Lectures in Quantum Field Theory – Lecture 3

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Summary for Lecture 3: Non Abelian Gauge Theories

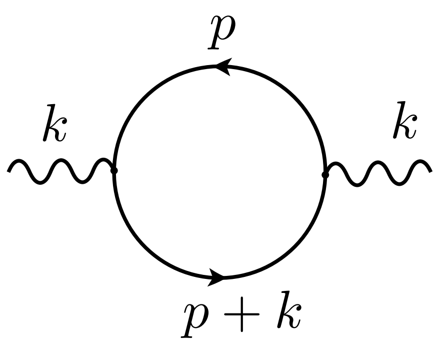
- Radiative corrections and renormalization
- Non Abelian gauge theories: Classical theory
- Non Abelian gauge theories: Quantization
- Feynman rules for a NAGT
- Example: Vacuum polarization in QCD

One-loop correction to the photon propagator

❒ We consider the theory described by the Lagrangian

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial\cdot A)^2 + \overline{\psi}(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi$$

❒ In first order the contribution to the photon propagator is



that we write in the form

$$G^{(1)}_{\mu\nu}(k) \equiv G^0_{\mu\mu'}\; i\,\Pi^{\mu'\nu'}(k) G^0_{\nu'\nu}(k)$$

where

$$i\,\Pi_{\mu\nu} = -(+ie)^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr}\left(\gamma_\mu \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\varepsilon}\gamma_\nu \frac{i(p\!\!\!/ + k\!\!\!/ + m)}{(p+k)^2 - m^2 + i\varepsilon}\right)$$

One-loop correction to the photon propagator ...

- Evaluating the trace we get

$$i\,\Pi_{\mu\nu} = -4e^2 \int \frac{d^4p}{(2\pi)^4} \frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)}$$

- Simple power counting indicates that this integral is quadratically divergent.
- The integral being divergent we have first to regularize it and then to define a renormalization procedure to cancel the infinities.
- We will use the method of dimensional regularization. For a value of d small enough the integral converges. We define $\epsilon = 4 - d$, and we will have a divergent result in the limit $\epsilon \to 0$.

$$i\,\Pi_{\mu\nu}(k,\epsilon) = -\,4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)]}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)}$$

$$= -\,4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)}$$

$$\boxed{\begin{array}{c}[e] = \frac{4-d}{2} = \frac{\epsilon}{2} \\ e \to e\mu^{\frac{\epsilon}{2}}\end{array}}$$

where

$$N_{\mu\nu}(p,k) = 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)$$

One-loop correction to the photon propagator ...

- Now we use the Feynman parameterization to rewrite the denominator as a single term

$$\frac{1}{ab} = \int_0^1 \frac{dx}{\left[ax + b(1-x)\right]^2}$$

to get

$$i\,\Pi_{\mu\nu}(k,\epsilon) = -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\,\frac{N_{\mu\nu}(p,k)}{\left[(p+kx)^2 + k^2x(1-x) - m^2 + i\varepsilon\right]^2}$$

- For dimension d sufficiently small this integral converges and we can change variables, $p \to p - kx$, to get

$$i\,\Pi_{\mu\nu}(k,\epsilon) = -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\,\frac{N_{\mu\nu}(p-kx,k)}{\left[p^2 - C + i\epsilon\right]^2}$$

where

$$C = m^2 - k^2x(1-x)$$

One-loop correction to the photon propagator . . .

- $N_{\mu\nu}$ is a polynomial of second degree in the loop momenta. As the denominator in only depends on p^2 we can show

$$\int \frac{d^dp}{(2\pi)^d} \, \frac{p^\mu}{\left[p^2 - C + i\epsilon\right]^2} = 0$$

$$\int \frac{d^dp}{(2\pi)^d} \, \frac{p^\mu p^\nu}{\left[p^2 - C + i\epsilon\right]^2} = \frac{1}{d} g^{\mu\nu} \int \frac{d^dp}{(2\pi)^d} \, \frac{p^2}{\left[p^2 - C + i\epsilon\right]^2}$$

- This means that we only have to calculate integrals of the form

$$\begin{aligned} I_{r,m} =& \int \frac{d^dp}{(2\pi)^d} \, \frac{(p^2)^r}{\left[p^2 - C + i\epsilon\right]^m} \\ =& i C^{r-m+\frac{d}{2}} \, \frac{(-1)^{r-m}}{(4\pi)^{\frac{d}{2}}} \, \frac{\Gamma(r+\frac{d}{2})}{\Gamma(\frac{d}{2})} \, \frac{\Gamma(m-r-\frac{d}{2})}{\Gamma(m)} \\ =& i \frac{(-1)^{r-m}}{(4\pi)^2} \left(\frac{4\pi}{C}\right)^{\frac{\epsilon}{2}} C^{2+r-m} \, \frac{\Gamma(2+r-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})} \, \frac{\Gamma(m-r-2+\frac{\epsilon}{2})}{\Gamma(m)} \end{aligned}$$

that has poles for $m - r - 2 \leq 0$ due to the properties of the Γ function.

- For the relevant terms we have to expand in powers if ϵ. For instance

$$\begin{aligned}\mu^\epsilon I_{0,2} =& \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\frac{\epsilon}{2}} \frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(2)}\\ =& \frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)\end{aligned}$$

where we have used the expansion of the Γ function

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon), \quad \text{and} \quad \Delta_\epsilon = \frac{2}{\epsilon} - \gamma + \ln 4\pi$$

and γ is the Euler constant

- Putting everything together we finally get

$$\Pi_{\mu\nu} = -\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\Pi(k^2,\epsilon)$$

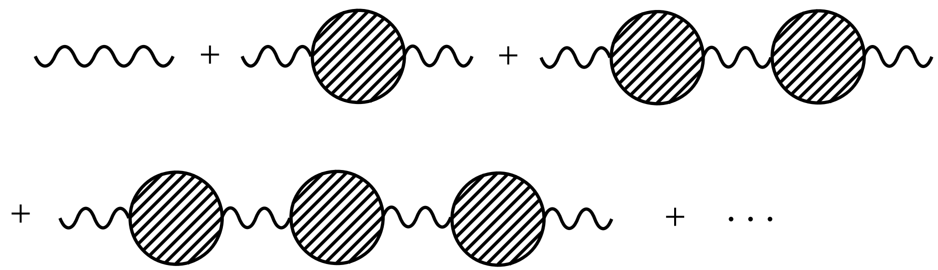
where

$$\Pi(k^2,\epsilon) \equiv \frac{2\alpha}{\pi}\int_0^1 dx\; x(1-x)\left[\Delta_\epsilon - \ln\frac{m^2 - x(1-x)k^2}{\mu^2}\right]$$

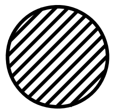
Summing all 1-Particle Irreducible diagrams

- Consider the sum of all 1-PI contributions to photon propagator

$$G_{\mu\nu} =$$

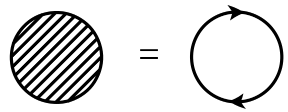


where



$\equiv i\,\Pi_{\mu\nu}(k) =$ sum of all one-particle irreducible (*proper*) diagrams to *all* orders

which we just calculated in lowest order



- Now we separate the propagator in transverse and longitudinal parts

$$iG^0_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2} + \frac{k_\mu k_\nu}{k^4} = P^T_{\mu\nu}\frac{1}{k^2} + \frac{k_\mu k_\nu}{k^4} \equiv iG^{0T}_{\mu\nu} + iG^{0L}_{\mu\nu}$$

$$P^T_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right), \quad k^\mu P^T_{\mu\nu} = 0, \quad P^{T\,\nu}_{\mu} P^T_{\nu\rho} = P^T_{\mu\rho}$$

Summing all 1-Particle Irreducible diagrams

- The same is true for the full propagator

$$G_{\mu\nu} = G^T_{\mu\nu} + G^L_{\mu\nu}, \quad G^T_{\mu\nu} = P^T_{\mu\nu} G_{\mu\nu}$$

- We have obtained, in first order, that the vacuum polarization tensor is transversal, that is

$$i\,\Pi_{\mu\nu}(k) = -ik^2 P^T_{\mu\nu}\,\Pi(k)$$

- This can be shown to be true to all orders (Ward-Takahashi identities). So

$$\begin{aligned} iG^T_{\mu\nu} =& P^T_{\mu\nu}\frac{1}{k^2} + P^T_{\mu\mu'}\frac{1}{k^2}(-i)k^2 P^{T\mu'\nu'}\,\Pi(k)(-i)P^T_{\nu'\nu}\frac{1}{k^2} \\ &+ P^T_{\mu\rho}\frac{1}{k^2}(-i)k^2 P^{T\rho\lambda}\,\Pi(k)(-i)P^T_{\lambda\tau}\frac{1}{k^2}(-i)k^2 P^{T\tau\sigma}\,\Pi(k)(-i)P^T_{\sigma\nu}\frac{1}{k^2} + \cdots \\ =& P^T_{\mu\nu}\frac{1}{k^2}\left[1 - \Pi(k) + \Pi^2(k^2) + \cdots\right] \end{aligned}$$

- Summing the geometric series,

$$iG^T_{\mu\nu} = P^T_{\mu\nu}\frac{1}{k^2\left[1+\Pi(k)\right]}$$

Renormalization

- All that we have done up to this point is formal because the function $\Pi(k)$ diverges.

- The most satisfying way to solve this problem is the following. The correct Lagrangian is obtained by adding corrections to the classical Lagrangian, order by order in perturbation theory, so that we keep the definitions of charge and mass as well as the normalization of the wave functions. The terms that we add to the Lagrangian are called *counter-terms*

$$\mathcal{L}_{\text{total}} = \mathcal{L}(e, m, ...) + \Delta\mathcal{L}$$

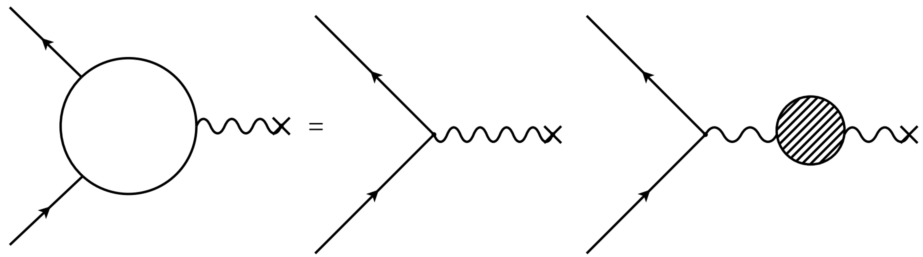
- Counter-terms are defined from the normalization conditions that we impose on the fields and other parameters of the theory. We define the normalization of the photon field as ($G^{RT}_{\mu\nu}$ is the renormalized photon propagator)

$$\lim_{k\to 0} k^2 i G^{RT}_{\mu\nu} = 1 \cdot P^T_{\mu\nu}$$

- The justification for this definition comes from the definition of electric charge as we will now show

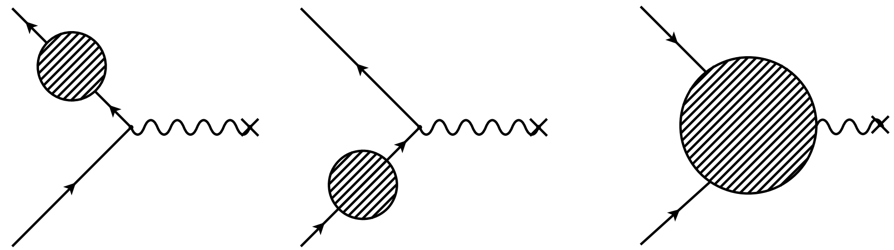
Corrections to Coulomb Scattering: Definition of electric charge

- Consider the corrections to Coulomb scattering



Ward-Takahashi

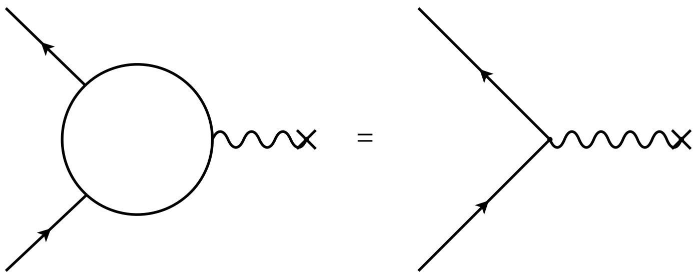
$\lim_{q\to 0}$



$= 0$

- Then the normalization condition, $\lim_{k\to 0} k^2 i G^{RT}_{\mu\nu} = 1 \cdot P^T_{\mu\nu}$, means that the experimental value of the electric charge is determined in the limit $q \to 0$ of the Coulomb scattering by the lowest order

$\lim_{q\to 0}$



Counter-term Lagrangian

- The counter-term Lagrangian has to have the same form as the classical Lagrangian to respect the symmetries of the theory. For the photon field it is traditional to write

$$\Delta\mathcal{L} = -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\delta Z_3\, F_{\mu\nu}F^{\mu\nu}$$

corresponding to the following Feynman rule

$$-i\,\delta Z_3 k^2 \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)$$

- We have then

$$i\Pi_{\mu\nu} = i\Pi^{\text{loop}}_{\mu\nu} - i\,\delta Z_3 k^2 \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) = -i\,\left[\Pi(k,\epsilon) + \delta Z_3\right] k^2\, P^T_{\mu\nu}$$

Therefore we should make the substitution in the photon propagator

$$\Pi(k,\epsilon) \to \Pi(k,\epsilon) + \delta Z_3$$

Renormalized photon propagator

- We obtain for the full photon propagator

$$iG^T_{\mu\nu} = P^T_{\mu\nu}\,\frac{1}{k^2}\,\frac{1}{1+\Pi(k,\epsilon)+\delta Z_3}$$

- The normalization condition implies

$$\Pi(k,\epsilon)+\delta Z_3 = 0$$

from which one determines the constant δZ_3. We get

$$\begin{aligned}\delta Z_3 &= -\,\Pi(0,\epsilon) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left[\Delta_\epsilon - \ln\frac{m^2}{\mu^2}\right]\\ &= -\,\frac{\alpha}{3\pi}\left[\Delta_\epsilon - \ln\frac{m^2}{\mu^2}\right]\end{aligned}$$

- The renormalized photon propagator can then be written as

$$iG_{\mu\nu}(k) = \frac{P^T_{\mu\nu}}{k^2[1+\Pi^R(k^2)]} + i\,G^L_{\mu\nu}, \qquad \Pi^R(k^2) \equiv \Pi(k^2,\epsilon) - \Pi(0,\epsilon)$$

Counter-terms and power counting

- All that we have shown in the previous sections can be interpreted as follows. The initial Lagrangian $\mathcal{L}(e, m, \cdots)$ has to be modified by quantum corrections

$$\mathcal{L}_{\text{total}} = \mathcal{L}(e, m, \cdots) + \Delta\mathcal{L}, \quad \Delta\mathcal{L} = \Delta\mathcal{L}^{[1]} + \Delta\mathcal{L}^{[2]} + \cdots$$

where $\Delta\mathcal{L}^{[i]}$ is the $i^{\text{th}} - loops$ correction.

- Up to first order

$$\mathcal{L}(e, m, \cdots) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}(\partial \cdot A)^2 + i\overline{\psi}\not{\partial}\psi - m\overline{\psi}\psi - e\overline{\psi}\not{A}\psi$$

$$\begin{aligned}\Delta\mathcal{L}^{(1)} = &-\frac{1}{4}(Z_3 - 1) F_{\mu\nu} F^{\mu\nu} + (Z_2 - 1)(i\overline{\psi}\not{\partial}\psi - m\overline{\psi}\psi) \\ &+ Z_2 \delta m \overline{\psi}\psi - e(Z_1 - 1)\overline{\psi}\not{A}\psi\end{aligned}$$

- This Lagrangian will give finite Green functions up to first order.

- The question arises, how do we know that there are no other divergent diagrams, or how can we tell if a theory is renormalizable?

Counter-terms and power counting ...

- Let us consider a Feynman diagram G, with L *loops*, I_B bosonic and I_F fermionic internal lines. If there are vertices with derivatives, δ_v is the number of derivatives in that vertex.

- We define then the *superficial degree of divergence* of the diagram (note that $L = I_B + I_F + 1 - V$) by,

$$
\begin{aligned}
\omega(G) =& 4L + \sum_v \delta_v - I_F - 2I_B \\
=& 4 + 3I_F + 2I_B + \sum_v (\delta_v - 4)
\end{aligned}
$$

$$
\begin{array}{lcl}
\displaystyle\int \frac{d^4q}{(2\pi)^4} & \to & 4L \\
\partial_\mu \Leftrightarrow k_\mu & \to & \delta_v \\
\displaystyle\frac{i}{\not{q} - m} & \to & -I_F \\
\displaystyle\frac{i}{q^2 - m^2} & \to & -2I_B
\end{array}
$$

- For large values of the momenta the diagram will be divergent as

$$
\Lambda^\omega(G) \quad \text{if} \quad \omega(G) > 0, \quad \text{or} \quad \ln\Lambda \quad \text{if} \quad \omega(G) = 0
$$

- The expression is more useful in terms of the external lines. We define ω_v to be the dimension, in terms of mass, of the vertex v. One can shown

$$
\sum_v \omega_v = \sum_v \delta_v + 3I_F + 2I_B + \frac{3}{2}E_F + E_B
\qquad
\omega(G) = 4 - \frac{3}{2}E_F - E_B + \sum_v (\omega_v - 4)
$$

Counter-terms and power counting ... $\omega(G)=4-\frac{3}{2}E_F-E_B+\sum_v(\omega_v-4)$

- We can then classify theories in three classes,

  - *Non-renormalizable Theories*

    They have at least one vertex with $\omega_v > 4$. The superficial degree of divergence increases with the number of vertices, that is, with the order of perturbation theory. For an order high enough all the Green functions will diverge

  - *Renormalizable Theories*

    All the vertices have $\omega_v \leq 4$ and at least one has $\omega_v = 4$. If all vertices have $\omega_v = 4$ then

    $$\omega(G) = 4 - \frac{3}{2}E_F - E_B$$

    and all the diagrams contributing to a given Green function have the same degree of divergence. Only a *finite* number of Green functions are divergent.

  - *Super-Renormalizable Theories*

    All the vertices have $\omega_v < 4$. Only a finite number of diagrams are divergent

Divergent diagrams in QED

$$\omega(G) = 4 - \frac{3}{2}E_F - E_B$$

- Coming back to our question of knowing which are the divergent diagram in QED, we can now summarize the situation

E_F	E_B	$\omega(G)$	Effective degree of divergence
0	2	2	0 (Current Conservation (CC))
0	3		0 (Furry's Theorem)
0	4	0	Convergent (CC)
2	0	1	0 (Current Conservation)
2	1	0	0

- All the other diagrams are superficially convergent. We have therefore a situation where there are only a finite number of divergent diagrams, exactly the ones that we considered before.

- Successes of the renormalization program in QED
  - Calculation of the anomalous magnetic moment of the electron to 1 part in 10^{11}. Needing 8^{th} order in perturbation theory
  - Cancellation of infrared divergences in all processes in QED

Non Abelian Classical Theory

- This is a generalization of what we have done in QED. We start with the Lagrangian

$$\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}$$

- Ψ is a vector in a space of dimension n where acts a representation of a Non-Abelian group G. Under infinitesimal local transformations

$$\delta\Psi = i\varepsilon^a(x)\Omega^a\Psi, \quad a = 1, \dots m$$

where Ω^a are m (dimension of G) hermitian $n \times n$ matrices that obey the commutation relations of G

$$\boxed{[\Omega^a, \Omega^b] = if^{abc}\Omega^c}$$

and f^{abc} are the structure constants of G

Non Abelian Classical Theory

- To make the Lagrangian invariant under *local* gauge transformations, like in QED, we introduce the covariant derivative

$$\partial_\mu \to D_\mu \Psi = (\partial_\mu + ig A_\mu^a \Omega^a)\Psi$$

where the vector fields A_μ^a $(a = 1, 2, \ldots, m)$, the analog of the photon, are called *gauge fields*

- The transformation law for A_μ^a is obtained requiring that $D_\mu \Psi$ transforms as Ψ. It is convenient to introduce the compact matrix notation,

$$\underline{\varepsilon} \equiv \varepsilon^a \Omega^a, \quad \underline{A}_\mu \equiv A_\mu^a \Omega^a, \quad \boxed{\delta\Psi = i\ \underline{\varepsilon}\ \Psi}$$

- The variation of $D_\mu \Psi$ is

$$\begin{aligned}\delta(D_\mu \Psi) =& \partial_\mu(\delta\Psi) + ig\ \delta(\underline{A}_\mu\ \Psi)\\ =& i\ \underline{\varepsilon}\ \partial_\mu \Psi + i\partial_\mu\ \underline{\varepsilon}\ \Psi - g\ \underline{A}_\mu\ \underline{\varepsilon}\ \Psi + ig\delta\underline{A}_\mu\ \Psi\end{aligned}$$

but

$$\delta(D_\mu \Psi) = i\ \underline{\varepsilon}\ D_\mu \Psi = i\ \underline{\varepsilon}\ \partial_\mu \Psi - g\ \underline{\varepsilon}\ \underline{A}_\mu\ \Psi \ \to \ \boxed{\delta\underline{A}_\mu = i\left[\underline{\varepsilon}, \underline{A}_\mu\right] - \frac{1}{g}\partial_\mu \underline{\varepsilon}}$$

Non Abelian Classical Theory

- In component form we have

$$\delta A^a_\mu = -f^{bca}\ \varepsilon^b\ A^c_\mu - \frac{1}{g}\ \partial_\mu \varepsilon^a$$

- The commutator of two covariant derivatives is

$$\begin{aligned}(D_\mu D_\nu - D_\nu D_\mu)\Psi =&(\partial_\mu + ig\ \underline{A}_\mu)(\partial_\nu + ig\ \underline{A}_\nu)\ \Psi - (\mu \leftrightarrow \nu)\\ =&ig\ (\partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu + ig\ \left[\underline{A}_\mu, \underline{A}_\nu\right])\ \Psi \equiv ig\ \underline{F}_{\mu\nu}\Psi\end{aligned}$$

where

$$\underline{F}_{\mu\nu} \equiv F^a_{\mu\nu}\ \Omega^a, \quad \underline{F}_{\mu\nu} \equiv \partial_\mu\ \underline{A}_\nu - \partial_\nu\ \underline{A}_\mu + ig\ \left[\underline{A}_\mu, \underline{A}_\nu\right]$$

- $\underline{F}_{\mu\nu}$ is the generalization to the non abelian case of the Maxwell tensor. It transforms as

$$\delta(\underline{F}_{\mu\nu}) = i\left[\underline{\varepsilon}, \underline{F}_{\mu\nu}\right], \quad \delta F^a_{\mu\nu} = -f^{bca}\varepsilon^b F^c_{\mu\nu}$$

Non Abelian Classical Theory: The Lagrangian

- The generalization of the Maxwell Lagrangian (called Yang-Mills theory)

$$\boxed{\mathcal{L}_{\rm YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}, \qquad F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\ f^{bca} A^b_\mu A^c_\nu}$$

is invariant

$$\delta \mathcal{L}_{\rm YM} = -\frac{1}{2} F^a_{\mu\nu} \delta F^{a\mu\nu} = \frac{1}{2} \varepsilon^b F^a_{\mu\nu} F^{c\mu\nu} f^{bca} = 0$$

- Therefore the Lagrangian

$$\boxed{\mathcal{L} = \overline{\Psi}(i \not{D} - m)\Psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}}$$

is invariant under *local gauge transformations*

- If $G = SU(3)$ this is the theory for the strong interactions, the so-called Quantum Chromodynamics (QCD), a part of the Standard Model, as we will see

- A mass term, $\mathcal{L}_{\rm mass} = -\frac{1}{2}\, m^2 A^a_\mu A^{a\mu}$, would not be gauge invariant, so photons and gluons are massless

Energy–momentum tensor

- The energy-momentum tensor is the analog of the electromagnetic case,

$$\theta^{\mu\nu} = F^{a\mu\rho} F^{a\nu}{}_{\rho} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma a} F^{a}_{\rho\sigma}$$

- Its conservation follows from the equation of motion, $\partial_\mu F^{a\mu\rho} = 0$, and the Bianchi Identity

$$\begin{aligned}
\partial_\mu \theta^{\mu\nu} =& \partial_\mu F^{a\mu\rho} F^{a\nu}{}_{\rho} + F^{a\mu\rho} \partial_\mu F^{a\nu}{}_{\rho} - \frac{1}{2} \partial^\nu F^{a\rho\mu} F^{a}_{\rho\mu} \\
=& \frac{1}{2} F^{a}_{\mu\rho} g^{\nu\sigma} \left(\partial_\mu F^{a}_{\sigma\rho} - \partial_\rho F^{a}_{\sigma\mu} + \partial_\sigma F^{a}_{\rho\mu} \right) \\
=& \frac{1}{2} F^{a}_{\mu\rho} g^{\nu\sigma} \boxed{\left(\partial_\mu F^{a}_{\sigma\rho} + \partial_\sigma F^{a}_{\rho\mu} + \partial_\rho F^{a}_{\mu\sigma} \right) = 0} \qquad \text{Bianchi Identity}
\end{aligned}$$

- Introducing the analog of electric and magnetic fields

$$E^i_a = F^{i0}_a \; ; \; B^k_a = -\frac{1}{2} \varepsilon_{ijk} F^{ij}_a \qquad i,j,k = 1,2,3$$

$$\boxed{\theta^{00} = \tfrac{1}{2} (\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a)} \qquad \boxed{\theta^{0i} = (\vec{E}^a \times \vec{B}^a)^i}$$

Hamiltonian formalism

- From the expression for θ^{00} we get the Hamiltonian

$$H = \int d^3x \frac{1}{2}(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a) \equiv \int d^3x \mathcal{H}$$

where $\mathcal{H}$ is the Hamiltonian density

- The main point we want to emphasize is that the relation between Hamiltonian and Lagrangian is not the usual one. For this we start with the action in the form (1st order formalism)

$$S = \int d^4x \left\{ -\frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu) F^{\mu\nu a} + \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \right\}$$

where A^a_μ and $F^a_{\mu\nu}$ independent variables. The equation of motion for $F^a_{\mu\nu}$ gives its definition.

- Using the definitions of $\vec{E}^a$ and $\vec{B}^a$ we get

$$\begin{aligned} S &= \int d^4x - (\partial^0 \vec{A}^a + \vec{\nabla} A^{0a} - g f^{abc} A^{0b} \vec{A}^c) \cdot \vec{E}^a - \frac{1}{2}(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a) \\ &= \int d^4x \left\{ -\partial^0 \vec{A}^a \cdot \vec{E}^a - \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + A^{0a}(\vec{\nabla} \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c) \right\} \end{aligned}$$

- The Lagrangian density can then be written as

$$\boxed{\mathcal{L} = -E^{ak}\partial^0 A^{ak} - \mathcal{H}(E^{ak}, A^{ak}) + A^{a0}C^a}$$

where

$$\boxed{\begin{aligned} \mathcal{H} &\equiv \tfrac{1}{2}(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a) \\ B^{ak} &\equiv -\tfrac{1}{2}\epsilon^{kmn}F^{amn} \\ C^a &= \vec{\nabla} \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c \end{aligned}}$$

- A_k^a are the *coordinates* and $-E_k^a$ the *conjugated momenta*, $\mathcal{H}(E_k^a, A_k^a)$ is the Hamiltonian density. The variables A^{0a} are Lagrange multipliers for the conditions

$$C^a = \vec{\nabla} \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c = 0$$

which are the equations of motion for $\nu = 0$ (Gauss's Law)

Hamilton and Generalized Hamilton Systems

- Consider a system with canonical variables (p_i, q_i) that generate the phase space Γ^{2n} $(i = 1, \ldots, n)$.
- Then the action for a (canonical) Hamilton System is

$$S = \int dt L(t)$$

where

$$\boxed{L(t) = \sum_{i=1}^{n} p_i \dot{q}_i - h(p, q)}$$

- We can also consider *Generalized Hamilton Systems* (GHS) where

$$\boxed{L(t) = \sum_{i=1}^{n} p_i \dot{q}_i - h(p, q) - \sum_{\alpha=1}^{m} \lambda^\alpha \varphi_\alpha(p, q)}$$

- The quantization of Generalized Hamilton Systems was studied by Dirac

Generalized Hamilton Systems

- The variables $\lambda^\alpha(\alpha = 1, ...m)$ are Lagrange multipliers and φ^α are constraints. For the system to be a generalized Hamilton system the following conditions should be verified

$$\begin{aligned} \{\varphi^\alpha, \varphi^\beta\} &= \sum_\alpha C^{\alpha\beta\gamma}(p,q)\varphi^\gamma \\ \{h, \varphi^\alpha\} &= C^{\alpha\beta}(p,q)\varphi^\beta \end{aligned}$$

{ , } is the Poisson Bracket

- Gauge theories

$$\{C^a(x), C^b(y)\}_{x_0=y_0} = -g f^{abc} C^c(x)\delta^3(\vec{x}-\vec{y})$$
$$\{\mathcal{H}, C^a(x)\} = 0$$

are a particular case with $C^{\alpha\beta} = 0$.

- We have therefore to learn how to quantize generalized Hamilton systems

Quantization: Systems with n degrees of freedom

- Consider the GHS described by

$$\boxed{L(t) = p_i \dot{q}_i - h(p,q) - \lambda^\alpha \varphi^\alpha(p,q)}$$

- This leads to the equations of motion

$$\begin{cases} \dot{q}_i = \dfrac{\partial h}{\partial p_i} + \lambda^\alpha \dfrac{\partial \varphi^\alpha}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial h}{\partial q_i} - \lambda^\alpha \dfrac{\partial \varphi^\alpha}{\partial q_i} \\ \varphi^\alpha(p,q) = 0 \qquad \alpha = 1, \ldots, m \end{cases}$$

- One can show that this GHS is equivalent to a normal HS defined in a space $\Gamma^{*2(n-m)}$, that is, to a system with $n-m$ degrees of freedom. This is constructed as follows. Let be m conditions

$$\chi^\alpha(p,q) = 0, \quad \alpha = 1, \ldots, m, \quad \text{satisfying} \quad \left\{\chi^\alpha, \chi^\beta\right\} = 0$$

and

$$\det \left| \{\varphi^\alpha, \chi^\beta\} \right| \neq 0$$

- Then the subspace Γ^{2n} defined by the conditions

$$\boxed{\chi^\alpha(p,q)=0, \quad \varphi^\alpha(p,q)=0, \qquad \alpha=1,\ldots,m}$$

is the subspace $\Gamma^{*2(n-m)}$ that we want.

- The canonical variables p^* and q^* in $\Gamma^{*2(n-m)}$ can be found as follows:
  - As $\{\chi^\alpha,\chi^\beta\}=0$ we can reorder the variables q_i to make χ^α to coincide with the first m coordinate variables

$$\underbrace{q}_{n} \equiv (\underbrace{\chi^\alpha}_{m}, \underbrace{q^*}_{n-m})$$

  - $p=(p^\alpha,p^*)$ are the corresponding conjugated momenta. Then

$$\det\left|\{\varphi^\alpha,\chi^\beta\}\right| \neq 0, \quad \to \quad \det\left|\frac{\partial\varphi^\alpha}{\partial p^\beta}\right| \neq 0$$

  - The conditions $\varphi^\alpha(p,q)=0$ can then be solved for

$$p^\alpha = p^\alpha(p^*,q^*)$$

Equivalence between the GHS Γ^{2n} and the HS $\Gamma^{*2(n-m)}$

- The subspace Γ^* is given by the conditions

$$\begin{cases} \chi^\alpha \equiv & q^\alpha = 0 \\ p^\alpha = & p^\alpha(p^*, q^*) \end{cases}$$

- The variables p^* and q^* are canonical and the Hamiltonian is

$$h^*(p^*, q^*) = h(p,q)\Big|_{(\chi=0\ ;\ \varphi=0)}$$

- With equations of motion

$$\dot{q}^* = \frac{\partial h^*}{\partial p^*}, \qquad \dot{p}^* = -\frac{\partial h^*}{\partial q^*}, \qquad 2(n-m) \text{ equations}$$

- The fundamental result can be formulated in the form of theorem

The two representations, Γ and Γ^*, are equivalent as they lead to the same equations of motion.

Quantization of Γ^* and Γ

- For Γ^* we have to equivalent ways to quantize:
  - Canonical quantization, with

$$\boxed{\left[p_i^*, q_j^*\right] = -i\ \delta_{ij}}$$

  - Path integral quantization, where the evolution operator is

$$\boxed{U(q_f^*, q_i^*) = \int \prod_t \frac{dp^* dq^*}{(2\pi)} e^{i\int [p^* \dot{q}^* - h(p^*, q^*)]dt}}$$

- In practice this is not very useful because it is not possible to invert the relations $\varphi^\alpha = 0$ to get $p^\alpha = p^\alpha(p^*, q^*)$. It is more convenient to use variables (p, q) with restrictions. This can only be done in the path integral

$$\prod_t \frac{dp^* dq^*}{(2\pi)} \rightarrow \prod_t \frac{dpdq}{2\pi} \prod_t \delta(q^\alpha)\delta(p^\alpha - p^\alpha(p^*, q^*))$$

leading to

$$\boxed{U(q_f, q_i) = \int \prod_t \frac{dpdq}{2\pi} \prod_t \delta(q^\alpha)\delta(p^\alpha - p^\alpha(p^*, q^*)) e^{i\int dt(p\dot{q} - h(p,q))}}$$

Quantization of Γ

- We can rewrite this expression in terms of the constraints. We have

$$\delta(q^\alpha) = \delta(\chi^\alpha), \quad \delta(p^\alpha - p^\alpha(p^*, q^*)) = \delta(\varphi^\alpha) \det \left| \frac{\partial \varphi_\alpha}{\partial p_\beta} \right|$$

and therefore

$$\boxed{\prod_t \delta(q^\alpha)\delta(p^\alpha - p^\alpha(p^*, q^*)) = \prod_t \delta(\varphi^\alpha)\delta(\chi^\alpha) \det |\{\varphi_\alpha, \chi_\beta\}|}$$

- Finally using

$$\delta(\varphi^\alpha) = \int \frac{d\lambda}{2\pi} e^{-i\int dt \lambda^\alpha \varphi_\alpha}$$

we get

$$\boxed{U(q_f, q_i) = \int \prod_t \frac{dpdq}{2\pi} \frac{d\lambda}{2\pi} \prod_{t,x} \delta(\chi^\alpha) \det |\{\varphi^\alpha, \chi_\beta\}| \, e^{iS(p,q,\lambda)}}$$

where we recover the original action

$$S(p, q, \lambda) = \int [p\dot{q} - h(p, q) - \lambda\varphi] dt$$

	Systems finite # degrees of freedom	Field Theory
Classical	t	x^μ
	q	$\phi(x)$
	$\dot q$	$\partial_\mu \phi(x)$
	$S = \int dt L(q, \dot q)$	$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$
	$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0$	$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$
Quantum	$[p_i, q_j] = -i\ \delta_{ij}$	$[\pi(\vec x, t), \varphi(\vec x', t)] = -i\delta(\vec x - \vec x')$
	$U(q_f, q_i) = \int \prod_t \frac{dpdq}{(2\pi)} e^{i\int [p\dot q - h(p,q)]dt}$	$Z[J] = \int \mathcal{D}(\phi) e^{i\int d^4x[\mathcal{L}(\phi,\partial_\mu\phi) + J(x)\phi(x)]}$

❒ It is the correspondence in the last line that we are going to explore in the case of gauge theories.

QED as a simple example

- Consider the electromagnetic field coupled to a conserved current $J^\mu = (p, \vec{J})$, $\partial_\mu J^\mu = 0$. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu$$

- The action in the first order formalism is

$$S = \int d^4x \left[-\vec{E} \cdot (\vec{\nabla} A^0 + \dot{\vec{A}}) - \vec{B} \cdot \vec{\nabla} \times \vec{A} + \frac{\vec{B}^2 - \vec{E}^2}{2} - \rho A^0 + \vec{J} \cdot \vec{A} \right]$$

- Varying with respect to $\vec{E}$, $\vec{B}$, A^0 and $\vec{A}$, we get the usual Maxwell equations ($\vec{E} = -(\vec{\nabla} A^0 + \dot{\vec{A}})$ and $\vec{B} = \vec{\nabla} \times \vec{A}$)

$$\boxed{\begin{array}{ll} \vec{\nabla} \cdot \vec{E} = \rho & \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} - \dfrac{\partial E}{\partial t} = \vec{J} & \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \end{array}}$$

- Substituting back in the action

$$\boxed{S = \int d^4x \left\{ -\vec{E} \cdot \dot{\vec{A}} - \left(\frac{\vec{E}^2 + (\vec{\nabla} \times A)^2}{2} - \vec{J} \cdot \vec{A} \right) + A^0 (\vec{\nabla} \cdot \vec{E} - \rho) \right\}}$$

- It is clear that we have a GHS with A^0 playing the role of a Lagrange multiplier for one constraint $\vec{\nabla} \cdot \vec{E} = \rho$ (Gauss' Law)

- The constraint is linear in the fields. This is the great simplification of QED. In fact if we choose a condition $\chi = 0$ (choice of gauge) that is linear in the fields, then $\det\{\varphi, \chi\}$ does not depend on $\vec{E}$ and $\vec{A}$ and can be absorbed in the normalization

- This is obtained, for instance, in the class of Lorentz gauges

$$\chi = \partial_\mu A^\mu - c(\vec{x}, t)$$

where $c(\vec{x}, t)$ is an arbitrary function that does not depend on the fields

- The generating functional for the Green functions is then

$$Z[J^\mu] = \int \mathcal{D}(\vec{E},\vec{A},A^0) \prod_x \delta(\partial_\mu A^\mu - c(x)) e^{iS}$$

where

$$S = \int d^4x \left\{ -\vec{E}\cdot\dot{\vec{A}} - \left[\frac{E^2 + (\vec{\nabla}\times A)^2}{2} + (\vec{J}\cdot\vec{A}) \right] + A^0(\vec{\nabla}\cdot\vec{E} - \rho) \right\}$$
$$= \int d^4x \left\{ -\frac{E^2}{2} - \vec{E}\cdot(\vec{\nabla}A^0 + \dot{\vec{A}}) - \frac{(\vec{\nabla}\times A)^2}{2} - J_\mu A^\mu \right\}$$

- The integration in $\vec{E}$ is Gaussian and can be done

$$\boxed{Z[J^\mu] = \int \mathcal{D}(A_\mu) \prod_x \delta(\partial_\mu A^\mu - c(x)) e^{iS}}$$

where we are neglecting normalization factors everywhere

QED as a simple example ...

- After integration in $\vec{E}$ the action is

$$S = \int d^4x \left[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - J_\mu A^\mu \right]$$
$$= \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \right]$$

- As the functions $c(x)$ are arbitrary we can average over them with the weight

$$\exp\left(-\frac{1}{2\xi} \int d^4x c^2(x) \right)$$

getting the familiar result

$$\boxed{Z[J^\mu] = \int \mathcal{D}(A_\mu) e^{i \int d^4x \left[-\frac{1}{4} F^2 - \frac{1}{2\xi}(\partial \cdot A)^2 - J \cdot A \right]}}$$

- If we had chosen a non-linear gauge condition then $\det|\{q, \chi\}|$ would depend on $\vec{E}$ and $\vec{A}$ and we could not absorb it in the irrelevant normalization (we choose in the end $Z[0] = 1$). This is the case of NAGT to which we now turn

- We have seen that the classical action of NAGT is

$$\boxed{\begin{aligned} S =& 2\int d^4x \,\mathrm{Tr}\left[\underline{\vec{E}}\cdot\partial^0\underline{\vec{A}} + \frac{1}{2}(\underline{\vec{E}}^2 + \underline{\vec{B}}^2) - \underline{A}^0(\vec{\nabla}\cdot\underline{\vec{E}} + g[\underline{\vec{A}},\underline{\vec{E}}])\right] \\ =& \int d^4x \left[-E^a_k\partial^0 A^a_k - \mathcal{H}(E_k, A_k) + A^{a0}C^a\right] \end{aligned}}$$

where A^{0a} are the Lagrange multipliers for the constraints

$$\boxed{C^a = \vec{\nabla}\cdot\vec{E}^a - g f^{abc}\vec{A}^b\cdot\vec{E}^c}$$

- We introduce the equal time Poisson brackets

$$\left\{-E^i_a(x), A^j_b(y)\right\}_{x^0=y^0} = \delta^{ij}\delta_{ab}\delta^3(\vec{x}-\vec{y})$$

we can show that we have a GHS

$$\{C^a(x), C^b(y)\}_{x_0=y_0} = -g f^{abc}C^c(x)\delta^3(\vec{x}-\vec{y})$$

$$\{H, C^a(x)\} = 0,\ \ H = \int d^3x\mathcal{H}(E_k, A_k) = \frac{1}{2}\int d^3x\left[(E^{ka})^2 + (B^{ka})^2\right]$$

We summarize:

- NAGT are example of generalized Hamilton systems. The coordinates are A_k^a, the conjugate momenta $-E_k^a$ and A^{0a} are Lagrange multipliers for the constraints (Gauss's Law)

$$C^a(x) = \vec{\nabla} \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c = 0, \quad a = 1, \dots, r$$

- To quantize these GHS, we have *to impose* an equal number (r) of auxiliary conditions that we call gauge choice, or gauge fixing (what we called before $\chi^\alpha = 0$)
- This choice is arbitrary and the physical results (S matrix elements) should not depend on it
- We notice that $C^a(x)$ already is quadratic in the fields and momenta. So, even a linear gauge fixing condition will in general lead to a non trivial determinant that can not be absorbed in the normalization constant

Non Abelian Gauge Theories (NAGT): Fixing the gauge

- We choose the gauge fixing condition to be

$$F^a[A_\mu] = 0 \qquad\qquad a = 1, \dots, r$$

- Now we have to calculate the Poisson bracket of the gauge fixing $F^a[A_\mu]$ with the constraint C^b. This is a non trivial calculation with the result

$$\{F^a[A_\mu](x), C^b(y)\} \propto \mathcal{M}_F^{ab}(x,y)$$

where

$$\boxed{\mathcal{M}_F^{ab}(x,y) = -g\frac{\delta F^a[\delta A_\mu(x)]}{\delta\alpha^b(y)} = \frac{\delta F^a}{\delta A_\mu^c(x)} D_\mu^{cb}\delta^4(x-y)}$$

and

$$\delta A_\mu^c = -f^{bdc}\,\alpha^b\,A_\mu^d - \frac{1}{g}\,\partial_\mu\alpha^c = -\frac{1}{g}(D_\mu\alpha)^c$$

- We finally arrive at the generating functional for the Green functions

$$\boxed{Z_F[J^a_\mu] \equiv \int \mathcal{D}(A_\mu)\Delta_F[A_\mu]\prod_{x,a}\delta(F^a[A^b_\mu(x)])e^{i(S[A_\mu]+\int d^4x J^a_\mu A^{\mu a})}}$$

where we have introduced the usual notation

$$\Delta_F[A_\mu] \equiv \det \mathcal{M}_F$$

- For the applications we still have to solve two problems. In fact to be able to formulate the Feynman rules we should exponentiate $\Delta_F[A_\mu]$ and $\delta(F^a[A_\mu])$

- We will address the second problem in first place. Like in QED we start by defining a more general gauge condition

$$F^a[A^b_\mu] - c^a(x) = 0$$

where $c^a(x)$ are arbitrary functions that do not depend on the fields

Non Abelian Gauge Theories (NAGT): Generating Functional

- Now we take the average with the weight

$$\exp\left\{-\frac{i}{2}\int d^4x\ c_a^2(x)\right\}$$

- We get then

$$\begin{aligned} Z_F[J_\mu^a] =& \mathcal{N}\int \mathcal{D}(A_\mu)\Delta_F[A_\mu]e^{i\left(S[A_\mu]+\int d^4x\left(-\frac{1}{2}F_a^2+J^{\mu a}A_\mu^a\right)\right)} \\ =& \mathcal{N}\int \mathcal{D}(A_\mu)\Delta_F[A_\mu]e^{i\int d^4x[\mathcal{L}(x)-\frac{1}{2}F_a^2+J^{\mu a}A_\alpha^a]} \end{aligned}$$

- To be able to formulate the Feynman rules we still have to deal with the determinant $\Delta_F[A_\mu]$. This will lead to the so-called Fadeev-Popov *ghosts* to which we now turn

A Mathematical Detour: Grassmann variables

- Consider anticommuting classical variables, ω, $\overline{\omega}$ (Grassmann variables), defined by

$$\omega\overline{\omega}+\overline{\omega}\omega=0,\ \omega^2=\overline{\omega}^2=0,\int d\omega\,\omega=\int d\overline{\omega}\,\overline{\omega}=1,\int d\omega\,\overline{\omega}=\int d\overline{\omega}\,\omega=0$$

- Now we have

$$\int d\overline{\omega}\,d\omega\ e^{-\overline{\omega}\omega}=\int d\overline{\omega}\,d\omega\ (1-\overline{\omega}\omega)=\int d\overline{\omega}\,d\omega\ (1+\omega\overline{\omega})=1$$

- Next we take two pairs of variables

$$\begin{aligned}\int d\overline{\omega}_1\,d\omega_1\ d\overline{\omega}_2\,d\omega_2\ e^{-\overline{\omega}_i A_{ij}\omega_j}&=\int d\overline{\omega}_1\,d\omega_1\ d\overline{\omega}_2\,d\omega_2\ (1+\cdots\\&+\overline{\omega_1}\omega_1\overline{\omega_2}\omega_2A_{11}A_{22}+\overline{\omega_1}\omega_2\overline{\omega_1}\omega_2A_{12}A_{21})\\&=(A_{11}A_{22}-A_{12}A_{21})=\det A\end{aligned}$$

- In general (here z_i and $\overline{z}_i$ are complex commuting variables)

$$\boxed{\int\prod_{i=1}^n d\overline{\omega}_i\,d\omega_i\ e^{-\overline{\omega}_iA_{ij}\omega_j}=\det A}\qquad\boxed{\int\prod_{i=1}^n d\overline{z}_i\,dz_i\ e^{-\overline{z}_iA_{ij}z_j}\propto(\det A)^{-1}}$$

Fadeev-Popov ghosts

- Now we go to the final step in quantizing our NAGT. The starting point is the generating functional for the Green Functions

$$\boxed{Z_F[J^a_\mu] = \mathcal{N} \int \mathcal{D}(A_\mu)\Delta_F[A] e^{i\int d^4x[\mathcal{L}(x) - \frac{1}{2\xi}(F^a)^2 + J^a_\mu A^{\mu a}]}}$$

where

$$\boxed{\Delta_F[A] = \det \mathcal{M}_F, \quad \mathcal{M}^{ab}_F(x,y) = \frac{\delta F^a[A(x)]}{\delta A^c_\mu(y)} D^{cb}_\mu}$$

- In this form the Feynman rules would be complicated as the term $\det \mathcal{M}_F$ would lead to non-local interactions.

- But we have just seen that we can exponentiate the determinant using anticommuting fields. We take

$$\boxed{\int \mathcal{D}(\overline{\omega}, \omega) e^{-\int d^4x \, \overline{\omega} \mathcal{M}_F \omega} = \det \mathcal{M}_F}$$

where the only requirement is that $\overline{\omega}$ and ω are anticommuting fields

Fadeev-Popov ghosts ...

- Using this result and changing for convenience $\mathcal{M}_F \to i\mathcal{M}_F$ (an irrelevant normalization change) we get

$$\boxed{Z_F[J^a_\mu] = \mathcal{N} \int \mathcal{D}(A_\mu, \overline{\omega}, \omega) e^{i\int d^4x[\mathcal{L}_{eff} + J^a_\mu A^{\mu a}]}}$$

- The NAGT is now described by and *effective* Lagrangian $\mathcal{L}_{\rm eff}$ given by

$$\boxed{\mathcal{L}_{\rm eff} = \mathcal{L} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm G}}$$

where

$$\boxed{\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}, \quad \mathcal{L}_{\rm GF} = -\frac{1}{2\xi}(F^a)^2, \quad \mathcal{L}_{\rm G} = -\overline{\omega}^a \mathcal{M}_F^{ab} \omega^b}$$

- The first term is the classical Lagrangian for the pure NAGT, and the second term, $\mathcal{L}_{\rm GF}$ is the gauge fixing Lagrangian. The third term, $\mathcal{L}_{\rm G}$, that resulted from the exponentiation of the determinant, is new and needs some further explanation

Fadeev-Popov ghosts . . .

- The fields ω and $\overline{\omega}$ are, by construction, auxiliary fields. As we will see they are scalars but also anti-commuting. There is no problem with the spin-statistics theorem in QFT as they are not physical fields. They are called *Fadeev-Popov ghosts*

- Let us look in more detail at their action

$$S_{\rm G} = -\int d^4x d^4y \overline{\omega}^a(x) \mathcal{M}_F^{ab}(x,y)\omega^b(y) = -\int d^4x \int d^4y\, \overline{\omega}^a(x) \frac{\delta F^a(x)}{\delta A_\mu^c(y)} D_\mu^{cb}\omega_b(y)$$

or

$$\mathcal{L}_{\rm G}(x) = \boxed{-\int d^4y\, \overline{\omega}^a(x) \frac{\delta F^a(x)}{\delta A_\mu^b(y)} D_\mu^{bc}\omega_c(y)}$$

- As the ghost Lagrangian depends on the gauge fixing, to proceed we have to be more specific. We choose the Lorentz gauge

$$F^a = \partial_\mu A^{a\mu}$$

Fadeev-Popov ghosts . . .

❒ We therefore get

$$
\begin{aligned}
\mathcal{L}_{\rm G}(x) &= -\int d^4y\, \overline{\omega}^a(x)\partial_x^\mu \left[\delta^4(x-y)\right] D_\mu^{ab}\omega^b(y) \\
&= \partial^\mu\overline{\omega}^a(x) D_\mu^{ab}\omega^b(x) \\
&= \partial^\mu\overline{\omega}^a(x)\partial_\mu\omega^b(x) - g f^{abc} A_\mu^c(x)\partial^\mu\overline{\omega}^a(x)\omega^b(x)
\end{aligned}
$$

where we have used the covariant derivative in the adjoint representation

$$
D_\mu^{ab} = \partial_\mu\delta^{ab} - g f^{abc} A_\mu^c
$$

❒ We summarize

- The ghosts are scalar fields but they are also anticommuting by construction
- The ghosts, like the gauge fields are in the adjoint representation of the gauge group
- The specific form of the $\mathcal{L}_{\rm G}$ depends on the gauge fixing chosen

Feynman rules in the Lorentz gauge

- We are now in position to write the Feynman rules in the Lorentz gauge, $F^a[A] = \partial_\mu A^{\mu a}(x)$. The effective Lagrangian is

$$\boxed{\mathcal{L}_{\text{eff}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - \frac{1}{2\xi} (\partial_\mu A^{a\mu})^2 + \partial^\mu \overline{\omega}^a D^{ab}_\mu \omega^b}$$

where

$$D^{ab}_\mu \omega^b = (\partial_\mu \delta^{ab} - g f^{abc} A^c_\mu) \omega^b$$

- The group constants f^{abc} are defined with the conventions

$$[t^a, t^b] = i f^{abc} t^c, \quad \text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$$

- We can therefore separate the free (kinetic) and interaction parts

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}}$$

Feynman rules in the Lorentz gauge . . .

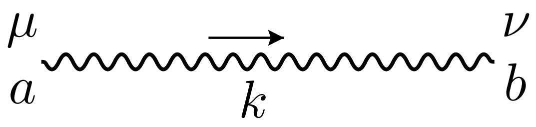
Summary
Renormalization QED
Non Abelian Classical
Quantization GHS
Quantization NAGT
• Method
• Gauge Fixing
• Functional Z_F
• Grassmann variables
• Ghosts
• Feynman rules
• Matter
• Group Factors
Vacuum Pol in QCD

- The kinetic Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{kin}} &= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial_\mu A^{\mu a})^2 + \partial_\mu \overline{\omega}^a \partial^\mu \omega^a \\ &= \frac{1}{2} A^{\mu a}\left[\Box g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial_\mu \partial_\nu\right]\delta^{ab} A^{\nu b} - \overline{\omega}^a \Box\, \delta^{ab} \omega^b\end{aligned}$$

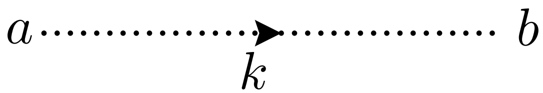
- We get the Feynman rules for the propagators

i) Gauge fields



$$-i\delta_{ab}\left[\frac{g^{\mu\nu}}{k^2 + i\epsilon} - (1-\xi)\frac{k^\mu k^\nu}{(k^2+i\epsilon)^2}\right]$$

ii) Ghosts



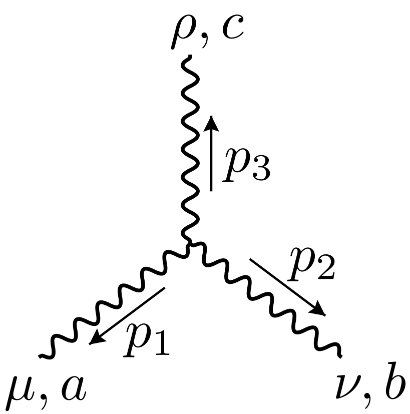
$$\frac{i}{k^2 + i\epsilon}\,\delta_{ab}$$

Feynman rules in the Lorentz gauge ...

- For the interaction Lagrangian we get

$$\mathcal{L}_{\text{int}} = -g f^{abc} \partial_\mu A^a_\nu A^{\mu b} A^{\nu c} - \frac{1}{4} g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e} + g f^{abc} \partial^\mu \overline{\omega}^a A^b_\mu \omega^c$$

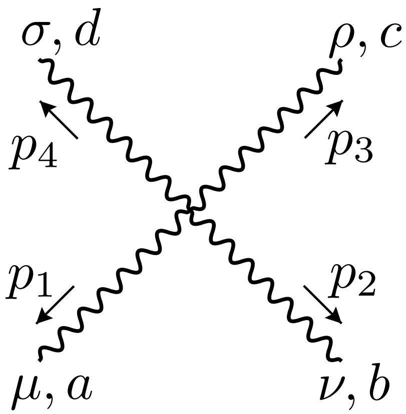
- Triple gauge interaction



$$-g f^{abc} \Big[g^{\mu\nu} (p_1 - p_2)^\rho + g^{\nu\rho} (p_2 - p_3)^\mu + g^{\rho\mu} (p_3 - p_1)^\nu \Big]$$

$$p_1 + p_2 + p_3 = 0$$

- Quartic gauge interaction

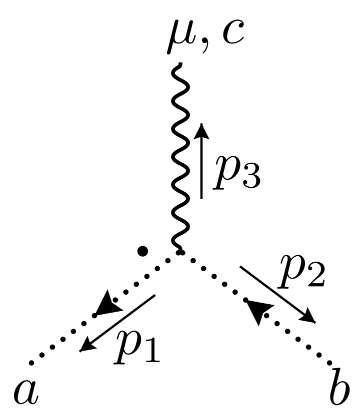


$$-ig^2 \Big[f_{eab} f_{ecd} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) + f_{eac} f_{edb} (g_{\mu\sigma} g_{\rho\nu} - g_{\mu\nu} g_{\rho\sigma}) + f_{ead} f_{ebc} (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}) \Big]$$

$$p_1 + p_2 + p_3 + p_4 = 0$$

Feynman rules in the Lorentz gauge ...

- Interaction Ghosts–Gauge fields



$$g\ f^{abc} p_1^{\mu}$$

$$p_1 + p_2 + p_3 = 0$$

- Comments
  - Ghost lines are oriented, they carry *ghost number*
  - The *dot* refers to the leg that has the derivative, the outgoing leg
  - Other rules are as usual, not forgetting the minus sign for each loop of ghosts

Feynman rules for the interaction with matter

- The interaction with matter is derived from the covariant derivatives
- We take scalar fields $\phi_i, \; i = 1, ...M$, and spinor fields $\psi_j, \; j = 1, ...N$ in representations of dimension M and N. The Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{Matter}} =& (D_\mu \phi)^\dagger D^\mu \phi - m_\phi^2 \phi^\dagger \phi - V(\phi) + i\overline{\psi} D^\mu \gamma_\mu \psi - m_\psi \overline{\psi}\psi \\ \equiv & \; \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}} \; .\end{aligned}$$

- The free kinetic part is the usual one. The interaction Lagrangian can be obtained from the covariant derivative

$$D^\mu_{ij} = \partial_\mu \delta_{ij} - igA^a_\mu T^a_{ij}$$

where T^a_{ij} are the generators in the representations of ϕ and ψ, satisfying

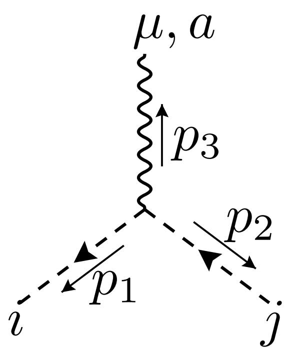
$$[T^a, T^b] = if^{abc} T^c, \quad \text{Tr}(T^a T^b) = \delta^{ab} T(R)$$

- The interaction Lagrangian is

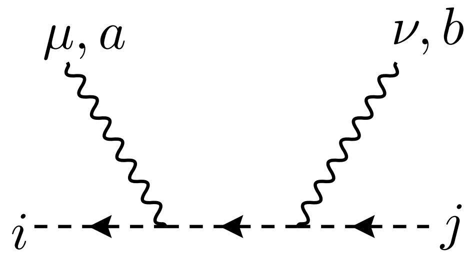
$$\boxed{\mathcal{L}_{\text{int}} = ig\phi^*_i (\overrightarrow{\partial} - \overleftarrow{\partial})^\mu \phi_j T^a_{ij} A_{\mu a} + g^2 \phi^*_i T^a_{ij} T^b_{jk} \phi_k A^a_\mu A^{\mu b} + g\overline{\psi}_i \gamma^\mu \psi_j T^a_{ij} A^a_\mu}$$

Feynman rules for the interaction with matter: Vertices

- Scalars

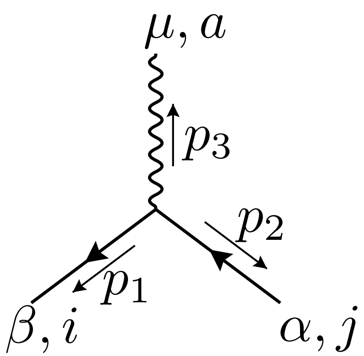


$$ig(p_1 - p_2)^\mu T^a_{ij}$$



$$ig^2 g_{\mu\nu}\{T^a, T^b\}_{ij}$$

- Fermions



$$ig(\gamma^\mu)_{\beta\alpha} T^a_{ij}$$

Feynman rules: Group and Symmetry Factors

- The generators satisfy

$$[T^a, T^b] = i f^{abc} T^c, \quad \mathrm{Tr}(T^a T^b) = \delta^{ab} T(R), \quad T(R) r = d(R) C_2(R)$$

where $T(R)$ characterizes the representation and C_2 is the Casimir

$$\sum_{a,k} T^a_{ik} T^a_{kj} = \delta_{ij} C_2(R)$$

- For $SU(N)$

$$r = N^2 - 1 \; ; \; d(N) = N \; ; \; d(\mathrm{adj}) \equiv d(G) = r$$

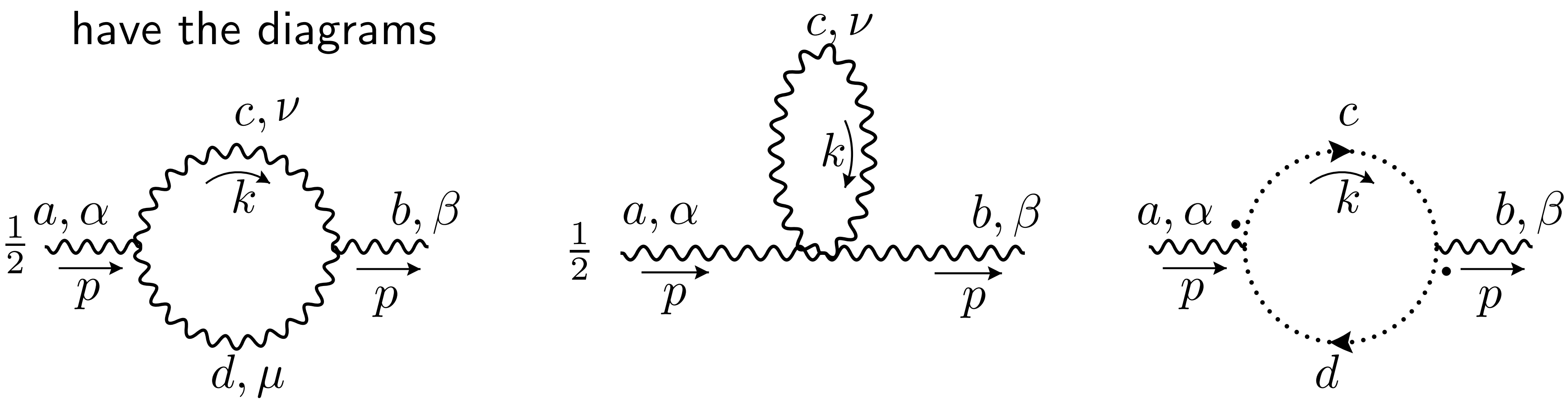
$$T(N) = \frac{1}{2} \; ; \; C_2(N) = \frac{N^2-1}{2N} \; ; \; T(G) = C_2(G) = N$$

- Symmetry Factors

> Each diagram has to be multiplied by its *Symmetry Factor*. This is the # of different ways the external lines can be connected to the vertices divided by the permutation factor of each vertex and a permutation factor for equal vertices.

Vacuum Polarization in QCD: Renormalization constant δZ_A

- As an example we outline the calculation of the renormalization gauge boson self-energy, the so-called vacuum polarization. In the pure gauge theory we have the diagrams



- The amplitude for the first diagram in the $\xi = 1$ gauge is,

$$\boxed{\mathcal{M}^I_{\rm AA} = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\Gamma^{\nu\alpha\mu}_{cad}(k,-p,p-k)\Gamma^{\beta\nu\mu}_{bcd}(p,-k,-p+k)}{[k^2][(k-p)^2]}}$$

- As we just want to evaluate the renormalization constant δZ_A (the analog of δZ_3 for the photon) we just keep the divergent part. We use the $\overline{\rm MS}$ scheme, where we look for the terms proportional to

$$\Delta_\epsilon = \frac{2}{\epsilon} - \gamma + \ln 4\pi, \quad \gamma \text{ is the Euler constant} \tag{1}$$

Vacuum Polarization in QCD: Renormalization constant δZ_A ...

- The result for this diagram is (as usual we define the tensor $i\Pi_{\alpha\beta}$ as the result of the diagram),

$$\Pi^{I}_{\alpha\beta}(\xi=1) = -\frac{g^2}{96\pi^2} C_A \delta_{ab} \left(22 p_\alpha p_\beta - 19 p^2 g_{\alpha\beta}\right) \Delta_\epsilon \tag{2}$$

  where C_A is the Casimir of the adjoint representation

- The amplitude for the second diagram is

$$\boxed{\mathcal{M}^{II}_{\mathrm{AA}} = -\frac{1}{2} i \int \frac{d^4k}{(2\pi)^4} \frac{\Gamma^{\alpha\beta\rho\sigma}_{abcc}\ g_{\rho\sigma}}{k^2} = 0}$$

  a well known result for dimensional regularization with massless fields

- Finally the amplitude for the third diagram, the ghost loop, is

$$\boxed{\mathcal{M}^{III}_{\mathrm{AA}} = (-1) i^2 \int \frac{d^4k}{(2\pi)^4} \frac{\Gamma^{\alpha}_{cda}\Gamma^{\beta}_{dcb}}{[k^2][(k-p)^2]}}$$

- This gives

$$\Pi^{III}_{\alpha\beta}(\xi=1) = \frac{g^2}{96\pi^2} C_A \delta_{ab} \left(2p_\alpha p_\beta + p^2 g_{\alpha\beta}\right) \Delta_\epsilon$$

- Adding everything we get $(\xi = 1)$

$$\boxed{\Pi_{\alpha\beta}(\xi=1) = \frac{5g^2 C_A}{24\pi^2} \delta_{ab} \left(p^2 g_{\alpha\beta} - p_\alpha p_\beta\right) \Delta_\epsilon}$$

showing the transversality property of the vacuum polarization. This is a well known consequence of the gauge invariance and can be shown to hold to all orders in perturbation theory (Ward Identities)

- From this, using the usual definitions, we get

$$\boxed{\delta Z_A = \frac{5g^2 C_A}{24\pi^2} \frac{1}{\epsilon}}$$