec{y},t)\}=0\end{aligned}$$

- These relations can be generalized to unequal times

$$\begin{aligned}\{\psi_\alpha(x),\psi^\dagger_\beta(y)\} =&\int \widetilde{dp}\left[\left[(\not{p}+m)\gamma^0\right]_{\alpha\beta} e^{-ip\cdot(x-y)} - \left[(-\not{p}+m)\gamma^0\right]_{\alpha\beta} e^{ip\cdot(x-y)}\right]\\ =&\left[(i\not{\partial}_x+m)\gamma^0\right]_{\alpha\beta}\ i\Delta(x-y)\end{aligned}$$

where the $\Delta(x-y)$ function was defined before for the scalar field.

- The fact that γ^0 appears is due to the fact that in the above relation we took $\psi^\dagger$ and not $\overline{\psi}$. In fact, if we multiply on the right by γ^0 we get

$$\begin{aligned}\{\psi_\alpha(x),\overline{\psi}_\beta(y)\} =&(i\not{\partial}_x+m)_{\alpha\beta} i\Delta(x-y)\\ \{\psi_\alpha(x),\psi_\beta(y)\} =&\{\overline{\psi}_\alpha(x),\overline{\psi}_\beta(y)\}=0\end{aligned}$$

Microscopic causality

- We can easily verify the covariance of the above relations. We use

$$U(a,b)\psi(x)U^{-1}(a,b) = S^{-1}(a)\psi(ax+b)$$

$$U(a,b)\overline{\psi}(x)U^{-1}(a,b) = \overline{\psi}(ax+b)S(a)$$

$$S^{-1}\gamma^\mu S = a^\mu{}_\nu \gamma^\nu$$

- We get

$$U(a,b)\{\psi_\alpha(x),\overline{\psi}_\beta(y)\}U^{-1}(a,b) =$$

$$= S^{-1}_{\alpha\tau}(a)\{\psi_\tau(ax+b),\overline{\psi}_\lambda(ay+b)\}S_{\lambda\beta}(a)$$

$$= S^{-1}_{\alpha\tau}(a)(i\partial\!\!\!/_{ax}+m)_{\tau\lambda} i\Delta(ax-ay)S_{\lambda\beta}(a)$$

$$= (i\partial\!\!\!/+m)_{\alpha\beta} i\Delta(x-y)$$

where we have used the invariance of $\Delta(x-y)$ and the result

$$S^{-1} i\partial\!\!\!/_{ax} S = i\partial\!\!\!/_x$$

Microscopic causality

- For $(x-y)^2<0$ the anti-commutators vanish, because $\Delta(x-y)$ also vanishes. This result allows us to show that any two observables built as bilinear products of $\overline{\psi}$ e ψ commute for two spacetime points for which $(x-y)^2<0$.

- We have

$$\begin{aligned}
&\left[\overline{\psi}_\alpha(x)\psi_\beta(x),\overline{\psi}_\lambda(y)\psi_\tau(y)\right]= \\
&\quad=\overline{\psi}_\alpha(x)\{\psi_\beta(x),\overline{\psi}_\lambda(y)\}\psi_\tau(y)-\{\overline{\psi}_\alpha(x),\overline{\psi}_\lambda(y)\}\psi_\beta(x)\psi_\tau(y) \\
&\qquad+\overline{\psi}_\lambda(y)\overline{\psi}_\alpha(x)\{\psi_\beta(x),\psi_\tau(y)\}-\overline{\psi}_\lambda(y)\{\psi_\tau(y),\overline{\psi}_\alpha(x)\}\psi_\beta(x) \\
&\quad=0
\end{aligned}$$

for $(x-y)^2<0$.

- In this way the microscopic causality is satisfied for the physical observables, such as the charge density or the momentum density.

Feynman propagator

- For the Dirac field, as in the case of the charged scalar field, there are two ways of increasing the charge by one unit in x' and decrease it by one unit in x (note that the electron has negative charge). These ways are

$$\theta(t'-t)\,\langle 0|\psi_\beta(x')\psi^\dagger_\alpha(x)|0\rangle$$

$$\theta(t-t')\,\langle 0|\psi^\dagger_\alpha(x)\psi_\beta(x')|0\rangle$$

In one case an electron of positive energy is created at $\vec{x}$ in the instant t, propagates until $\vec{x}'$ where is annihilated at time $t' > t$. In the other case a positron of positive energy is created in x' and annihilated at x with $t > t'$.

- The Feynman propagator is obtained summing the two amplitudes. Due the exchange of ψ_β and $\overline{\psi}_\alpha$ there must be a minus sign between these two amplitudes. Multiplying by γ^0, we get for the Feynman propagator,

$$\begin{aligned} S_F(x'-x)_{\alpha\beta} =& \theta(t'-t)\,\langle 0|\psi_\alpha(x')\overline{\psi}_\beta(x)|0\rangle \\ & -\theta(t-t')\,\langle 0|\overline{\psi}_\beta(x)\psi_\alpha(x')|0\rangle \\ \equiv & \langle 0|T\psi_\alpha(x')\overline{\psi}_\beta(x)|0\rangle \end{aligned}$$

Feynman propagator

❒ We have defined the time ordered product for fermion fields,

$$T\eta(x)\chi(y) \equiv \theta(x^0 - y^0)\eta(x)\chi(y) - \theta(y^0 - x^0)\chi(y)\eta(x) \; .$$

❒ Inserting in the definitio the expansions for ψ and $\overline{\psi}$ we get,

$$S_F(x'-x)_{\alpha\beta} = \int \widetilde{dk} \Big[(\not{k} + m)_{\alpha\beta}\theta(t'-t)e^{-ik\cdot(x'-x)} + (-\not{k} + m)_{\alpha\beta}\theta(t-t')e^{ik\cdot(x'-x)} \Big]$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} + m)_{\alpha\beta}}{k^2 - m^2 + i\varepsilon} e^{-ik\cdot(x'-x)}$$

$$= \int \frac{d^4k}{(2\pi)^4} S_F(k)_{\alpha\beta} e^{-ik\cdot(x'-x)}$$

where $S_F(k)$ is the Feynman propagator in momenta space.

❒ We can also verify that Feynman's propagator is the Green function for the Dirac equation, that is (see Problems),

$$(i\not{\partial} - m)_{\lambda\alpha} S_F(x' - x)_{\alpha\beta} = i\delta_{\lambda\beta}\delta^4(x' - x)$$

Canonical formalism for Electromagnetic Field

- The free electromagnetic field is described by the classical Lagrangian,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- The free field Maxwell equations are

$$\partial_\alpha F^{\alpha\beta} = 0$$

that correspond to the usual equations in 3-vector notation,

$$\vec{\nabla} \cdot \vec{E} = 0 \quad ; \quad \vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t}$$

- The other Maxwell equations are a consequence of the anti-symmetry of $F_{\mu\nu}$ and can written as,

$$\partial_\alpha \widetilde{F}^{\alpha\beta} = 0 \quad ; \quad \widetilde{F}^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\mu\nu} F_{\mu\nu}$$

corresponding to

$$\vec{\nabla} \cdot \vec{B} = 0 \quad ; \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Canonical formalism for Electromagnetic Field

- Classically, the quantities with physical significance are the fields $\vec{E}$ e $\vec{B}$, and the potentials A^μ are auxiliary quantities that are not unique due to the gauge invariance of the theory.
- In quantum theory the potentials A_μ are the ones playing the leading role as, for instance in the minimal prescription. We have therefore to formulate the quantum fields theory in terms of A^μ and not of $\vec{E}$ and $\vec{B}$.
- When we try to apply the canonical quantization to the potentials A^μ we immediately run into difficulties.
- For instance, if we define the conjugate momentum as,

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial (\dot{A}_{\mu)}}$$

we get

$$\pi^k = \frac{\partial \mathcal{L}}{\partial (\dot{A}_k)} = -\dot{A}^k - \frac{\partial A^0}{\partial x^k} = E^k \quad \pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$$

Canonical formalism for Electromagnetic Field

- Therefore the conjugate momentum to the coordinate A^0 vanishes and does not allow us to use directly the canonical formalism. The problem has its origin in the fact that the photon, that we want to describe, has only two degrees of freedom (positive or negative helicity) but we are using a field A^μ with four degrees of freedom. In fact, we have to impose constraints on A^μ in such a way that it describes the photon. This problem can be addressed in three different ways:

- **1) Radiation Gauge**

  Historically, this was the first method to be used. It is based in the fact that it is always possible to choose a gauge, called the *radiation gauge*, where

$$A^0 = 0 \quad ; \quad \vec{\nabla} \cdot \vec{A} = 0$$

  that is, the potential $\vec{A}$ is transverse. These conditions in reduce the number of degrees of freedom to two, the transverse components of $\vec{A}$. It is then possible to apply the canonical formalism to these transverse components and quantize the electromagnetic field in this way. The problem with this method is that we loose explicit Lorentz covariance. It is then necessary to show that this is recovered in the final result. This method is followed in many text books, for instance in Bjorken and Drell.

Canonical formalism for Electromagnetic Field

- **2) Quantization of systems with constraints**

  It can be shown that the electromagnetism is an example of an Hamilton generalized system, that is a system where there are constraints among the variables. The way to quantize these systems was developed by Dirac for systems of particles with n degrees of freedom. The generalization to quantum field theories is done using the formalism of path integrals. We will study this method in Chapter 6, as it will be shown, this is the only method that can be applied to non-abelian gauge theories, like the Standard Model.

- **3) Undefined metric formalism**

  There is another method that works for the electromagnetism, called the formalism of the *undefined metric*, developed by Gupta and Bleuler. In this formalism, that we will study below, Lorentz covariance is kept, that is we will always work with the 4-vector A_μ, but the price to pay is the appearance of states with negative norm. We have then to define the Hilbert space of the physical states as a sub-space where the norm is positive. We see that in all cases, in order to maintain the explicit Lorentz covariance, we have to complicate the formalism. We will follow the book of Silvan Schweber.

Undefined metric formalism

- To solve the difficulty of the vanishing of π^0, we will start by modifying the Maxwell Lagrangian introducing a new term,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial \cdot A)^2$$

where ξ is a dimensionless parameter.

- The equations of motion are now,

$$\Box A^\mu - \left(1 - \frac{1}{\xi}\right)\partial^\mu(\partial \cdot A) = 0$$

and the conjugate momenta

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} - \frac{1}{\xi}g^{\mu 0}(\partial \cdot A)$$

that is

$$\left\{ \begin{array}{ll} \pi^0 & = -\frac{1}{\xi}(\partial \cdot A) \\ \pi^k & = E^k \end{array} \right.$$

Undefined metric formalism

- We remark that the above Lagrangian and the equations of motion, reduce to Maxwell theory in the gauge $\partial \cdot A = 0$. This why we say that our choice corresponds to a class of Lorenz gauges with parameter ξ. With this abuse of language (in fact we are not setting $\partial \cdot A = 0$, otherwise the problems would come back) the value of $\xi = 1$ is known as the *Feynman gauge* and $\xi = 0$ as the *Landau gauge*.

- From the equations of motion we get

$$\Box(\partial \cdot A) = 0$$

implying that $(\partial \cdot A)$ is a massless scalar field. Although it would be possible to continue with a general ξ, from now on we will take the case of the so-called *Feynman gauge*, where $\xi = 1$. Then the equation of motion coincide with the Maxwell theory in the Lorenz gauge.

- As we do not have anymore $\pi^0 = 0$, we can impose the canonical commutation relations at equal times:

$$[\pi^\mu(\vec{x}, t), A_\nu(\vec{y}, t)] = -i g^\mu{}_\nu \, \delta^3(\vec{x} - \vec{y})$$
$$[A_\mu(\vec{x}, t), A_\nu(\vec{y}, t)] = [\pi_\mu(\vec{x}, t), \pi_\nu(\vec{y}, t)] = 0$$

Undefined metric formalism

- Knowing that $[A_\mu(\vec{x},t), A_\mu(\vec{y},t)] = 0$ at equal times, we can conclude that the space derivatives of A_μ also commute at equal times. Then, noticing that

$$\pi^\mu = -\dot{A}^\mu + \text{ space derivatives}$$

we can write instead of the previous commutation relations

$$[A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = [\dot{A}_\mu(\vec{x},t), \dot{A}_\mu(\vec{y},t)] = 0$$

$$[\dot{A}_\mu(\vec{x},t), A_\nu(\vec{y},t)] = i g_{\mu\nu}\delta^3(\vec{x}-\vec{y})$$

- If we compare these relations with the corresponding ones for the real scalar field, where the only one non-vanishing is,

$$[\dot{\varphi}(\vec{x},t), \varphi(\vec{y},t)] = -i\delta^3(\vec{x}-\vec{y})$$

we see ($g_{\mu\nu} = \text{diag}(+,-,-,-)$) that the relations for space components are equal but they differ for the time component. This sign will be the source of the difficulties previously mentioned.

Undefined metric formalism

- If, for the moment, we do not worry about this sign, we expand $A_\mu(x)$

$$A^\mu(x) = \int \widetilde{dk} \sum_{\lambda=0}^{3} \left[a(k,\lambda)\varepsilon^\mu(k,\lambda)e^{-ik\cdot x} + a^\dagger(k,\lambda)\varepsilon^{\mu *}(k,\lambda)e^{ik\cdot x} \right]$$

where $\varepsilon^\mu(k,\lambda)$ are a set of four independent 4-vectors that we assume to real, without loss of generality.

- We will now make a choice for these 4-vectors. We choose $\varepsilon^\mu(1)$ and $\varepsilon^\mu(2)$ orthogonal to k^μ and n^μ, such that

$$\varepsilon^\mu(k,\lambda)\varepsilon_\mu(k,\lambda') = -\delta_{\lambda\lambda'} \text{ for } \lambda,\lambda' = 1,2$$

After, we choose $\varepsilon^\mu(k,3)$ in the plane $(k^\mu, n^\mu) \perp$ to n^μ such that

$$\varepsilon^\mu(k,3)n_\mu = 0 \quad ; \quad \varepsilon^\mu(k,3)\varepsilon_\mu(k,3) = -1$$

- Finally we choose $\varepsilon^\mu(k,0) = n^\mu$. The vectors $\varepsilon^\mu(k,1)$ and $\varepsilon^\mu(k,2)$ are called transverse polarizations, while $\varepsilon^\mu(k,3)$ and $\varepsilon^\mu(k,0)$ longitudinal and scalar polarizations, respectively.

Undefined metric formalism

- In the frame where $n^\mu = (1,0,0,0)$ and $\vec{k}$ is along the z axis we have

$$\varepsilon^\mu(k,0) \equiv (1,0,0,0) \; ; \; \varepsilon^\mu(k,1) \equiv (0,1,0,0)$$

$$\varepsilon^\mu(k,2) \equiv (0,0,1,0) \; ; \; \varepsilon^\mu(k,3) \equiv (0,0,0,1)$$

- In general we can show that

$$\varepsilon(k,\lambda)\cdot\varepsilon^*(k,\lambda') = g^{\lambda\lambda'} \,, \quad \sum_\lambda g^{\lambda\lambda}\varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda) = g^{\mu\nu}$$

- Inserting the plane wave expansion we get

$$[a(k,\lambda), a^\dagger(k',\lambda')] = -g^{\lambda\lambda'}2k^0(2\pi)^3\delta^3(\vec{k}-\vec{k}')$$

showing, once more, that the quanta associated with $\lambda = 0$ has a commutation relation with the wrong sign.

- Before addressing this problem, we can verify we get for arbitrary times

$$[A_\mu(x), A_\nu(y)] = -ig_{\mu\nu}\Delta(x,y)$$

showing the covariance of the theory. The function $\Delta(x-y)$ is the same that was introduced before for scalar fields.

Undefined metric formalism

- Therefore, up to this point, everything is as if we had 4 scalar fields. There is, however, the problem of the sign difference in one of the commutators.
- Let us now see what are the consequences of this sign. For that we introduce the vacuum state defined by

$$a(k,\lambda)\left|0\right\rangle = 0 \qquad \lambda = 0,1,2,3$$

- To see the problem with the sign we construct the one-particle state with scalar polarization, that is

$$\left|1\right\rangle = \int \widetilde{dk}\ f(k) a^\dagger(k,0)\left|0\right\rangle$$

and calculate its norm

$$\begin{aligned}\left\langle 1|1\right\rangle &= \int \widetilde{dk_1}\widetilde{dk_2} f^*(k_1) f(k_2)\left\langle 0|a(k_1,0)a^\dagger(k_2,0)|0\right\rangle \\ &= -\left\langle 0|0\right\rangle \int \widetilde{dk}\ |f(k)|^2\end{aligned}$$

The state $\left|1\right\rangle$ has a negative norm.

Undefined metric formalism

- The same calculation for the other polarization would give well behaved positive norms. We therefore conclude that the Fock space of the theory has indefinite metric. What happens to the probabilistic interpretation of QM?

- To solve this problem we note that we are not working anymore with the classical Maxwell theory because we modified the Lagrangian. What we would like to do is to impose the condition $\partial \cdot A = 0$, but that is impossible as an equation for operators. We can, however, require that condition on a weaker form, as a condition only to be verified by the physical states.

- More specifically, we require that the part of $\partial \cdot A$ that contains the annihilation operator (positive frequencies) annihilates the physical states,

$$\partial^\mu A_\mu^{(+)} |\psi\rangle = 0$$

The states $|\psi\rangle$ can be written in the form

$$|\psi\rangle = |\psi_T\rangle \, |\phi\rangle$$

where $|\psi_T\rangle$ is obtained from the vacuum with creation operators with transverse polarization and $|\phi\rangle$ with scalar and longitudinal polarization.

Undefined metric formalism

- This decomposition depends, of course, on the choice of polarization vectors. To understand the consequences it is enough to analyze the states $|\phi\rangle$ as $\partial^\mu A_\mu^{(+)}$ contains only scalar and longitudinal polarizations,

$$i\partial \cdot A^{(+)} = \int \widetilde{dk}\; e^{-ik\cdot x} \sum_{\lambda=0,3} a(k,\lambda)\; \varepsilon(k,\lambda)\cdot k$$

- Therefore the previous condition becomes

$$\sum_{\lambda=0,3} k\cdot\varepsilon(k,\lambda)\; a(k,\lambda)\,|\phi\rangle = 0$$

- This does not determine completely $|\phi\rangle$. In fact, there is much arbitrariness in the choice of the transverse polarization vectors, to which we can always add a term proportional to k^μ because $k\cdot k=0$. This arbitrariness must reflect itself on the choice of $|\phi\rangle$. The condition is equivalent to,

$$\left[a(k,0)-a(k,3)\right]|\phi\rangle = 0\ .$$

Undefined metric formalism

- We can construct $|\phi\rangle$ as a linear combination of states $|\phi_n\rangle$ with n scalar or longitudinal photons:

$$|\phi\rangle = C_0\,|\phi_0\rangle + C_1\,|\phi_1\rangle + \cdots + C_n\,|\phi_n\rangle + \cdots$$

$$|\phi_0\rangle \equiv |0\rangle$$

- The states $|\phi_n\rangle$ are eigenstates of the operator number for scalar or longitudinal photons,

$$N'\,|\phi_n\rangle = n\,|\phi_n\rangle$$

where

$$N' = \int \widetilde{dk}\ \left[a^\dagger(k,3)a(k,3) - a^\dagger(k,0)a(k,0)\right]$$

- Then

$$n\,\langle\phi_n|\phi_n\rangle = \langle\phi_n|N'|\phi_n\rangle = 0$$

Undefined metric formalism

- This means that

$$\langle \phi_n | \phi_n \rangle = \delta_{n0}$$

that is, for $n \neq 0$, the state $|\phi_n\rangle$ has zero norm. We have then for the general state $|\phi\rangle$,

$$\langle \phi | \phi \rangle = |C_0|^2 \geq 0$$

and the coefficients $C_i, i = 1, \cdots n \cdots$ are arbitrary.

- We have to show that this arbitrariness does not affect the physical observables. The Hamiltonian is

$$\begin{aligned}
H =& \int d^3x : \pi^\mu \dot{A}_\mu - \mathcal{L} : \\
=& \frac{1}{2} \int d^3x : \sum_{i=1}^{3} \left[\dot{A}_i^2 + (\vec{\nabla} A_i)^2 \right] - \dot{A}_0^2 - (\vec{\nabla} A_0)^2 : \\
=& \int \widetilde{dk}\; k^0 \left[\sum_{\lambda=1}^{3} a^\dagger(k,\lambda) a(k,\lambda) - a^\dagger(k,0) a(k,0) \right]
\end{aligned}$$

Undefined metric formalism

- It is easy to check that if $|\psi\rangle$ is a physical state we have

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}=\frac{\left\langle\psi_T|\int\widetilde{dk}\;k^0\sum_{\lambda=1}^{2}a^\dagger(k,\lambda)a(k,\lambda)|\psi_T\right\rangle}{\langle\psi_T|\psi_T\rangle}$$

  and the arbitrariness on the physical states completely disappears when we take average values. Besides that, only the physical transverse polarizations contribute to the result. One can show that the arbitrariness in $|\phi\rangle$ is related with a gauge transformation within the class of Lorenz gauges.

- It is important to note that although for the average values of the physical observables only the transverse polarizations contribute, the scalar and longitudinal polarizations are necessary for the consistency of the theory. In particular they show up when we consider complete sums over the intermediate states.

- Invariance for translations is readily verified. For that we write,

$$P^\mu=\int\widetilde{dk}\;k^\mu\sum_{\lambda=0}^{3}(-g^{\lambda\lambda})a^\dagger(k,\lambda)a(k,\lambda)$$

Undefined metric formalism

- Then

$$
\begin{aligned}
i[P^\mu, A^\nu] =& \int \widetilde{dk}\, \widetilde{dk}'\, ik^\mu \sum_{\lambda,\lambda'} (-g^{\lambda\lambda}) \left\{ \left[a^\dagger(k,\lambda)a(k,\lambda), a(k',\lambda')\right] \varepsilon^\nu(k',\lambda')e^{-ik\cdot x} \right. \\
&\left. + \left[a^\dagger(x,\lambda)a(k,\lambda), a^\dagger(k',\lambda')\right] \varepsilon^{*\nu}(k',\lambda')e^{ik'\cdot x} \right\} \\
=& \int \widetilde{dk}\; ik^\mu \sum_\lambda \left[a(k,\lambda)\varepsilon^\nu(k,\lambda)e^{-ik\cdot x} - a^\dagger(k,\lambda)\varepsilon^\nu(k,\lambda)e^{ik\cdot x}\right] \\
=& \partial^\mu A^\nu
\end{aligned}
$$

  showing the invariance under translations.

- In a similar way, it can be shown the invariance for Lorentz transformations (see Problems). For that we have to show that

$$
M^{jk} = \int d^3x : \left[x^j T^{0k} - x^k T^{0j} + E^j A^k - E^k A^j\right] :
$$

$$
M^{0i} = \int d^3x : \left[x^0 T^{0i} - x^i T^{00} - (\partial \cdot A) A^i - E^i A^0\right] :
$$

Undefined metric formalism

- Where ($\xi = 1$)

$$T^{0i} = -(\partial \cdot A)\,\partial^i A^0 - E^k \partial^i A^k$$

$$T^{00} = \sum_{i=1}^{3} \left[\dot{A}_i^2 + (\vec{\nabla} A_j)^2\right] - \dot{A}_0^2 - (\vec{\nabla} A_0)^2$$

- Using these expressions one can show that the photon has helicity ± 1, corresponding therefore to spin one. For that we start by choosing the direction of $\vec{k}$ along the axis 3 (z axis) and take the polarization vector defined before. A one-photon physical state will then be (not normalized),

$$|k, \lambda\rangle = a^\dagger(k, \lambda)\,|0\rangle \qquad \lambda = 1, 2$$

- Let us now calculate the angular momentum along the axis 3. This is given by

$$M^{12}\,|k, \lambda\rangle = M^{12} a^\dagger(k, \lambda)\,|0\rangle = [M^{12}, a^\dagger(k, \lambda)]\,|0\rangle$$

where we have used the fact that the vacuum state satisfies $M^{12}\,|0\rangle = 0$.

Undefined metric formalism

- The operator M^{12} has one part corresponding the orbital angular momenta and another corresponding to the spin. The contribution of the orbital angular momenta vanishes (angular momenta in the direction of motion) as one can see calculating the commutator. In fact the commutator with the orbital angular momenta is proportional to k^1 or k^2, which are zero by hypothesis. Let us then calculate the spin part. Using the notation,

$$A^\mu = A^{\mu(+)} + A^{\mu(-)}$$

where $A^{\mu(+)}(A^{\mu(-)})$ correspond to the $+$ $(-)$ frequencies, we get

$$: E^1A^2 - E^2A^1 := E^{1(+)}A^{2(+)} + E^{1(-)}A^{2(+)} + A^{2(-)}E^{1(+)} + E^{1(-)}A^{2(-)} - (1 \leftrightarrow 2)$$

- Then

$$\begin{aligned}
&\left[: E^1A^2 - E^2A^1 :, a^\dagger(k,\lambda)\right] = \\
&= E^{1(+)}\left[A^2(+), a^\dagger(k,\lambda)\right] + \left[E^{1(+)}, a^\dagger(k,\lambda)\right] A^{2(+)} \\
&\quad + E^1(-)\left[A^2(+), a^\dagger(k,\lambda)\right] + A^{2(-)}\left[E^{1(+)}, a^\dagger(k,\lambda)\right] - (1 \leftrightarrow 2) \\
&= E^1\left[A^{2(+)}, a^\dagger(k,\lambda)\right] + A^2\left[E^{1(+)}, a^\dagger(k,\lambda)\right] - (1 \leftrightarrow 2)
\end{aligned}$$

Undefined metric formalism

❐ Now (recall that $\lambda = 1, 2$)

$$
\begin{aligned}
[A^{2(+)}, a^\dagger(k,\lambda)] =& \int \widetilde{dk}' \sum_{\lambda'} \varepsilon^2(k',\lambda') \; \left[a(k',\lambda'), a^\dagger(k,\lambda)\right] e^{-ik'\cdot x} \\
=& \varepsilon^2(k,\lambda) e^{-ik\cdot x}
\end{aligned}
$$

$$
\begin{aligned}
[E^{1(+)}, a^\dagger(k,\lambda)] =& \int \widetilde{dk}' \sum_{\lambda'} \left(ik'^0 \varepsilon^0(k',\lambda') + ik'^1 \varepsilon^0(k',\lambda') \right) \left[a(k',\lambda'), a^\dagger(k,\lambda)\right] e^{-ik'\cdot x} \\
=& ik^0 \varepsilon^1(k,\lambda) e^{-ik\cdot x}
\end{aligned}
$$

❐ Therefore

$$
\begin{aligned}
&\int d^3x \left[: E^1 A^2 - E^2 A^1 :, a^\dagger(k,\lambda)\right] \\
&= \int d^3x e^{-ik\cdot x} \left[E^1 \varepsilon^2(k,\lambda) + A^2 ik^0 \varepsilon^1(k,\lambda) - E^2 \varepsilon^1(k,\lambda) + A^1 ik^0 \varepsilon^2(k,\lambda)\right] \\
&= \int d^3x e^{-ik\cdot x} \left[\varepsilon^1(k,\lambda) \overleftrightarrow{\partial}_0 A^2(x) - \varepsilon^2(k,\lambda) \overleftrightarrow{\partial}_0 A^1(x)\right]
\end{aligned}
$$

where we have used the fact that $E^i = -\dot{A}^i, \quad i = 1, 2$, for our choice of frame and polarization vectors.

Undefined metric formalism

- On the other hand

$$a(k,\lambda) = -i \int d^3x e^{ik\cdot x} \overleftrightarrow{\partial}_0 \, \varepsilon^\mu(k,\lambda) A_\mu(x)$$

$$a^\dagger(k,\lambda) = i \int d^3x e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \, \varepsilon^\mu(k,\lambda) A_\mu(x)$$

- For our choice we get

$$a^\dagger(k,1) = -\, i \int d^3x e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \, A^1(x)$$

$$a^\dagger(k,2) = -\, i \int d^3x e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \, A^2(x)$$

- Therefore

$$[M^{12}, a^\dagger(k,\lambda)] = i\varepsilon^1(k,\lambda) a^\dagger(k,2) - i\varepsilon^2(k,\lambda) a^\dagger(k,1)$$

Undefined metric formalism

- We find that the state $a^\dagger(k,\lambda)\left|0\right\rangle, \lambda=1,2$ is not an eigenstate of the operator M^{12}.

- However the linear combinations,

$$
\begin{aligned}
a^\dagger_R(k) =& \frac{1}{\sqrt{2}}\left[a^\dagger(k,1)+ia^\dagger(k,2)\right]\\
a^\dagger_L(k) =& \frac{1}{\sqrt{2}}\left[a^\dagger(k,1)-ia^\dagger(k,2)\right]
\end{aligned}
$$

which correspond to right and left circular polarization, verify

$$
[M^{12},a^\dagger_R(k)]=a^\dagger_R(k)\ ;\ [M^{12},a^\dagger_L(k)]=-a^\dagger_L(k)
$$

- This shows that the photon has spin 1 with right or left circular polarization (negative or positive helicity).

Feynman propagator

- The Feynman propagator is defined as the vacuum expectation value of the time ordered product of the fields, that is

$$\begin{aligned}G_{\mu\nu}(x,y) \equiv& \langle 0|T A_\mu(x) A_\nu(y)|0\rangle \\ =&\theta(x^0-y^0)\,\langle 0|A_\mu(x)A_\nu(y)|0\rangle + \theta(y^0-x^0)\,\langle 0|A_\nu(y)A_\mu(x)|0\rangle\end{aligned}$$

- Inserting the expansions for $A_\mu(x)$ and $A_\nu(y)$ we get

$$\begin{aligned}G_{\mu\nu}(x-y) =& -g_{\mu\nu}\int \widetilde{dk}\,\left[e^{-ik\cdot(x-y)}\theta(x^0-y^0) + e^{ik\cdot(x-y)\theta(y^0-x^0)}\right] \\ =& -g_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\varepsilon}e^{-ik\cdot(x-y)} \\ \equiv& \int \frac{d^4k}{(2\pi)^4} G_{\mu\nu}(k) e^{-ik\cdot(x-y)}\end{aligned}$$

- $G_{\mu\nu}(k)$ is the Feynman propagator on the momentum space

$$G_{\mu\nu}(k) \equiv \frac{-ig_{\mu\nu}}{k^2+i\varepsilon}$$

Feynman propagator

- It is easy to verify that $G_{\mu\nu}(x-y)$ is the Green's function of the equation of motion, that for $\xi = 1$ is the wave equation, that is

$$\Box_x G_{\mu\nu}(x-y) = i g_{\mu\nu} \delta^4(x-y)$$

- These expressions for $G_{\mu\nu}(x-y)$ and $G_{\mu\nu}(k)$ correspond to the particular case of $\xi = 1$, the so-called *Feynman gauge*. For the general case, $\xi \neq 0$

$$\left[\Box_x g^{\mu}_{\rho} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial_\rho\right] A^\rho(x) = 0$$

- For this case the equal times commutation relations are more complicated (see Problems). Using those relations one can show that the Feynman propagator is still the Green's function of the equation of motion, that is

$$\left[\Box_x g^{\mu}_{\rho} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial_\rho\right] \langle 0|T A^\rho(x) A^\nu(y)|0\rangle = i g^{\mu\nu} \delta^4(x-y)$$

- Using this equation we can then obtain in an arbitrary ξ gauge (of the Lorenz type),

$$G_{\mu\nu}(k) = -i \frac{g_{\mu\nu}}{k^2 + i\varepsilon} + i(1-\xi) \frac{k_\mu k_\nu}{(k^2 + i\varepsilon)^2} \ .$$

Discrete Symmetries

- We know from the study of the Dirac equation the transformations like space inversion (Parity) and charge conjugation, are symmetries of the Dirac equation.

- More precisely, if $\psi(x)$ is a solution of the Dirac equation, then

$$\psi'(x) = \psi'(-\vec{x}, t) = \gamma_0 \psi(\vec{x}, t)$$

$$\psi^c(x) = C\overline{\psi}^T(x)$$

  are also solutions (if we take the charge $-e$ for ψ^c). Similar operations could also be defined for scalar and vector fields.

- With second quantization the fields are no longer functions, they become operators. We have therefore to find unitary operators $\mathcal{P}$ and $\mathcal{C}$ that describe those operations within this formalism. There is another discrete symmetry, time reversal, that in second quantization will be described by an anti-unitary operator $\mathcal{T}$.

- We will exemplify with the scalar field how to get these operators. We will leave the Dirac and Maxwell fields as exercises.

Parity

- To define the meaning of the Parity operation we have to put the system in interaction with the measuring system, considered to be classical. This means that we will consider the system described by

$$\mathcal{L} \longrightarrow \mathcal{L} - j_\mu(x) A^\mu_{ext}(x)$$

where we have considered that the interaction is electromagnetic. $j_\mu(x)$ is the electromagnetic current that has the form,

$$j_\mu(x) = ie : \varphi^* \overleftrightarrow{\partial}_\mu \varphi : \quad \text{scalar field}$$

$$j_\mu(x) = e : \overline{\psi} \gamma_\mu \psi : \quad \text{Dirac field}$$

- In a Parity transformation we invert the coordinates of the measuring system, therefore the classical fields are now

$$A^\mu_{ext} = (A^0_{ext}(-\vec{x}, t)), -\vec{A}_{ext}(-\vec{x}, t) = A_\mu^{ext}(-\vec{x}, t)$$

- For the dynamics of the new system to be identical to that of the original system, which should be the case if Parity is conserved, it is necessary that the equations of motion remain the same.

Parity

- This is true if

$$\mathcal{P}\mathcal{L}(\vec{x},t)\mathcal{P}^{-1} = \mathcal{L}(-\vec{x},t)$$

$$\mathcal{P}j_\mu(\vec{x},t)\mathcal{P}^{-1} = j^\mu(-\vec{x},t)$$

- These are the conditions that a theory should obey in order to be invariant under Parity.

- Furthermore $\mathcal{P}$ should leave the commutation relations unchanged, so that the quantum dynamics is preserved. For each theory that conserves Parity should be possible to find an unitary operator $\mathcal{P}$ that satisfies these conditions.

- Now we will find such an operator $\mathcal{P}$ for the scalar field. It is easy to verify that the condition

$$\mathcal{P}\varphi(\vec{x},t)\mathcal{P}^{-1} = \pm\varphi(-\vec{x},t)$$

satisfies all the requirements. The sign $\pm$ is the *intrinsic* parity of the particle described by the field φ, ($+$ for scalar and $-$ for pseudo-scalar).

Parity

- In terms of the expansion of the momentum, the requirement is

$$\mathcal{P} a(k) \mathcal{P}^{-1} = \pm a(-k) \quad ; \quad \mathcal{P} a^\dagger(k) \mathcal{P}^{-1} = \pm a^\dagger(-k)$$

where $-k$ means that we have changed $\vec{k}$ into $-\vec{k}$ (but k^0 remains intact, that is, $k^0 = +\sqrt{|\vec{k}|^2 + m^2}$).

- It is easier to solve this requirement in the momentum space. As $\mathcal{P}$ should be unitary, we write

$$\mathcal{P} = e^{iP}$$

- Then

$$\begin{aligned}\mathcal{P} a(k) \mathcal{P}^{-1} =& a(k) + i[P, a(k)] + \cdots + \frac{i^n}{n!}[P, [\cdots, [P, a(k)] \cdots] + \cdots \\ =& -a(-k)\end{aligned}$$

where we have chosen the case of the pseudo-scalar field.

Parity

- The last relation suggests the form

$$[P, a(k)] = \frac{\lambda}{2}[a(k) + \varepsilon a(-k)]$$

where λ and $\varepsilon = \pm 1$ are to be determined.

- We get

$$[P, [P, a(k)]] = \frac{\lambda^2}{2}[a(k) + \varepsilon a(-k)]$$

and therefore

$$\begin{aligned}\mathcal{P} a(k) \mathcal{P}^{-1} =& a(k) + \frac{1}{2}\left[i\lambda + \frac{(i\lambda)^2}{2!} + \cdots + \frac{(i\lambda)^4}{n!} + \cdots \right] (a(k) + \varepsilon a(-k)) \\ =& \frac{1}{2}[a(k) - \varepsilon a(-k)] + \frac{1}{2} e^{i\lambda}[a(k) + \varepsilon a(-k)] \\ =& - a(-k)\end{aligned}$$

- We solve this if we choose $\lambda = \pi$ and $\varepsilon = +1$ ($\lambda = \pi$ and $\varepsilon = -1$ for the scalar case).

Parity

- It is easy to check that (for $\lambda = \pi$ and $\varepsilon = +1$)

$$P_{ps} = -\frac{\pi}{2}\int \widetilde{dk}\ \left[a^\dagger(k)a(k) + a^\dagger(k)a(-k)\right] = P_{ps}^\dagger$$

$$\mathcal{P}_{ps} = \exp\left\{-i\frac{\pi}{2}\int \widetilde{dk}\ \left[a^\dagger(k)a(k) + a^\dagger(k)a(-k)\right]\right\}$$

- For the scalar field

$$\mathcal{P}_{s} = \exp\left\{-i\frac{\pi}{2}\int \widetilde{dk}\ \left[a^\dagger(k)a(k) - a^\dagger(k)a(-k)\right]\right\}$$

- For the case of the Dirac field, the condition is now

$$\mathcal{P}\psi(\vec{x},t)\mathcal{P}^{-1} = \gamma^0\psi(-\vec{x},t)$$

- Repeating the same steps we get

$$\mathcal{P}_{Dirac} = \exp\left\{-i\frac{\pi}{2}\int \widetilde{dp}\ \sum_s \left[b^\dagger(p,s)b(p,s) - b^\dagger(p,s)b(-p,s)\right.\right.$$
$$\left.\left. + d^\dagger(p,s)d(p,s) + d^\dagger(p,s)d(-p,s)\right]\right\}$$

- The case of the Maxwell field is left as an exercise.

Charge conjugation

- The conditions for charge conjugation invariance are now

$$\mathcal{C}\mathcal{L}(x)\mathcal{C}^{-1} = \mathcal{L} \quad ; \quad \mathcal{C}j_\mu\mathcal{C}^{-1} = -j_\mu$$

where j_μ is the electromagnetic current.

- These conditions are verified for the charged scalar fields if

$$\mathcal{C}\varphi(x)\mathcal{C}^{-1} = \varphi^*(x) \quad ; \quad \mathcal{C}\varphi^*(x)\mathcal{C}^{-1} = \varphi(x)$$

and for the Dirac field if

$$\mathcal{C}\psi_\alpha(x)\mathcal{C}^{-1} = C_{\alpha\beta}\,\overline{\psi}_\beta(x)$$

$$\mathcal{C}\overline{\psi}_\alpha(x)\mathcal{C}^{-1} = -\psi_\beta(x)C^{-1}_{\beta\alpha}$$

where C is the charge conjugation matrix.

- Finally from the invariance of $j_\mu A^\mu$ we obtain the condition for the electromagnetic field,

$$\mathcal{C}A_\mu\mathcal{C}^{-1} = -A_\mu$$

Charge conjugation

- By using a method similar to the one used in the case of the Parity we can get the operator $\mathcal{C}$ for the different theories. For the scalar field we get

$$\mathcal{C}_s = \exp\left\{ i\frac{\pi}{2} \int \widetilde{dk}\ (a_+^\dagger - a_-^\dagger)(a_+ - a_-) \right\}$$

- For the Dirac field

$$\mathcal{C} = \mathcal{C}_1 \mathcal{C}_2$$

with

$$\mathcal{C}_1 = \exp\left\{ -i \int \widetilde{dp}\ \sum_s \phi(p,s) \left[b^\dagger(p,s) b(p,s) - d^\dagger(p,s) d(p,s) \right] \right\}$$

$$\mathcal{C}_2 = \exp\left\{ i\frac{\pi}{2} \int \widetilde{dp}\ \sum_s \left[b^\dagger(p,s) - d^\dagger(p,s) \right] \left[b(p,s) - d(p,s) \right] \right\}$$

- Where

$$v(p,s) = e^{i\phi(p,s)}\ u^c(p,s)\ , \quad u(p,s) = e^{i\phi(p,s)}\ v^c(p,s)$$

and the phase $\phi(p,s)$ is arbitrary.

Time reversal

- Classically the meaning of the time reversal invariance it is clear. We change the sign of the time, the velocities change direction and the system goes from what was the final state to the initial state.

- This exchange between the initial and final state has as consequence, in quantum mechanics, that the corresponding operator must be anti-linear or anti-unitary. In fact $\langle f|i\rangle = \langle i|f\rangle^*$ and therefore if we want $\langle \mathcal{T}\varphi_f|\mathcal{T}\varphi_i\rangle = \langle \varphi_i|\varphi_f\rangle$ then $\mathcal{T}$ must include the complex conjugation operation.

- We can write

$$\mathcal{T} = \mathcal{U} K$$

where $\mathcal{U}$ is unitary and K is the instruction to tale the complex conjugate of all *c-numbers*. Then

$$\begin{aligned}\langle T\varphi_f|T\varphi_i\rangle &= \langle \mathcal{U}K\varphi_f|\mathcal{U}K\varphi_i\rangle \\ &= \langle \mathcal{U}\varphi_f|\mathcal{U}\varphi_i\rangle^* \\ &= \langle \varphi_f|\varphi_i\rangle^* = \langle \varphi_i|\varphi_f\rangle\end{aligned}$$

as we wanted.

Time reversal

- A theory will be invariant under time reversal if

$$\mathcal{T}\mathcal{L}(\vec{x},t)\mathcal{T}^{-1}=\mathcal{L}(\vec{x},-t)\,,\quad \mathcal{T}j_\mu(\vec{x},t)\mathcal{T}^{-1}=j^\mu(\vec{x},-t)$$

- For the scalar field this condition will be verified if

$$\mathcal{T}\varphi(\vec{x},t)\mathcal{T}^{-1}=\pm\varphi(\vec{x},-t)$$

- For the electromagnetic field we must have.

$$\mathcal{T}A^\mu(\vec{x},t)\mathcal{T}^{-1}=A_\mu(\vec{x},-t)$$

making $j^\mu A_\mu$ invariant.

- For the case of the Dirac field the transformation is

$$\mathcal{T}\psi_\alpha(\vec{x},t)\mathcal{T}^{-1}=T_{\alpha\beta}\psi_\beta(\vec{x},-t)$$

In order that the last equation is satisfied, the T matrix must satisfy

$$T\gamma_\mu T^{-1}=\gamma_\mu^T=\gamma^{\mu *}$$

Time reversal

- This has a solution, in the Dirac representation,

$$T = i\gamma^1\gamma^3$$

- Applying the same type of reasoning already used for $\mathcal{P}$ and $\mathcal{C}$ we can find $\mathcal{T}$, or equivalently, $\mathcal{U}$. For the Dirac field, noticing that

$$\begin{aligned} Tu(p,s) &= u^*(-p,-s)e^{i\alpha_+(p,s)} \\ Tv(p,s) &= v^*(-p,-s)e^{i\alpha_-(p,s)} \end{aligned}$$

we can write $\mathcal{U} = \mathcal{U}_1\mathcal{U}_2$ and obtain

$$\begin{aligned} \mathcal{U}_1 &= \exp\left\{-i\int \widetilde{dp} \sum_s \left[\alpha_+ b^\dagger(p,s)b(p,s) - \alpha_- d^\dagger(p,s)d(p,s)\right]\right\} \\ \mathcal{U}_2 &= \exp\left\{-i\frac{\pi}{2}\int \widetilde{dp} \sum_s \left[b^\dagger(p,s)b(p,s) + b^\dagger(p,s)b(-p-s)\right.\right. \\ &\qquad \left.\left. - d^\dagger(p,s)d(p,s) - d^\dagger(p,s)d(-p,-s)\right]\right\} \end{aligned}$$

The $\mathcal{TCP}$ theorem

- It is a fundamental theorem in Quantum Field Theory that the product $\mathcal{TCP}$ is an invariance of any theory that satisfies the following general conditions:
  - The theory is local and covariant for Lorentz transformations.
  - The theory is quantized using the usual relation between spin and statistics, that is, commutators for bosons and anti-commutators for fermions.
- This theorem due to Lüdus, Zumino, Pauli e Schwinger has an important consequence that if one of the discrete symmetries is not preserved then another one must also be violated to preserve the invariance of the product.
- For a proof of the theorem see, for instance, the books of Bjorken and Drell and Itzykson and Zuber