Lectures in Quantum Field Theory – Lecture 2

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Summary for Lecture 2: QED as an Example

- QED as a gauge theory
- Propagators and Green functions
- Feynman rules for QED
  - Electrons and positrons in external lines
  - Photons in internal lines
  - Photons in external lines
  - Higher orders
- Example 1: Compton scattering
- Example 2: $e^- + e^+ \to \mu^- + \mu^+$ in QED

Local invariance of Dirac equation

- We start with Dirac Lagrangian

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi$$

- It is invariant under global phase transformations

$$\psi' = e^{i\alpha}\psi, \quad \alpha \text{ infinitesimal} \to \delta\psi = i\alpha\psi, \quad \delta\overline{\psi} = -i\alpha\overline{\psi}$$

- What happens if the transformations are local, $\alpha = \alpha(x)$?

$$\delta\psi = i\alpha(x)\psi \qquad ; \qquad \delta\overline{\psi} = -i\alpha(x)\overline{\psi}$$

- We have then

$$\delta\mathcal{L} = -\overline{\psi}\gamma^\mu\psi\; \partial_\mu\alpha(x)$$

and the Lagrangian is no longer invariant

Local invariance of Dirac equation . . .

- We see that the problem is connected with the fact that $\partial_\mu \psi$ do not transform as ψ. We are then led to the concept of *covariant derivative* D_μ that transforms as the fields,

$$\delta D_\mu \psi = i\alpha(x) D_\mu \psi$$

- For the Dirac field we define, in analogy with minimal prescription,

$$D_\mu = \partial_\mu + ieA_\mu$$

- The vector field A_μ is a field that ensures that we can choose the phase locally. Its transformation is chosen to compensate the term proportional to $\partial_\mu \alpha$

$$\delta A_\mu = -\frac{1}{e}\, \partial_\mu \alpha(x)$$

- We are then led to the introduction of the electromagnetic field A_μ satisfying the usual gauge invariance.

QED Lagrangian

- This new vector field A_μ needs a kinetic term. The only term quadratic that is invariant under the local gauge transformations is

$$F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu, \quad \delta F_{\mu\nu} = 0$$

- A mass term of the form $A^\mu A_\mu$ is not gauge invariant, so the field A_μ (photon) is massless
- The final Lagrangian is

$$\boxed{\mathcal{L}_{\texttt{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi \equiv \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{interaction}}}$$

where

$$\mathcal{L}_{\text{free}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}(i\not{\partial} - m)\psi, \quad \mathcal{L}_{\text{interaction}} = -e\overline{\psi}\gamma_\mu \psi A^\mu$$

- This Lagrangian is invariant under local gauge transformations and describes the interactions of electrons (and positrons) with photons. The theory is called *Quantum Electrodynamics* (QED)

The non-relativistic propagator

- We will follow the method of Richard Feynman to arrive at the rules for calculations in QED.
- As a warm up exercise we start with the non-relativistic Schrödinger equation

$$\left(i\frac{\partial}{\partial t} - H \right) \psi(\vec{x},t) = 0, \qquad H = H_0 + V$$

where H_0 is the free particle Hamiltonian

$$H_0 = -\frac{\nabla^2}{2m}$$

- We can rewrite the equation in the form

$$\left(i\frac{\partial}{\partial t} - H_0 \right) \psi = V\psi$$

- For arbitrary V this equation can normally only be solved in perturbation theory

The non-relativistic propagator . . .

- For the scattering problems we are interested we will develop a perturbative expansion using the technique of the Green's Functions (GF). We introduce the GF for the free Schrödinger equation with retarded boundary condition

$$\left(i\frac{\partial}{\partial t'} - H_0(\vec{x}')\right) G_0(x',x) = \delta^4(x'-x), \quad G_0(x',x) = 0 \quad \text{for} \quad t' < t$$

- If $\phi_i(\vec{x},t)$ is a solution of the free Schrödinger equation,

$$\left(i\frac{\partial}{\partial t} - H_0\right) \phi_i(\vec{x},t) = 0$$

the most general solution of the original equation

$$\left(i\frac{\partial}{\partial t} - H_0\right) \psi = V\psi$$

is

$$\psi(\vec{x}',t') = \phi_i(\vec{x}',t') + \int d^4x\, G_0(x',x)V(x)\psi(x)$$

The non-relativistic propagator ...

- We can use this integral equation to establish a perturbative series. Consider that the interaction is localized, that is $V(\vec{x},t) \to 0$ as $t \to -\infty$. Then due to the retarded GF properties we have

$$\lim_{t' \to -\infty} \psi(\vec{x}', t') = \phi_i(\vec{x}', t')$$

that is in the remote past we have a plane wave.

- Now if V is *small* (in some sense) we can solve the integral equation perturbatively

$$\begin{aligned}
\psi(\vec{x}', t') = &\phi_i(\vec{x}', t') + \int d^4x_1\ G_0(x', x_1)V(x_1)\phi_i(x_1) \\
&+ \int d^4x_1 d^4x_2\ G_0(x', x_1)V(x_1)G_0(x_1, x_2)V(x_2)\phi_i(x_2) \\
&+ \int d^4x_1 d^4x_2 d^4x_3\ G_0(x', x_1)V(x_1)G_0(x_1, x_2)V(x_2)G_0(x_2, x_3)V(x_3)\phi_i(x_3) \\
&+ \cdots
\end{aligned}$$

The non-relativistic propagator ...

- We can look at the perturbative series in another way, in terms of the *full* GF of the theory with interactions, $G(x', x)$

$$\left(i\frac{\partial}{\partial t} - H_0(x') - V(x') \right) G(x', x) \equiv \delta^4(x' - x)$$

- It satisfies

$$G(x', x) = G_0(x', x) + \int d^4y \, G_0(x', y) V(y) G(y, x)$$

- This leads to the perturbative series (*small* V)

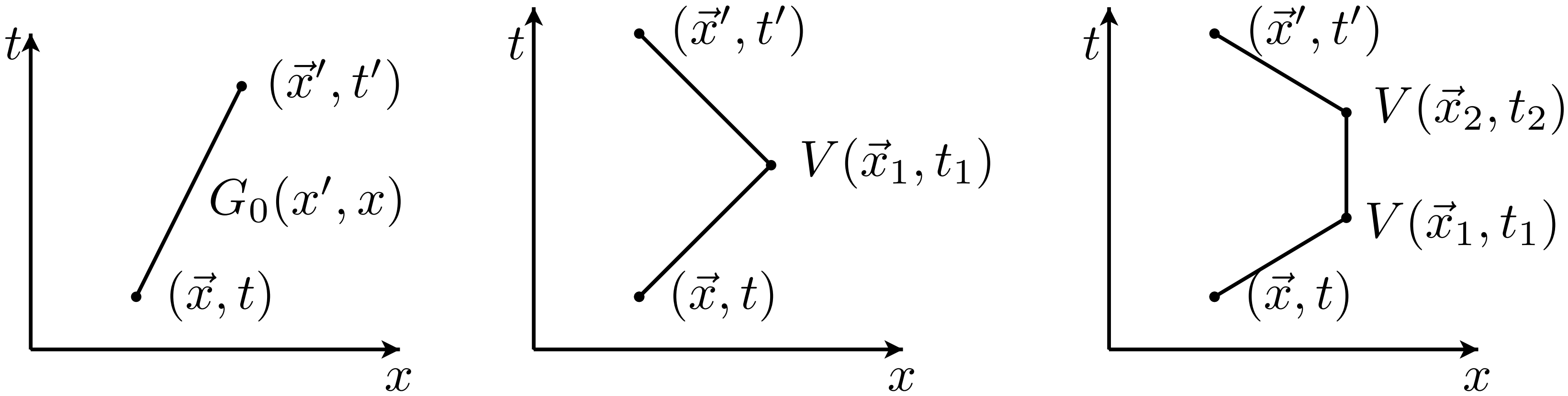
$$\begin{aligned} G(x', x) =& G_0(x', x) + \int d^4x_1 \; G_0(x', x_1) V(x_1) G_0(x_1, x) \\ & + \int d^4x_1 d^4x_2 \; G_0(x', x_2) V(x_2) G_0(x_2, x_1) V(x_1) G_0(x_1, x) \\ & + \quad \cdots \end{aligned}$$

Green Functions as Propagators

- The last equation allows for suggestive graphical interpretation. We notice that the retarded character of G_0 implies $x'^0 > \cdots x_3^0 > x_2^0 > x_1^0 > x^0$.

- So we have the situation of the following diagrams for the first 3 terms

$$
\begin{aligned}
G(x',x) =& G_0(x',x) + \int d^4x_1\ G_0(x',x_1)V(x_1)G_0(x_1,x) \\
&+ \int d^4x_1 d^4x_2\ G_0(x',x_2)V(x_2)G_0(x_2,x_1)V(x_1)G_0(x_1,x) \\
&+ \quad \cdots
\end{aligned}
$$



Scattering processes and the S Matrix

- We are interested in scattering processes. This means that in the past we have a solution of the free equation, a plane wave with momentum $\vec{k}_i$

$$\phi_i(\vec{x},t) = \frac{1}{(2\pi)^{3/2}}\ e^{i\vec{k}_i\cdot\vec{x}-i\omega_i t}$$

- In the future (detector) we have another plane wave with momentum $\vec{k}_f$

$$\phi_f(\vec{x}',t') = \frac{1}{(2\pi)^{3/2}}\ e^{i\vec{k}_f\cdot\vec{x}'-i\omega_f t'}$$

- The relevant quantity is S matrix element (transition amplitude)

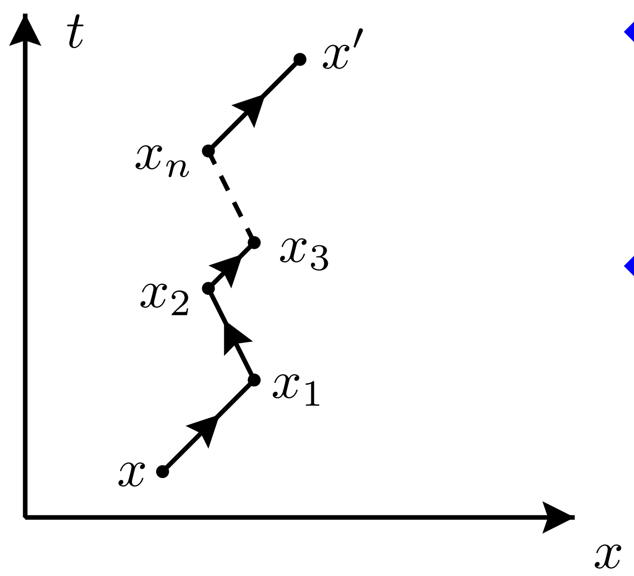
$$\begin{aligned}
S_{fi} &= \lim_{t'\to\infty}\int d^3x'\ \phi_f^*(\vec{x}',t')\psi(\vec{x}',t')\\
&= \lim_{t'\to\infty}\int d^3x'\ \phi_f^*(\vec{x}',t')\left[\phi_i(\vec{x}',t') + \int d^4x_1\ G_0(x',x_1)V(x_1)\phi_i(x_1)+\cdots\right]\\
&=\delta^3(\vec{k}_f-\vec{k}_i) + \lim_{t'\to\infty}\int d^3x' d^4x_1\ \phi_f^*(\vec{x}',t')G_0(x',x_1)V(x_1)\phi_i(x_1)+\cdots
\end{aligned}$$

The propagator for the Relativistic Theory

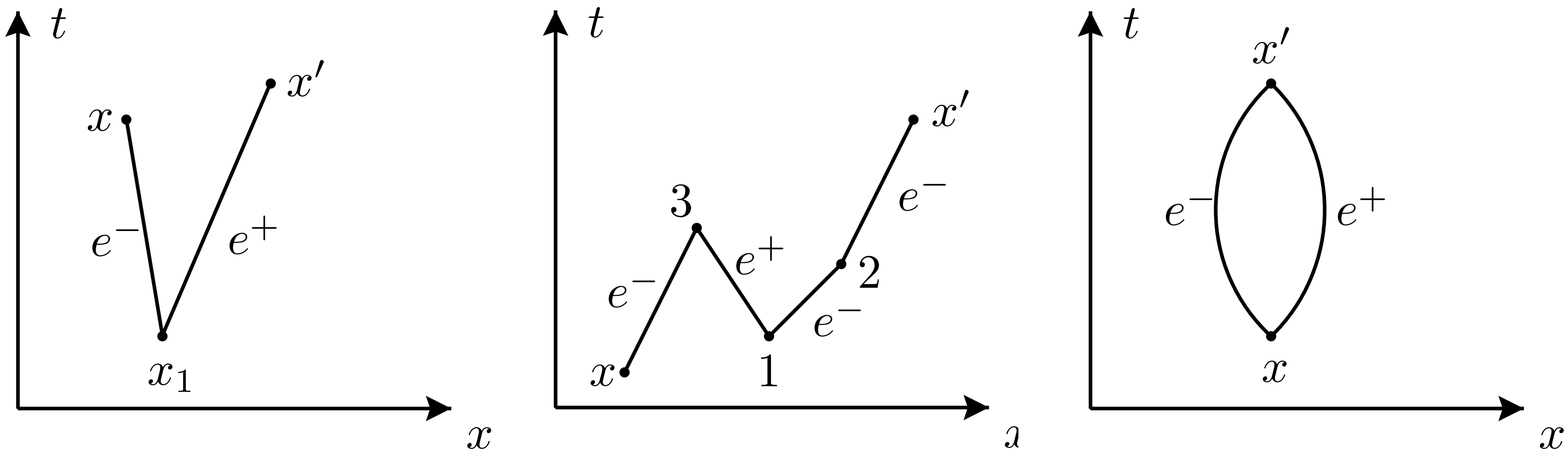
- The starting point is the interpretation of $G(x', x)$ as the probability amplitude to propagate the particle from x to x'

$$
\begin{aligned}
G(x', x) =& G_0(x', x) + \int d^4x_1 \; G_0(x', x_1) V(x_1) G_0(x_1, x) \\
& + \int d^4x_1 d^4x_2 \; G_0(x', x_2) V(x_2) G_0(x_2, x_1) V(x_1) G_0(x_1, x) \\
& \cdots
\end{aligned}
$$

- The contribution of order n corresponds to the diagram



- A particle is created at x, propagates to x_1, interacts with the potential $V(x_1)$, propagates to x_2 and so on.
- This interpretation is suited to the relativistic theory because of the space-time emphasis instead of the Hamiltonian evolution.



- The existence of a positron is associated with the absence of an electron of negative energy
- Therefore we can interpret the destruction of an positron at 3 as being the creation of an electron of negative energy at that point
- This suggests (Feynman) the possibility that the amplitude to create a positron at 1 and destroy it at 3 be related to the amplitude to create an electron of negative energy at 3 and destroy it at 1
- Then electrons of positive energy propagate to the future and electrons of negative energy (positrons) propagate back in time

The propagator for the Relativistic Theory: Green Function

- Let us then look for the GF of the Dirac equation in interaction with the electromagnetic field

$$(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi(x) = 0$$

- It is the solution of the equation

$$(i\partial\!\!\!/' - eA\!\!\!/ - m)S'_F(x', x) = i\delta^4(x' - x)$$

- The full GF can only be obtained in perturbation theory. For the free theory we have

$$(i\partial\!\!\!/' - m)S_F(x', x) = i\delta^4(x' - x)$$

- Noticing that $S_F(x', x) = S_F(x' - x)$ and applying the Fourier transform

$$S_F(x' - x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x'-x)} S_F(p)$$

The propagator for the Relativistic Theory: Green Function . . .

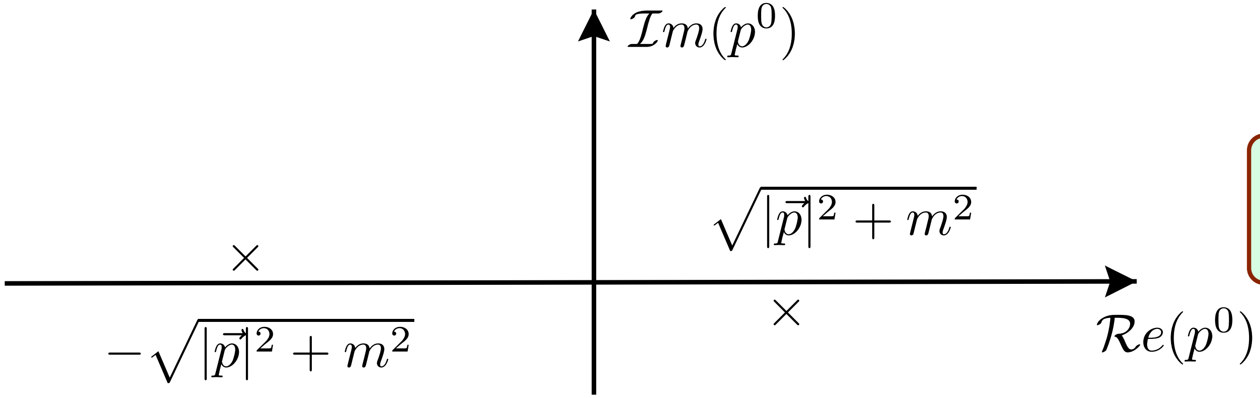
- Substituting in the equation we get for $S_F(p)$

$$(\not{p} - m) S_F(p) = i \quad \to \quad S_F(p) = \frac{i(\not{p} + m)}{p^2 - m^2}, \quad p^2 \neq m^2$$

- To complete the definition we need a prescription on how to deal with the singularity. This is related with the boundary conditions we want to impose on the GF, positive energies propagate into the future and negative energies back in time.

- The inverse Fourier transform is calculated using the residue theorem

$$S_F(x' - x) = \int \frac{dp^0}{2\pi} \int \frac{d^3p}{(2\pi)^3}\, e^{-ip^0 (x'-x)^0} e^{i\vec{p}\cdot(\vec{x}'-\vec{x})} \frac{i}{(p^0)^2 - (|\vec{p}|^2 + m^2)}$$



$t' > t$	lower semi-plane
$t' < t$	upper semi-plane

- The localization of the poles is obtained giving a negative infinitesimal part to m^2

$$m^2 \to m^2 - i\varepsilon$$

- With this prescription (due to Feynman) the propagator is

$$S_F(p) = i\frac{(\not p + m)}{p^2 - m^2 + i\varepsilon}, \qquad \to \qquad p_0 = \pm\left(\sqrt{|\vec{p}|^2 + m^2} - i\varepsilon\right)$$

- We can do now the integration in p^0 to obtain

$$S_F(x' - x) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E}\,\left[(\not p + m)\; e^{-ip\cdot(x'-x)}\;\theta(t'-t)\right.$$
$$\left. +(-\not p + m)\; e^{ip\cdot(x'-x)}\;\theta(t-t')\right]$$

- We define the normalized plane waves

$$\psi_p^r(x) = \frac{1}{\sqrt{2E}}\,(2\pi)^{-3/2}\, w^r(\vec{p})\; e^{-i\varepsilon_r p\cdot x}$$

- Then we obtain

$$\begin{aligned} S_F(x'-x) =&\theta(t'-t)\int d^3p\,\sum_{r=1}^{2}\psi_p^r(x')\overline{\psi}_p^r(x)\\ &-\theta(t-t')\int d^3p\,\sum_{r=3}^{4}\psi_p^r(x')\overline{\psi}_p^r(x)\end{aligned}$$

- This expresses $S_F(x'-x)$ as a sum of eigenfunctions of the free Dirac operator. From this expression is clear that the negative energy solutions $(r=3,4)$ are propagated back in time $(t'<t)$, while the positive energy solutions are propagated in the future $(t'>t)$

S Matrix elements

- As we will be interested in scattering problems, we will be focusing in the elements of the S matrix. To find these we start by noticing the solution of the Dirac equation with interactions,

$$(i\not{\partial} - m)\Psi = e\not{A}\Psi$$

can be written, in analogy with the non-relativistic case,

$$\Psi(x) = \psi(x) - ie \int d^4y\, S_F(x-y)\not{A}(y)\Psi(y)$$

- Using the expression for $S_F(x-y)$ we get

$$\lim_{t\to+\infty} \Psi(x) - \psi(x) = \int d^3p \sum_{r=1}^{2} \psi_p^r(x) \left[-ie \int d^4y\, \overline{\psi}_p^r(y)\not{A}(y)\Psi(y) \right]$$

$$\lim_{t\to-\infty} \Psi(x) - \psi(x) = \int d^3p \sum_{r=3}^{4} \psi_p^r(x) \left[+ie \int d^4y\, \overline{\psi}_p^r(y)\not{A}(y)\Psi(y) \right]$$

S Matrix elements ...

- This again shows that positive energies are scattered into the future and negative energy solutions in the past.

- Using now the S matrix definition

$$S_{fi} = \lim_{t \to \varepsilon_f \infty} \int d^3x \; \psi_f^\dagger(x) \Psi_i(x)$$

we get

$$\boxed{S_{fi} = \delta_{fi} - ie\varepsilon_f \int d^4y \; \overline{\psi}_f(y) \not{A}(y) \Psi_i(y)}$$

where $\varepsilon_f = +1$ for positive energies in the future (final state) and $\varepsilon_f = -1$ for negative energies into the past (initial state). ψ_f is a plane wave with the appropriate quantum numbers for the final state.

- This is the main result the we use in the following

Initial and Final states

- The description of initial and final states is as follows

Initial state

$$\text{electron} \quad \to \quad \psi_i = \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{V}} u(p_i, s_i)\, e^{-ip_i \cdot x}$$

$$\text{positron} \quad \to \quad \psi_i = \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{V}} v(p_f, s_f)\, e^{ip_f \cdot x}$$

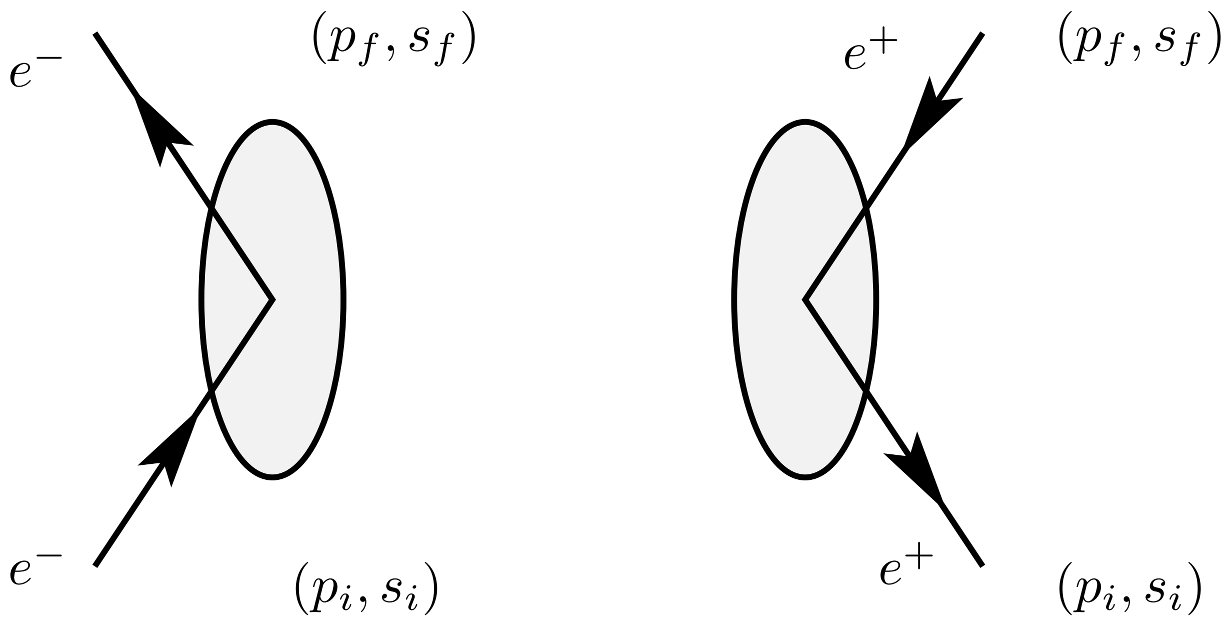
Final state

$$\text{electron} \quad \to \quad \psi_f = \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{V}} u(p_f, s_f)\, e^{-ip_f \cdot x}$$

$$\text{positron} \quad \to \quad \psi_f = \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{V}} v(p_i, s_i)\, e^{ip_i \cdot x}$$

Initial and Final states

- The conventions are spelled out in the following figure



- We have chosen the normalization in a box of volume V

$$\int_V d^3x\ \psi_i^\dagger \psi_i = \frac{1}{V}\frac{1}{2E_i} u^\dagger(p_i, s_i) u(p_i, s_i) \int_V d^3x = \frac{1}{V}\int_V d^3x = 1$$

How to find Feynman Rules for QED

- We are going here to start from the central result

$$\boxed{S_{fi} = -ie\varepsilon_f \int d^4y \overline{\psi}_f(y) \not{A}(y) \Psi_i(y) \qquad (i \neq f)}$$

and derive a set of rules (*Feynman Rules*) that will show us how to calculate in QED

- For that we will consider:
  - Electrons in external legs: Coulomb scattering for e^-:
    $e^- +$ Nuclei(Z) $\to e^- +$ Nuclei(Z)
  - Positrons in external legs: Coulomb scattering for e^+:
    $e^+ +$ Nuclei(Z) $\to e^+ +$ Nuclei(Z)
  - Photons in internal lines: $e^- \mu^- \to e^- \mu^-$
  - Higher order processes: $e^- \mu^- \to e^- \mu^-$
  - Photons in external legs: Compton scattering: $\gamma + e^- \to \gamma + e^-$

Coulomb Scattering for Electrons

- We consider Coulomb scattering by a fixed Nuclei(Z), that is, by a classical electromagnetic Coulomb potential (not quantized)

$$A^0(x) = \frac{-Ze}{4\pi|\vec{x}|}, \quad \vec{A}(x) = 0, \quad e < 0$$

- In lowest order we approximate $\Psi_i(x)$ by a plane wave

$$\Psi_i(x) = \frac{1}{\sqrt{2E_i}} \frac{1}{\sqrt{V}} u(p_i, s_i) e^{-ip_i \cdot x}$$

- For the final state we take

$$\overline{\psi}_f(x) = \frac{1}{\sqrt{2E_f}} \frac{1}{\sqrt{V}} \overline{u}(p_f, s_f) e^{ip_f \cdot x}$$

- The S matrix amplitude between the initial and final state is

$$S_{fi} = \frac{ie^2 Z}{4\pi} \frac{1}{V} \frac{1}{\sqrt{2E_i 2E_f}} \overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i) \int d^4x \frac{e^{i(p_f - p_i)\cdot x}}{|\, \vec{x}\, |}$$

- The integration can done ($\vec{q} = \vec{p}_f - \vec{p}_i$ is the transferred momentum) and we get the final result

$$S_{fi} = iZe^2 \frac{1}{V} \frac{1}{\sqrt{4E_i E_f}} \frac{\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)}{|\, \vec{q}\, |^2} 2\pi\delta(E_f - E_i)$$

- We notice that we are assuming the nuclei fixed, so we have only energy conservation

- The number of final states in the interval d^3p_f is $V\frac{d^3p_f}{(2\pi)^3}$, and therefore the probability for the particle to go into one of these states is

$$P_{fi} = |\, S_{fi}\, |^2\, V \frac{d^3p_f}{(2\pi)^3}$$

Coulomb Scattering for Electrons: The Amplitude & ...

- Putting everything together we have

$$P_{fi} = \frac{Z^2(4\pi\alpha)^2}{2E_i V} \frac{|\,\overline{u}(p_f,s_f)\gamma^0 u(p_i,s_i)\,|^2}{|\,\vec{q}\,|^4} \frac{d^3 p_f}{(2\pi)^3 2E_f} \left[2\pi\delta(E_f-E_i)\right]^2$$

- The square of the delta function needs some clarification. We define a transition time T and then

$$(2\pi)\delta(E_f-E_i) = \lim_{T\to\infty} \int_{-T/2}^{T/2} dt e^{i(E_f-E_i)t}$$

- Then

$$2\pi\delta(0) = \lim_{T\to\infty} \int_{T/2}^{T/2} dt = \lim_{T\to\infty} T$$

- Therefore

$$\left[2\pi\delta(E_f-E_i)\right]^2 = 2\pi\delta(0) 2\pi\delta(E_f-E_i) = 2\pi T\delta(E_f-E_i)$$

Coulomb Scattering for Electrons: The Transition Rate

❒ Dividing by T we obtain the transition rate

$$R_{fi} = \frac{4Z^2\alpha^2}{2E_iV}\frac{|\,\overline{u}(p_f s_f)\gamma^0 u(p_i s_i)\,|^2}{|\,\vec{q}\,|^4}\frac{d^3p_f}{2E_f}\delta(E_f - E_i)$$

❒ To get the cross section we have to divide by the incident flux. Using

$$\vec{J}_{\rm inc} = \overline{\psi}_i(x)\vec{\gamma}\psi_i(x), \quad \text{with} \quad \psi_i = \frac{1}{\sqrt{V}}\frac{\sqrt{E_i+m}}{\sqrt{2E_i}}\begin{bmatrix} \chi(s) \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_i+m}\chi(s)\end{bmatrix} e^{-ip_i\cdot x}$$

we get

$$|\,\vec{J}_{\rm inc}\,| = \frac{1}{V}\frac{1}{2E_i}2\,|\,\vec{p}_i\,| = \frac{1}{V}\frac{|\vec{p}_i|}{E_i}$$

with the usual interpretation: density, $1/V$, times velocity, $\vec{p}_i/E_i$

- The differential cross section is then

$$\frac{d\sigma}{d\Omega} = \int \frac{Z^2\alpha^2}{|\vec{p}_i|} \frac{|\overline{u}(p_f)\gamma^0 u(p_i)|^2}{|\vec{q}|^4} \frac{p_f^2 dp_f}{E_f} \delta(E_f - E_i)$$

- Finally using $p_f dp_f = E_f dE_f$ we get

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{|\vec{q}|^4} |\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2$$

- In practice we normally do not have polarized beams and do not measure the polarization of the final state. So we want the *unpolarized* cross section given by

$$\frac{d\overline{\sigma}}{d\Omega} = \frac{1}{2} \sum_{s_i, s_f} \frac{d\sigma}{d\Omega}$$

- The spin sums can be transformed into traces, the *Casimir's trick*. We have for any matrix Γ

$$\sum_{s_i,s_f} |\, \overline{u}(p_f,s_f)\Gamma u(p_i,s_i)\,|^2 =$$

$$= \sum_{s_f} u_\sigma(p_f,s_f)\overline{u}_\alpha(p_f,s_f)\Gamma_{\alpha\beta} \sum_{s_i} u_\beta(p_i,s_i)\overline{u}_\delta(p_i,s_i)\overline{\Gamma}_{\delta\sigma}$$

$$= \text{Tr}\left[(\not{p}_f + m)\Gamma(\not{p}_i + m)\overline{\Gamma}\right], \quad \text{with} \quad \overline{\Gamma} \equiv \gamma^0\Gamma^\dagger\gamma^0$$

  where we used

$$\sum_{\pm s} u_\alpha(p,s)\overline{u}_\beta(p,s) = (\not{p} + m)_{\alpha\beta}$$

- So the final result is

$$\frac{d\overline{\sigma}}{d\Omega} = \frac{Z^2\alpha^2}{2\,|\,\vec{q}\,|^4}\text{Tr}\left[(\not{p}_f + m)\gamma^0(\not{p}_i + m)\gamma^0\right]$$

Theorems on traces of Dirac γ matrices

- Due to equivalence relation $\gamma'^\mu = U^{-1}\gamma^\mu U$ and the cyclic property, traces are independent of the representation of the γ matrices

- The trace of an odd number of γ matrices vanishes

- For 0 and 2 matrices we have

$$\begin{aligned} \text{Tr}1 &= 4 \\ \text{Tr}[\not{a}\not{b}] &= \text{Tr}[(\not{b}\not{a})] = \tfrac{1}{2}\text{Tr}[(\not{a}\not{b}+\not{b}\not{a})] = a\cdot b\ \text{Tr}1 = 4a\cdot b \end{aligned}$$

- We have the recurrence form (n even)

$$\begin{aligned} \text{Tr}\left[\not{a}_1\cdots\not{a}_n\right] =& a_1\cdot a_2\ \text{Tr}\left[\not{a}_3\cdots\not{a}_n\right] - a_1\cdot a_3\ \text{Tr}\left[\not{a}_2\not{a}_4\cdots\not{a}_n\right] \\ &+ a_1\cdot a_n\ \text{Tr}\left[\not{a}_2\cdots\not{a}_{n-1}\right] \end{aligned}$$

- An important corollary is

$$\begin{aligned} \text{Tr}\left[\not{a}_1\not{a}_2\not{a}_3\not{a}_4\right] =& a_1\cdot a_2\ \text{Tr}\left[\not{a}_3\not{a}_4\right] - a_1\cdot a_3\ \text{Tr}\left[\not{a}_2\not{a}_4\right] + a_1\cdot a_4\ \text{Tr}\left[\not{a}_2\not{a}_3\right] \\ =& 4\left[a_1\cdot a_2\ a_3\cdot a_4 - a_1\cdot a_3\ a_2\cdot a_4 + a_1\cdot a_4\ a_2\cdot a_3\right] \end{aligned}$$

Theorems on traces of Dirac γ matrices ...

- For traces with γ_5 (needed for the SM)

$$\text{Tr}\left[\gamma_5\right] = 0, \quad \text{Tr}\left[\gamma_5 \not{a}\not{b}\right] = 0, \quad \text{Tr}\left[\gamma_5 \not{a}\not{b}\not{c}\not{d}\right] = -4i\varepsilon_{\mu\nu\rho\sigma} a^\mu b^\nu c^\rho d^\sigma$$

- Sometimes it is useful to reduce he number of γ matrices before taking the trace. Useful results are

$$\begin{aligned}
&\gamma_\mu \gamma^\mu = 4 \\
&\gamma_\mu \not{a} \gamma^\mu = -2\not{a} \\
&\gamma_\mu \not{a}\not{b} \gamma^\mu = 4 a.b \\
&\gamma_\mu \not{a}\not{b}\not{c} \gamma^\mu = -2\not{c}\not{b}\not{a} \\
&\gamma_\mu \not{a}\not{b}\not{c}\not{d} \gamma^\mu = 2\left[\not{d}\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}\not{d}\right]
\end{aligned}$$

- In practice when the number of γ matrices is bigger than 4 we use specific software to evaluate the traces.

Coulomb Scattering for Electrons: The Mott Cross Section

- Finally we calculate the differential cross section for Coulomb scattering

$$\boxed{\frac{d\overline{\sigma}}{d\Omega} = \frac{Z^2\alpha^2}{2\mid\vec{q}\mid^4}\text{Tr}\left[(\not{p}_f + m)\gamma^0(\not{p}_i + m)\gamma^0\right]}$$

- The trace gives

$$\begin{aligned}\text{Tr}\left[(\not{p}_f + m)\gamma^0(\not{p}_i + m)\gamma^0\right] &= \text{Tr}\left[\not{p}_f\gamma^0\not{p}_i\gamma^0\right] + m^2\text{Tr}\left[\gamma^0\gamma^0\right]\\ &= 8E_iE_f - 4p_i\cdot p_f + 4m^2\end{aligned}$$

- Using (recall that $E = E_i = E_j$, and θ is the scattering angle)

$$p_i\cdot p_f = E^2 - \mid\vec{p}\mid^2\cos\theta = m^2 + 2\beta^2E^2\sin^2(\theta/2)\,, \quad \mid\vec{q}\mid^2 = 4\mid\vec{p}\mid^2\sin^2(\theta/2)$$

- We get the final result, the Mott cross section

$$\boxed{\frac{d\overline{\sigma}}{d\Omega} = \frac{Z^2\alpha^2}{4\mid\vec{p}\mid^2\beta^2\sin^4(\theta/2)}\left[1 - \beta^2\sin^2(\theta/2)\right]}$$

in the limit $\beta \to 0$ it reduces to Rutherford non-relativistic formula

Coulomb Scattering for Positrons

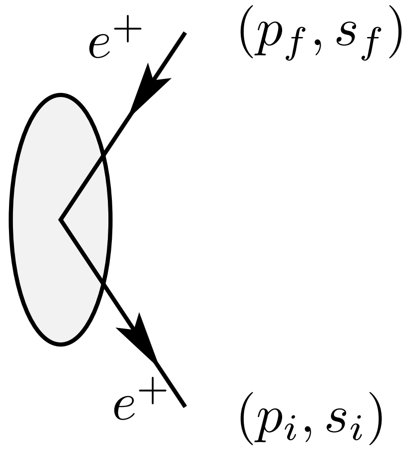
- We start now from

$$S_{fi} = ie \int d^4x \overline{\psi}_f(x) \not{A}(x) \psi_i(x)$$

where

$$\psi_i(x) = \frac{1}{\sqrt{2E_f}} \frac{1}{\sqrt{V}} v(p_f, s_f) e^{ip_f \cdot x}$$

$$\psi_f(x) = \frac{1}{\sqrt{2E_i}} \frac{1}{\sqrt{V}} v(p_i s_i) e^{ip_i \cdot x}$$



- Then the S matrix element is

$$S_{fi} = -i \frac{Ze^2}{4\pi} \frac{1}{V} \frac{1}{\sqrt{2E_i\ 2E_f}} \overline{v}(p_i, s_i) \gamma^0 v(p_f, s_f) \int \frac{d^4x}{|\vec{x}|} e^{i(p_f - p_i) \cdot x}$$

Coulomb Scattering for Positrons . . .

- We get now

$$\left(\frac{d\overline{\sigma}}{d\Omega}\right)_{e^+} = \frac{Z^2\alpha^2}{2\,|\,\vec{q}\,|^4} \sum_{s_f, s_i} |\, \overline{v}(p_i, s_i)\gamma^0 v(p_f, s_f)\,|^2$$

- Using the relation for v spinors

$$\sum_s v(p,s)\overline{v}(p,s) = (\not{p} - m)$$

we finally get

$$\left(\frac{d\overline{\sigma}}{d\Omega}\right)_{e^+} = \frac{Z^2\alpha^2}{2\,|\,\vec{q}\,|^4}\, \text{Tr}\left[(\not{p}_f - m)\gamma^0(\not{p}_i - m)\gamma^0\right]$$

- This is the same result as for electrons with $m \to -m$. As, in lowest order in α, the Mott cross section only depends in m^2, the cross section is the same for electrons and positrons

Scattering $e^- + \mu^- \to e^- + \mu^-$

- We want now to consider the situation when the electromagnetic field is not static but is also quantized. As the process $e^- + e^- \to e^- + e^-$ would bring an unnecessary complication due to identical particles we choose the process with the μ^-, a kind of heavy electron interacting in the same way as the e^-

- We start from the fundamental relation

$$S_{fi} = -ie \int d^4x \overline{\psi}_f(x) \gamma^\mu \psi_i(x) A_\mu(x)$$

where ψ_i and ψ_f refer to the electron

- We have to calculate $A_\mu(x)$. This is the field created by the muon. It is given by the solution of the equation (in the Lorentz gauge)

$$\Box A^\mu(x) = J^\mu(x)$$

- $J^\mu(x)$ is the current due to the muon, given by

$$J^\mu(x) = e \overline{\psi}_f^{\mu^-}(x) \gamma^\mu \psi_i^{\mu^-}(x), \quad e < 0$$

Scattering $e^- + \mu^- \to e^- + \mu^-$: Photon Propagator

- The solution of the equation for $A_\mu(x)$ is obtained with the GF technique leading to the photon propagator. We have

$$\Box D_F^{\mu\nu}(x-y) = i g^{\mu\nu} \delta^4(x-y)$$

- We get for the Fourier transform

$$D_F^{\mu\nu}(k) = i \frac{-g^{\mu\nu}}{k^2}$$

- We have to decide what to do at the pole $k^2 = 0$. A similar study, as done for the electrons, shows that the correct choice is

$$D_{F\mu\nu}(k) = -i \frac{g_{\mu\nu}}{k^2 + i\varepsilon}$$

- The solution for $A_\mu(x)$ is then (we neglect the solution of the free equation)

$$A^\mu(x) = -i \int d^4y D_F^{\mu\nu}(x-y) J_\nu(y)$$

Scattering $e^- + \mu^- \to e^- + \mu^-$: S Matrix

- Substituting we get the amplitude for the S matrix

$$S_{fi} = (-ie)^2 \int d^4x d^4y\, \overline{\psi}_f(x)\gamma_\mu \psi_i(x) D_F^{\mu\nu}(x-y) \overline{\psi}_f^{\mu^-}(y)\gamma_\nu \psi_i^{\mu^-}(y)$$

- After introducing the plane waves for initial and final states we get

$$\begin{aligned} S_{fi} =& \frac{-ie^2}{V^2}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\frac{1}{\sqrt{2E_i^{e^-}2E_f^{e^-}}}\frac{1}{\sqrt{2E_f^{\mu^-}2E_f^{\mu^-}}} \\ & \left[\overline{u}(p_4,s_e')\gamma_\mu u(p_2,s_e)\right]\frac{1}{(p_3-p_1)^2+i\varepsilon}\left[\overline{u}(p_3,s_{\mu^-}')\gamma^\mu u(p_1,s_{\mu^-})\right] \\ =& \frac{1}{V^2}\frac{1}{\sqrt{2E_i^{e^-}2E_f^{e^-}}}\frac{1}{\sqrt{2E_i^{\mu^-}2E_f^{\mu^-}}}\ (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\ M_{fi} \end{aligned}$$

- Where M_{fi} is given by

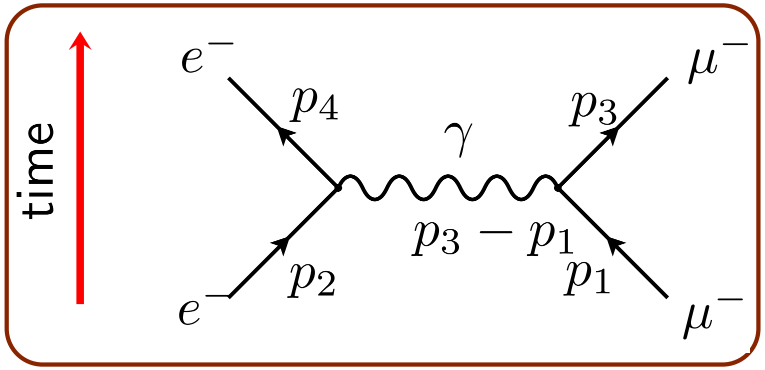
$$M_{fi} = \left[\overline{u}(p_4,s_e')(-ie\gamma^\mu)u(p_2,s_e)\right]\frac{-ig_{\mu\nu}}{(p_3-p_1)^2+i\varepsilon}\left[\overline{u}(p_3,s_{\mu^-}')(-ie\gamma^\nu)u(p_1,s_{\mu^-})\right]$$

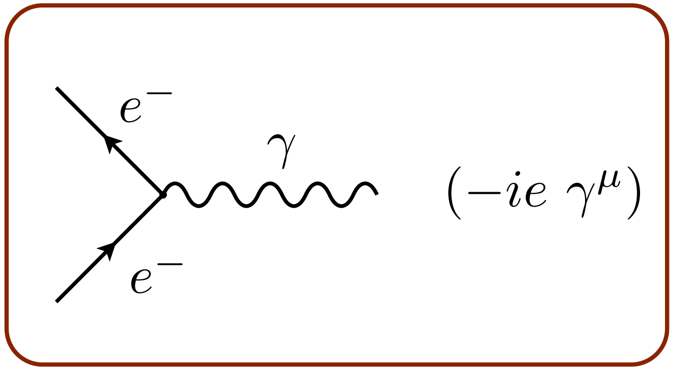
Feynman Diagram

- At this point Feynman had a genius idea that completely changed the way of making calculations in QFT. He made a one-to-one correspondence between the matrix element

$$M_{fi} = \left[\overline{u}(p_4, s'_e)(-ie\gamma^\mu)u(p_2, s_e)\right] \frac{-ig_{\mu\nu}}{(p_3 - p_1)^2 + i\varepsilon} \left[\overline{u}(p_3, s'_{\mu^-})(-ie\gamma^\nu)u(p_1, s_{\mu^-})\right]$$

and a diagram describing the process.





- To each fermion line entering the diagram we have a spinor u
- To each fermion line leaving the diagram we have a spinor $\overline{u}$
- The internal line corresponds to the virtual $(k^2 \neq 0)$ photon propagator
- Each vertex corresponds to the quantity $(-ie\gamma_\mu)$, as indicated on the right

On the Road to the Cross Section: δ^2 and Phase Space

- Like in the case of Coulomb scattering we have to deal with the square of the delta function. A generalization of

$$[2\pi\delta(E_f - E_i)]^2 \Rightarrow 2\pi T\delta(E_f - E_i)$$

gives

$$\left[(2\pi)^4\delta^4\left(\sum p_f - \sum p_i\right)\right]^2 \Rightarrow VT(2\pi)^4\delta^4\left(\sum p_f - \sum p_i\right)$$

where, as before, T is the interaction time and V is the volume of the box where we normalize the wave functions.

- To evaluate the cross section we have to sum over all the momenta states available. The number of states between $\vec{p}_3$ and $\vec{p}_3 + d\vec{p}_3$ and between $\vec{p}_4$ and $\vec{p}_4 + d\vec{p}_4$ is

$$V\frac{d^3p_3}{(2\pi)^3}V\frac{d^3p_4}{(2\pi)^3}$$

On the Road to the Cross Section: The Incident Flux

- The incident flux is

$$| \vec{J}_{\rm inc} |= \frac{1}{V}\left|\frac{\vec{p}_1}{p_1^0} - \frac{\vec{p}_2}{p_2^0}\right| = \frac{1}{V}| \vec{v}_{\rm relative} |$$

- For future use we note that the combination $V \mid \vec{J}_{\rm inc} \mid$ multiplied by the energy of the incoming particles is

$$\begin{aligned} V \mid \vec{J}_{\rm inc} \mid 2E_i^{e^-}\, 2E_i^{\mu^-} &=4 \mid p_1^0 \vec{p}_2 - p_2^0 \vec{p}_1 \mid \\ &=4\sqrt{(p_1 \cdot p_2)^2 - m_e^2 m_\mu^2} \end{aligned}$$

where the last expression shows that it is a Lorentz invariant. To derive this expression we have to assume that $\vec{p}_1$ and $\vec{p}_2$ are collinear, as is the situation in normal scattering experiments

The Cross Section

- We have now all the ingredients to evaluate the cross section. First we determine the transition rate by unit time and unit volume

$$\lim_{V,T\to\infty} \frac{1}{VT} \mid S_{fi} \mid^2 = w_{fi}$$

- Using the previous results we get

$$w_{fi} = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \frac{1}{V^4} \frac{1}{2p_1^0\, 2p_2^0\, 2p_3^0\, 2p_4^0} \mid M_{fi} \mid^2$$

- Finally we divide by the incident flux and by the number density of particles in the target (just $1/V$ with our normalization) and sum over the final states to get

$$\begin{aligned}
\sigma &= \int \frac{d^3p_3}{(2\pi)^3} \frac{d^3p_4}{(2\pi)^3}\; V^2\; \frac{V}{\mid \vec{J}_{\text{inc}} \mid} w_{fi} \\
&= \int \frac{d^3p_3}{(2\pi)^3} \frac{d^3p_4}{(2\pi)^3} \frac{1}{2p_1^0\, 2p_2^0\, 2p_3^0\, 2p_4^0} \frac{1}{V \mid \vec{J}_{\text{inc}} \mid} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \mid M_{fi} \mid^2
\end{aligned}$$

The Cross Section ...

$$\boxed{\sigma = \int \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} |M_{fi}|^2 (2\pi)^4 \delta^4(p_1 + q_2 - p_3 - p_4) \frac{d^3 p_3}{(2\pi)^3 2p_3^0} \frac{d^3 p_4}{(2\pi)^3 2p_4^0}}$$

- **Initial State**: The factor

$$\frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}$$

- **Final State**: The factor

$$(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \frac{d^3 p_3}{(2\pi)^3 2p_3^0} \frac{d^3 p_4}{(2\pi)^3 2p_4^0}$$

This factor is also Lorentz invariant because

$$\int \frac{d^3 p}{2E} = \int d^4 p \, \delta(p^2 - m^2) \theta(p^0)$$

- **Matrix Element**: $\mid M_{fi} \mid^2$

The Physics is in M_{fi} and this is evaluated through the Feynman diagrams and Feynman Rules.

Higher Order Corrections to $e^- \mu^- \to e^- \mu^-$

- To have the full Feynman rules we need to know how to evaluate higher orders in perturbation theory. We go back to the master equation

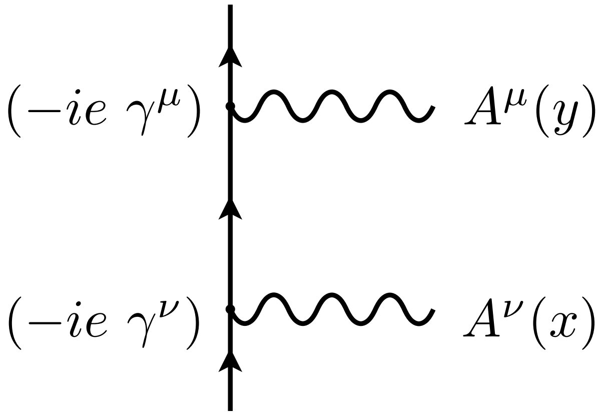
$$S_{fi} = -ie \int d^4y \overline{\psi}_f(y) \not{A}(y) \Psi_i(y)$$

- Instead of the plane wave we use now the next order to Ψ_i, that is

$$\Psi_i(y) = -ie \int d^4x S_F(y-x) \not{A}(x) \psi_i(x)$$

and

$$S_{fi}^{(2)} = \int d^4y d^4x \overline{\psi}_f(y)(-ie\gamma^\mu) S_F(y-x)(-ie\gamma^\nu)\psi_i(x) A_\mu(y) A_\nu(x)$$

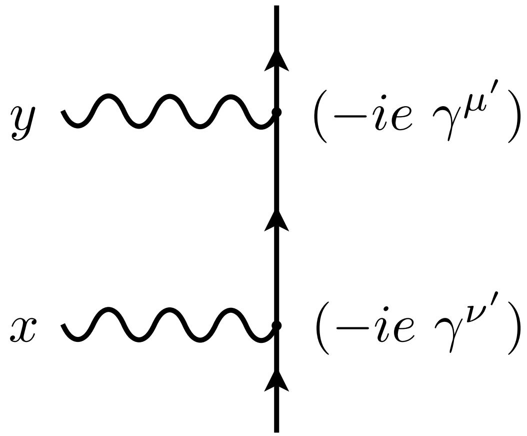


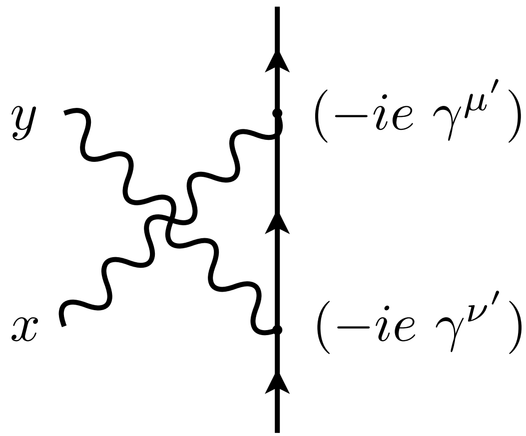
- The origin of the terms A_μ and A_ν is the current of the muon. So we should have

$$A_\mu(y)A_\nu(x)=\int d^4z d^4w \Big[D_{F\mu\mu'}(y-z)D_{F\nu\nu'}(x-w)+D_{F\mu\nu'}(y-w)D_{F\nu\mu'}(x-z)\Big]$$

$$\overline{\psi}_f^{\mu^-}(z)(-ie\gamma^{\mu'})S_F(z-w)(-ie\gamma^{\nu'})\psi_i^{\mu^-}(w)$$

- This corresponds to the diagrams





- Putting all together

$$S_{fi}^{(2)} = \int d^4y d^4x d^4z d^4w\ \overline{\psi}_f(y)(-ie\gamma^\mu)S_F(y-x)(-ie\gamma^\nu)\psi_i(x)$$
$$\left[D_{F\mu\mu'}(y-z)D_{F\nu\nu'}(x-w) + D_{F\mu\nu'}(y-w)D_{F\nu\mu'}(x-z)\right]$$
$$\overline{\psi}_f^{\mu^-}(z)(-ie\gamma^{\mu'})S_F(z-w)(-ie\gamma^{\nu'})\psi_i^{\mu^-}(w)$$

- Introducing $\psi_i, \psi_f \cdots$ and the Fourier transforms of the propagators we are lead to the final expression

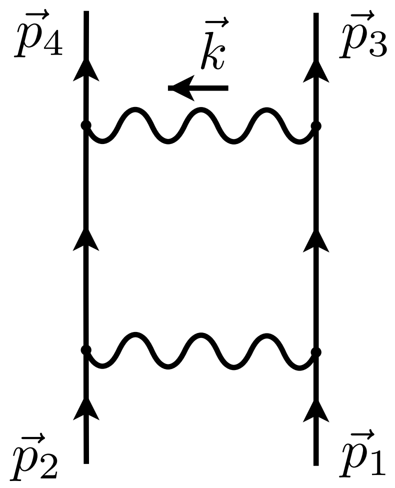
$$S_{fi}^{(2)} = \frac{1}{\sqrt{2E_i^{e^-}2E_f^{e^-}}}\frac{1}{\sqrt{2E_i^{\mu^-}2E_f^{\mu^-}}}\frac{1}{V^2}\ (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\ M_{fi}$$

- With

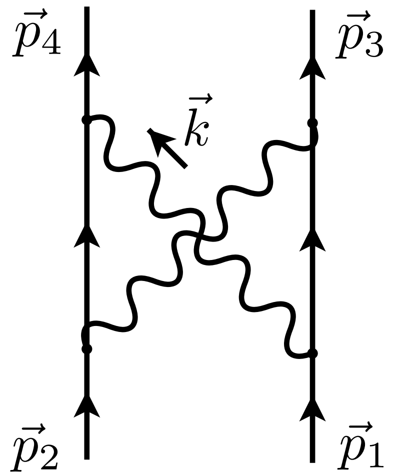
$$M_{fi} = M_{fi}^a + M_{fi}^b$$

Higher Order Corrections to $e^-\mu^- \to e^-\mu^- \ldots$

$$
\begin{aligned}
M^a_{fi} = \int \frac{d^4k}{(2\pi)^4} \Bigg[& \overline{u}(p_4)(-ie\gamma^\mu) \frac{i(\not{p}_4 - \not{k} + m_e)}{(p_4-k)^2 - m_e^2 + i\varepsilon} (-ie\gamma^\nu) u(p_2) \\
& \overline{u}(p_3)(-ie\gamma^{u'}) \frac{i(\not{p}_3 + \not{k} + m_\mu)}{(p_3+k)^2 - m_\mu^2 + i\varepsilon} (-ie\gamma^{\nu'}) u(p_1) \\
& (-ig^{\mu\mu'}) \frac{1}{k^2 + i\varepsilon} (-ig_{\nu\nu'}) \frac{1}{(p_2-p_4+k)^2 + i\varepsilon} \Bigg]
\end{aligned}
$$



$$
\begin{aligned}
M^b_{fi} = \int \frac{d^4k}{(2\pi)^4} \Bigg[& \overline{u}(p_4)(-ie\gamma^\mu) \frac{i(\not{p}_4 - \not{k} + m_e)}{(p_4-k)^2 - m_e^2 + i\varepsilon} (-ie\gamma^\nu) u(p_2) \\
& \overline{u}(p_3)(-ie\gamma^{\mu'}) \frac{i(\not{p}_1 - \not{k} + m_\mu)}{(p_1-k)^2 - m_\mu^2 + i\varepsilon} (-ie\gamma^{\nu'}) u(p_1) \\
& (-ig_{\mu\nu'}) \frac{1}{k^2 + i\varepsilon} (-ig_{\nu\mu'}) \frac{1}{(p_2-p_4+k)^2 + i\varepsilon} \Bigg]
\end{aligned}
$$



Photons in external lines: Compton Scattering

- To complete our Feynman rules we have to consider photons in external lines. The idea is to represent the photon in external lines by a plane wave. We have

$$A^\mu(x) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2k^0}} \left[\varepsilon^\mu(k)\, e^{-ik\cdot x} + \varepsilon^{*\mu}(k)\, e^{ik\cdot x} \right]$$

where the first term corresponds to the initial state and the second to the final state

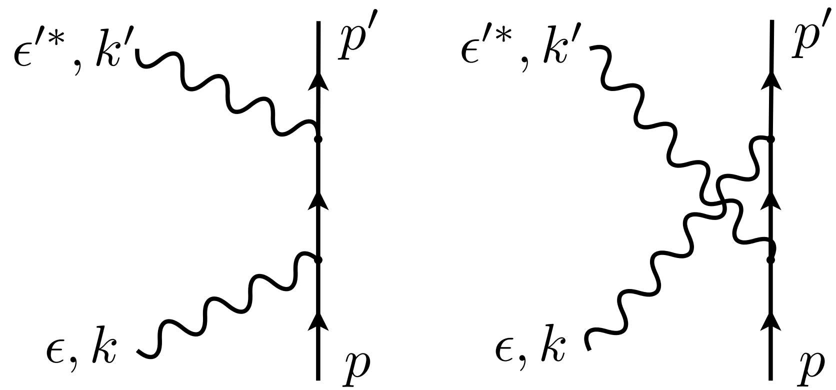
- The polarization vectors satisfy

$$k_\mu k^\mu = 0, \quad \varepsilon_\mu k^\mu = 0, \quad \varepsilon^*_\mu \varepsilon^\mu = -1$$

- Compton scattering

$$e^- + \gamma \to e^- + \gamma$$

We should have the diagrams:



Photons in external lines: Compton Scattering . . .

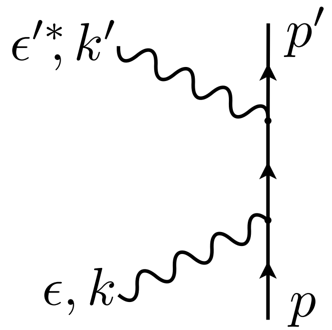
- The rules for the diagrams follow from the second order $S_{fi}^{(2)}$

$$S_{fi}^{(2)} = \int d^4y d^4x \overline{\psi}_f(y)(-ieQ_e\gamma^\mu)S_F(y-x)(-ieQ_e\gamma^\nu)\psi_i(x)A_\mu(y)A_\nu(x)$$

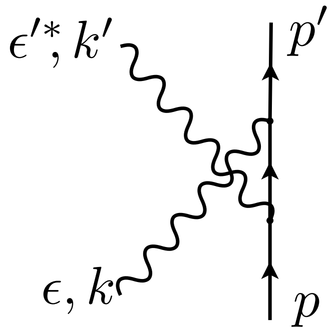
substituting $A_\mu(x)$ and $A_\nu(y)$ by plane waves. For instance for diagram a)

$$A_\mu(y) = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2k'^0}}\,\varepsilon'^*_\mu e^{ik'\cdot y}, \quad A_\nu(x) = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2k^0}}\,\varepsilon_\nu e^{-ik\cdot x}$$

- The amplitudes are then



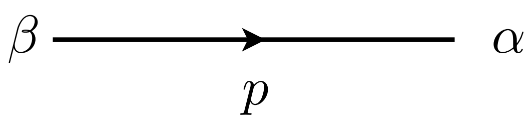
$$M^a_{fi} = \overline{u}(p')(ie\gamma^\mu)\frac{i(\not{p}+\not{k}+m_e)}{(p+k)^2-m_e^2}(ie\gamma^\nu)u(p)\ \varepsilon'^*_\mu(k')\varepsilon_\nu(k)$$



$$M^b_{fi} = \overline{u}(p')(ie\gamma^\nu)\frac{i(\not{p}-\not{k}+m_e)}{(p'-k)^2-m_e^2}(ie\gamma^\mu)u(p)\ \varepsilon'^*_\mu(k')\varepsilon_\nu(k)$$

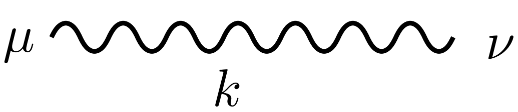
Summary of Feynman Rules for QED

1. For a given process draw all topologically distinct diagrams
2. For each electron entering the diagram a factor $u(p,s)$. If it leaves the diagram a factor $\overline{u}(p,s)$
3. For each positron leaving the diagram a factor $v(p,s)$. If it enters the diagram a factor $\overline{v}(p,s)$
4. For each photon in the initial state a polarization vector $\varepsilon_\mu(k)$. In the final state $\varepsilon^*_\mu(k)$
5. For each electron internal line the propagator



$$S_{F\alpha\beta}(p) = i\frac{(\not p + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon}$$

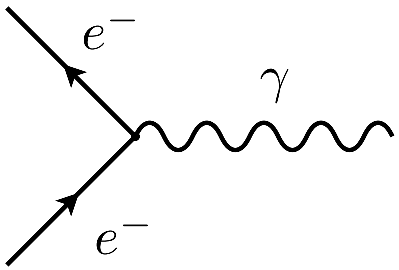
6. For each internal photon line the propagator (in the Feynman gauge)



$$D_{F\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2 + i\varepsilon}$$

Summary of Feynman Rules for QED ...

7. For each vertex the factor



$$(-ieQ_e\gamma^\mu)_{\alpha\beta}$$

8. For each internal momentum not fixed by energy-momentum conservation (in *loops*) a factor

$$\int \frac{d^4q}{(2\pi)^4}$$

9. For each *loop* of fermions a minus sign

10. A factor of -1 between diagrams that differ but odd permutations of fermions lines

Simple Processes in QED

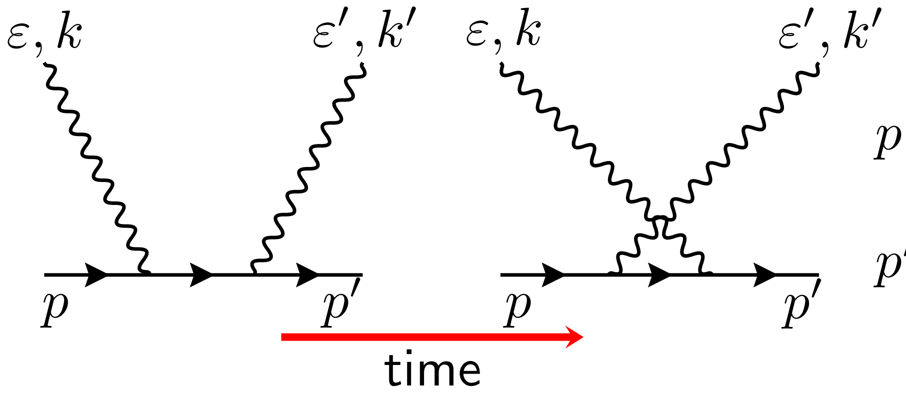
- If we restrict the processes to two particles in final state the number of processes is very small.

Process	Comment
$\gamma + e^- \to \gamma + e^-$	Compton Scattering
$e^- + e^+ \to \mu^- + \mu^+$	in QED
$\mu^- + e^- \to \mu^- + e^-$	in QED
$e^- + e^+ \to e^- + e^+$	Bhabha Scattering
e^-+ Nuclei(Z) $\to e^-+$ Nuclei(Z) $+\gamma$	Bremsstrahlung
$e^- + e^+ \to \gamma + \gamma$	Pair Annihilation
$e^- + e^- \to e^- + e^-$	Möller Scattering
$\gamma + \gamma \to e^- + e^+$	Pair Creation
$\gamma+$ Nuclei(Z) $\to$ Nuclei(Z) $+e^- + e^+$	Pair Creation

- We will discuss $\gamma + e^- \to \gamma + e^-$ and $e^- + e^+ \to \mu^- + \mu^+$ in QED

Compton Scattering

- Diagrams and kinematics



$$p = (m, \vec{0}) \qquad k = (k, 0, 0, k)$$

$$p' = (E', \vec{p}') \qquad k' = (k', k' \sin\theta, 0, k' \cos\theta)$$

- The amplitude is $M = M_1 + M_2$

$$M_1 = (ie)^2 \frac{i}{(p+k)^2 - m^2} \overline{u}(p') \gamma_\nu (\not{p} + \not{k} + m) \gamma_\mu u(p) \varepsilon^\mu(k) \varepsilon'^{\nu *}(k')$$

$$M_2 = (ie)^2 \frac{i}{(p-k')^2 - m^2} \overline{u}(p') \gamma_\mu (\not{p} - \not{k}' + m) \gamma_\nu u(p) \varepsilon^\mu(k) \varepsilon'^{\nu *}(k')$$

- We write $M_i \equiv -i \overline{u}(p', s') \Gamma_i u(p, s)$

$$\Gamma_1 = \frac{e^2}{2p \cdot k} \gamma_\nu (\not{p} + \not{k} + m) \gamma_\mu \varepsilon^\mu(k, \lambda) \varepsilon'^{\nu *}(k', \lambda')$$

$$\Gamma_2 = \frac{-e^2}{2p \cdot k'} \gamma_\mu (\not{p} - \not{k}' + m) \gamma_\nu \varepsilon^\mu(k, \lambda) \varepsilon'^{\nu *}(k', \lambda')$$

Spin Sums

- We want to calculate

$$\frac{1}{4}\sum_{s,s'}\sum_{\lambda,\lambda'}|M|^2 = \frac{1}{4}\sum_{s,s'}\sum_{\lambda,\lambda'}\left[|M_1|^2 + |M_2|^2 + M_1^\dagger M_2 + M_1 M_2^\dagger\right]$$

- We have $(i=1,2)$

$$\begin{aligned}\sum_{s,s'}|M_i|^2 &= \sum_{s,s'}\overline{u}(p',s')\Gamma_i u(p,s) u^\dagger(p,s)\Gamma_i^\dagger\gamma^0 u(p',s')\\ &= \sum_{s,s'}\overline{u}(p',s')\Gamma_i u(p,s)\overline{u}(p,s)\overline{\Gamma}_i u(p',s')\\ &= \text{Tr}\left[(\not{p}'+m)\Gamma_i(\not{p}+m)\overline{\Gamma}_i\right]\end{aligned}$$

$$\boxed{\overline{\Gamma}_i \equiv \gamma^0\Gamma_i^\dagger\gamma^0}$$

- Where we have used

$$\sum_s u_\alpha(p,s)\overline{u}_\beta(p,s) = (\not{p}+m)_{\alpha\beta}$$

Spin Sums ...

- For the interference terms

$$\sum_{s,s'} (M_1 M_2^\dagger + M_1^\dagger M_2) = \text{Tr}\left[(\not{p}' + m)\Gamma_1(\not{p} + m)\overline{\Gamma}_2\right] + \text{Tr}\left[(\not{p}' + m)\Gamma_2(\not{p} + m)\overline{\Gamma}_1\right]$$

- The sum over photon polarizations is

$$\sum_\lambda \varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda) = -g^{\mu\nu} + \text{terms proportional to } k$$

- Terms proportional to k do not contribute to the amplitude due to gauge invariance and therefore we will use the simplified form

$$\sum_\lambda \varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda) = -g^{\mu\nu}$$

Compton Cross Section

❒ In the rest frame of the electron the cross section is

$$d\sigma = \frac{1}{4mk}\,(2\pi)^4\delta^4(p+k-p'-k')\overline{|M|^2}\,\frac{d^3p'}{(2\pi)^3 2p'^0}\,\frac{d^3k'}{(2\pi)^3 2k'^0}$$

❒ Using the delta function we integrate over d^3p'. We get

$$\frac{d\sigma}{d\Omega_{k'}} = \frac{1}{4mk}\,\frac{1}{(2\pi)^2}\int dk'\,\frac{k'^2}{2k'2E'}\,\delta(m+k-E'-k')\overline{|M|^2}$$

❒ To use the last delta function we note that E' is related to k'. In fact from $\delta^3(\vec{p}+\vec{k}-\vec{p}\,'-\vec{k}')$ we have $\vec{p}\,'=\vec{k}-\vec{k}'$, and therefore

$$E' = \sqrt{\vec{p}\,'^2+m^2} = \sqrt{k^2+k'^2-2kk'\cos\theta+m^2}$$

❒ This implies

$$\delta(m+k-E'-k') = \frac{\delta\left(k' - \frac{k}{1+\frac{k}{m}(1-\cos\theta)}\right)}{\left|1+\frac{dE'}{dk'}\right|} \quad \text{with} \quad \frac{dE'}{dk'} = \frac{k'-k\cos\theta}{E'}$$

Compton Cross Section . . .

- And therefore

$$\left|1+\frac{dE'}{dk'}\right| = \frac{|E'+k'-k\cos\theta|}{E'} = \frac{m+k(1-\cos\theta)}{E'} = \frac{m}{E'}\frac{k}{k'}$$

- Putting all together

$$\frac{d\sigma}{d\Omega_{k'}} = \frac{1}{64\pi^2}\frac{1}{m^2}\left(\frac{k'}{k}\right)^2 \overline{|M|^2} \quad \text{where} \quad \overline{|M|^2} = \frac{1}{4}\sum_{s,s'}\sum_{\lambda,\lambda'}|M|^2$$

- Calculating the traces

$$\overline{|M_1|^2} = 8\left[2\, m^4 + m^2(-p\cdot p' - p'\cdot k + 2p\cdot k) + (p\cdot k)(p'\cdot k)\right]\frac{e^4}{(2p\cdot k)^2}$$

$$\overline{|M_2|^2} = 8\left[2m^4 + m^2(-p\cdot p' + p'\cdot k' - 2p\cdot k') + (p\cdot k')(p'\cdot k')\right]\frac{e^4}{(2p\cdot k')^2}$$

$$\overline{[M_1 M_2^\dagger + M_1^\dagger M_2]} = \frac{8e^4}{4(k\cdot p)(k'\cdot p)}\left[2(k\cdot p)(p\cdot p') - 2(k\cdot k')(p\cdot p') - 2(p\cdot p')(p\cdot k')\right.$$
$$\left. + m^2(-2k\cdot p - k\cdot p' + k\cdot k' - p\cdot p' + 2p\cdot k' + p'\cdot k') - m^4\right]$$

Compton Cross Section . . .

- Now we use the kinematics of the rest frame of the electron

$$
\begin{aligned}
p' &= p + k - k' & \qquad p \cdot k &= mk \\
p \cdot k' &= mk' & \qquad k \cdot k' &= kk'(1 - \cos\theta) = m(k - k')
\end{aligned}
$$

to obtain

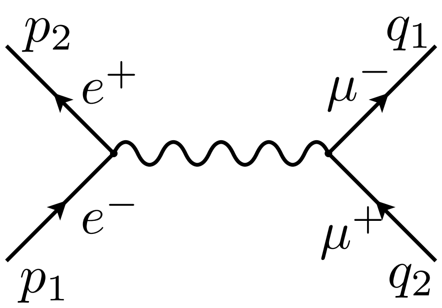
$$
\frac{1}{4}\sum_{s,s'}\sum_{\lambda,\lambda'}\{|M_1|^2+|M_2|^2+M_1 M_2^\dagger+M_1^\dagger M_2\} = 2e^4\left[\left(\frac{k}{k'}\right)+\left(\frac{k'}{k}\right)-\sin^2\theta\right]
$$

- Finally we put everything together to get the Klein-Nishina formula for the differential cross section of the Compton scattering.

$$
\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2\, m^2}\left(\frac{k'}{k}\right)^2\left[\left(\frac{k'}{k}\right)+\left(\frac{k}{k'}\right)-\sin^2\theta\right]}
$$

Scattering $e^- e^+ \to \mu^- \mu^+$ in QED

Diagram and kinematics



$$
\begin{aligned}
p_1 &= \sqrt{s}/2\ (1,0,0,1) \\
p_2 &= \sqrt{s}/2\ (1,0,0,-1) \\
q_1 &= \sqrt{s}/2\ (1,\beta\sin\theta,0,\beta\cos\theta) \\
q_2 &= \sqrt{s}/2\ (1,-\beta\sin\theta,0,-\beta\cos\theta)
\end{aligned}
$$

$$
\boxed{\beta = \sqrt{1-\frac{4m_\mu^2}{s}}}
$$

Amplitude

$$
\begin{aligned}
M &= \overline{v}(p_2)(-ie\gamma^\mu)u(p_1)\ \frac{-i\ g_{\mu\nu}}{(p_1+p_2)^2+i\varepsilon}\ \overline{u}(q_1)(-ie\gamma^\nu)v(q_2) \\
&= ie^2 \frac{1}{(p_1+p_2)^2+i\varepsilon}\ \overline{v}(p_2)\gamma^\mu u(p_1)\ \overline{u}(q_1)\gamma_\mu v(q_2)
\end{aligned}
$$

Spin averaged amplitude squared

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}} |M|^2 &= \frac{e^4}{4(p_1+p_2)^4} \text{Tr}\left[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\right] \text{Tr}\left[(\not{q}_1+m_\mu)\gamma_\mu(\not{q}_2-m_\mu)\gamma_\nu\right] \\
&= \frac{8e^4}{(p_1+p_2)^4}\left(p_1\cdot p_2 m_\mu^2 + p_1\cdot q_1 p_2\cdot q_2 + p_1\cdot q_2 p_2\cdot q_1 + q_1\cdot q_2 m_e^2 + 2m_e^2 m_\mu^2\right)
\end{aligned}
$$

Cross Section

- The general formula for the cross section is

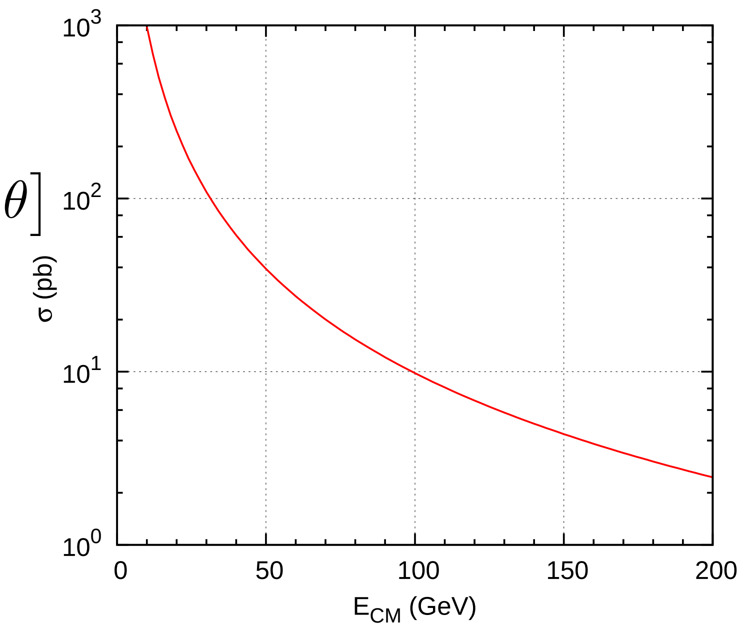
$$\sigma = \int \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_e^4}} \overline{|M|^2} (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2) \prod_{i=1}^{2} \frac{d^3 q_i}{(2\pi)^3 2 q_i^0}$$

- We get the differential cross section

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{32\pi^2 s} \frac{|\vec{q}_1|}{\sqrt{s}} \overline{|M|^2} \\ &= \frac{\alpha^2}{4s}\, \beta \left(\beta^2 \cos^2\theta + 1 + \frac{4m_\mu^2}{s} \right) \\ &= \frac{\alpha^2}{4s}\, \beta \left[1 + \cos^2\theta + (1-\beta^2)\sin^2\theta \right] \end{aligned}$$

- And finally the total cross section

$$\sigma = \frac{2\pi\alpha^2}{3s}\, \beta(3 - \beta^2)$$



- **Mathematica**
  - `FeynArts`
    Program to draw Feynman diagrams. Can be obtained from `http://www.feynarts.de`
  - `FeynCalc`
    Lorentz and Dirac algebra and calculations at *one–loop*. Can have as input FeynArts. Can be obtained from `http://www.feyncalc.org`
- **QGRAF**
  Very efficient program to generate Feynman diagrams for any theory to any loop order done by Paulo Nogueira. Can be downloaded from `http://cfif.ist.utl.pt/~paulo/qgraf.html`
- **Numerics: `C/C++` or `Fortran`**
  To do efficient numerics one has to use the power of `C/C++` or `Fortran`. A special useful package is `CUBA` with routines for numerical integration can be obtained from `http://www.feynarts.de/cuba/`
- **My CTQFT Home Page:** `http://porthos.ist.utl.pt/CTQFT/`
  Here you can find all the links and many programs for standard processes in `QED` and in the `SM`.