Lectures in Quantum Field Theory – Lecture 4

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Summary for Lecture 4: The Standard Model

Summary
Massless SM
Higgs Mechanism
Massive SM
$Z^0 \to f\overline{f}$
$e^- e^+ \to \mu^- \mu^+$
Muon Decay
Higgs & Unitarity

- Gauge group and particle content of the massless SM
- Spontaneous Symmetry Breaking and the Higgs mechanism
- The SM with masses after Spontaneous Symmetry Breaking
- Interactions dictated by the gauge symmetry group
- Example 1: The decay $Z \to f\overline{f}$ in the SM
- Example 2: $e^+e^- \to \mu^+\mu^-$ in the SM
- Example 3: Muon decay
- The need for the Higgs to correct the bad high energy behaviour in $W_L^+ W_L^- \to W_L^+ W_L^-$

The Standard Model

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Massless SM
• Gauge bosons
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• QCD
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- The Standard Model (SM) of strong and electroweak interactions has provided the cornerstone of elementary particle physics
- The basic principle of of the Standard Model is gauge invariance which puts together in the same basic framework the matter particles, their interactions and the gauge vector bosons which mediate them
- It is based on the gauge group $SU_c(3) \times SU_L(2) \times U_Y(1)$
- The theory consists of three sectors: Quantum Chromodynamics (QCD) which deals with the strong interaction, Quantum Electrodynamics (QED) responsible for the electromagnetic force, and the weak interaction sector
- QED and the weak forces are combined in what is called the Electroweak SM
- Apart from the recent important confirmation of neutrino anomalies and some theoretical unaesthetical aspects, the SM has been remarkably successful in account for all aspects over the past three decades

The Standard Model: Outline

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The plan of this lecture is as follows

- The massless Standard Model
  - Electroweak model $SU(2)_L \times U(1)_Y$
  - Quantum Chromodynamics $SU_c(3)$
- Spontaneous breaking of the gauge symmetry
- The Higgs mechanism
- The complete Standard Model with massive particles
- Examples of calculations in the SM

Gauge Bosons for Electroweak $SU(2)_L \times U(1)_Y$ Theory

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- There are four gauge bosons, three W^a_μ $(a = 1, 2, 3)$ transforming as the adjoint representation of $SU(2)_L$, and one B_μ for $U(1)_Y$. The corresponding field tensors are:

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\,\epsilon_{abc} W^b_\mu W^c_\nu$$
$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

g, g' are the coupling constants of the $SU(2)_L$, $U(1)_Y$ groups

- The kinetic Lagrangian for the bosons is given by

$$\mathcal{L}_G = -\frac{1}{4} W^a_{\mu\nu} W_a^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

- It is invariant under the (separate) local gauge transformations

$$W^a_\mu \frac{\sigma^a}{2} \to W'^a_\mu \frac{\sigma^a}{2} = \mathcal{U}_L\, W^a_\mu \frac{\sigma^a}{2} \mathcal{U}_L^{-1} - \frac{i}{g}\,\partial_\mu \mathcal{U}_L\, \mathcal{U}_L^{-1}, \quad \mathcal{U}_L = e^{-i\alpha^a \frac{\sigma^a}{2}}$$

$$B_\mu \to B'_\mu = B_\mu - \frac{i}{g'}\,\partial_\mu \mathcal{U}_Y\, \mathcal{U}_Y^{-1}, \quad \mathcal{U}_Y = e^{+i\alpha_Y}$$

$$\delta W^a_\mu = \epsilon^{abc}\,\alpha^b\, W^c_\mu - \frac{1}{g}\,\partial_\mu \alpha^a, \quad \delta B_\mu = -\frac{1}{g'}\,\partial_\mu \alpha_Y$$

Matter Fields for $SU(2)_L \times U(1)_Y$

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- The matter fields of the SM are all the known fermions which are classified in three generations. One can project each fermion in two helicity states, left and right

$$\psi_L = \frac{1}{2}(1-\gamma_5)\psi, \quad \psi_R = \frac{1}{2}(1+\gamma_5)\psi$$

- They transform differently under the $SU(2)_L$ group. Left handed components are assigned to doublet representation while right handed ones transform as singlets

$$L_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \; Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L, \; e_R^-, u_R, d_R,$$

where we have only shown the particles in the first generation. The other two generations are just copies of the first.

- We define the electric charge generator

$$Q = T_3 + Y, \quad T_3 = \tfrac{1}{2}\sigma_3$$

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Higgs Mechanism

Massive SM

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Higgs & Unitarity

- The quantum numbers for the first family are

Particle	ν_{eL}	e_L	u_L	d_L	e_R	u_R	d_R
T_3	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0
Y	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	-1	$\frac{2}{3}$	$-\frac{1}{3}$
Q	0	-1	$\frac{2}{3}$	$-\frac{1}{3}$	-1	$\frac{2}{3}$	$-\frac{1}{3}$

- Under finite local gauge transformations

$$\begin{aligned}\Psi_L \to \Psi'_L &= e^{-i\alpha^a \frac{\sigma^a}{2}}\, e^{i\alpha_Y Y}\, \Psi_L \\ \psi_R \to \psi'_R &= e^{i\alpha_Y Y}\, \psi_R\end{aligned}$$

- The principle of gauge invariance establishes that the piece of the Lagrangian describing the gauge interactions of the fermions is obtained from kinetic energy part of the Lagrangian, after substituting the derivative by the covariant derivative,

Matter Fields for $SU(2)_L \times U(1)_Y$ ...

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- We define

$$\partial_\mu \Psi_L \to \mathcal{D}_\mu \Psi_L = \left(\partial_\mu - ig \frac{\sigma_a}{2} W^a_\mu + ig' Y B_\mu \right) \Psi_L$$
$$\partial_\mu \psi_R \to \mathcal{D}_\mu \psi_R = \left(\partial_\mu + ig' Y B_\mu \right) \psi_R$$

- Using the transformations properties of the fields (matter and gauge) we can show that the covariant derivatives have the appropriate transformations properties (that is, they transform in the same way as the fields)

$$\mathcal{D}_\mu \Psi_L \to \mathcal{D}_\mu \Psi'_L = e^{-i\alpha^a \frac{\sigma^a}{2}}\, e^{i\alpha_Y Y}\, \mathcal{D}_\mu \Psi_L, \; \mathcal{D}_\mu \psi_R \to \mathcal{D}_\mu \psi'_R = e^{i\alpha_Y Y}\, \mathcal{D}_\mu \psi_R$$

- This assures that the Lagrangian

$$\mathcal{L}_{\rm EW} = -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu}_a - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \sum_{\rm doublets} i \overline{\Psi}_L \gamma^\mu D_\mu \Psi_L + \sum_{\rm singlets} i \overline{\psi}_R \gamma^\mu D_\mu \psi_R$$

is invariant under local $SU(2)_L \times U(1)_Y$ gauge transformations

- Mass terms are not allowed as they break gauge invariance

Quantum Chromodynamics

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- Quantum Chromodynamics is the gauge theory of $SU_c(3)$ that describes the strong force
- The gauge bosons, $A^a_\mu, (a = 1, \ldots, 8)$, called gluons, are in the adjoint representation of $SU_c(3)$. Only the quarks feel the strong force and they belong to the fundamental, triplet, representation of $SU_c(3)$
- The Lagrangian is

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \sum_f i\,\overline{q}_f \gamma^\mu\, D_\mu q_f$$

where the sum is over the quark flavours ($q_f = u, d, c, s, t, b$) and

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu A^c_\nu, \quad D_\mu q_f = \left(\partial_\mu - i g_3 \frac{\lambda_a}{2} A^a_\mu\right) q_f$$

f^{abc} are the structure constants and λ_a the Gell-Mann matrices

- We will leave all the discussion of the flavour in the SM to the lectures of Francisco Botella

Spontaneous Symmetry Breaking

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• Goldstone theorem
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- Most symmetries in Nature are not exact. We say they are broken. For instance the *isospin* symmetry is broken because the proton and neutron do not have equal masses

- There are two ways to implement the breaking

  - **Explicit Breaking**: There are *small* terms in the Lagrangian that break the symmetry
  - **Spontaneous Breaking**: The Lagrangian is invariant but the vacuum breaks the symmetry choosing some particular direction in field space

- We are interested in the second type, spontaneous breaking. To illustrate, let us consider the simplest case, a complex scalar field with the following Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 \;\equiv\; \partial_\mu \phi^* \partial^\mu \phi - V(\phi^* \phi)$$

- The Hamiltonian is

$$\mathcal{H} = \dot{\phi}^* \dot{\phi} + (\vec{\partial} \phi^*) \cdot (\vec{\partial} \phi) + V,$$

Bounded from below $\to \lambda > 0, \;\; \mu^2$arbitary

Mass Spectra

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- We want to study the mass spectra of the theory. For that we have look at the quadratic terms expanded around the vacuum
- For the potential

$$V = \mu^2 \phi^* \phi + \lambda \left(\phi^* \phi\right)^2$$

the minimization equations are

$$\frac{\partial V}{\partial \phi^*} = \phi(\mu^2 + 2\lambda|\phi|^2) = 0, \qquad \frac{\partial V}{\partial \phi} = \phi^*(\mu^2 + 2\lambda|\phi|^2) = 0$$

- We have two possibilities
  - $\phi = 0$
  - $\phi^* \phi = |\phi|^2 = -\dfrac{\mu^2}{2\lambda} \equiv v^2$

Minima of the Scalar Potential

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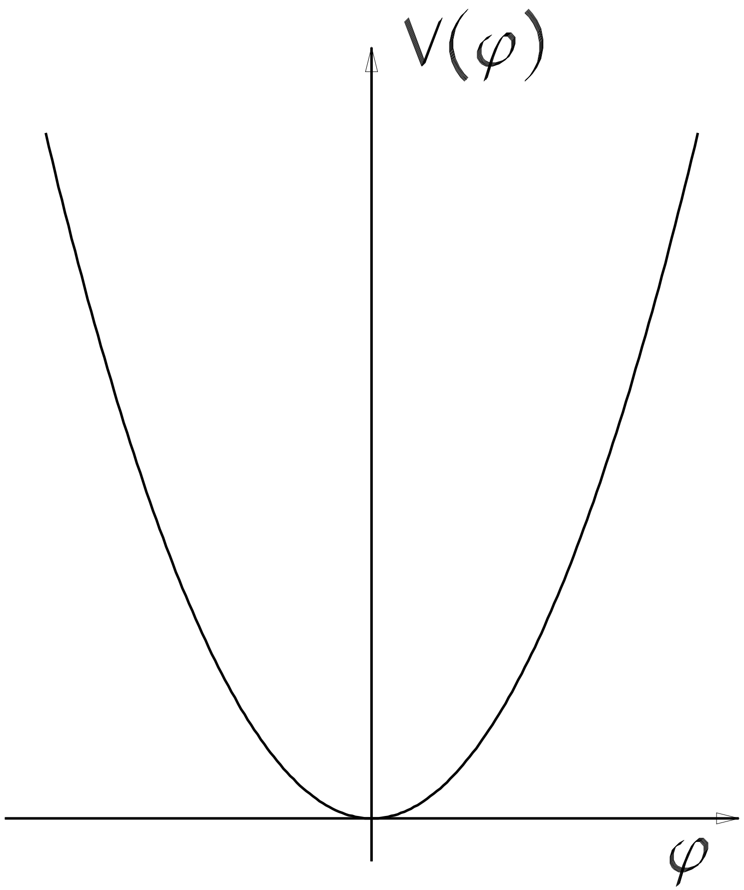
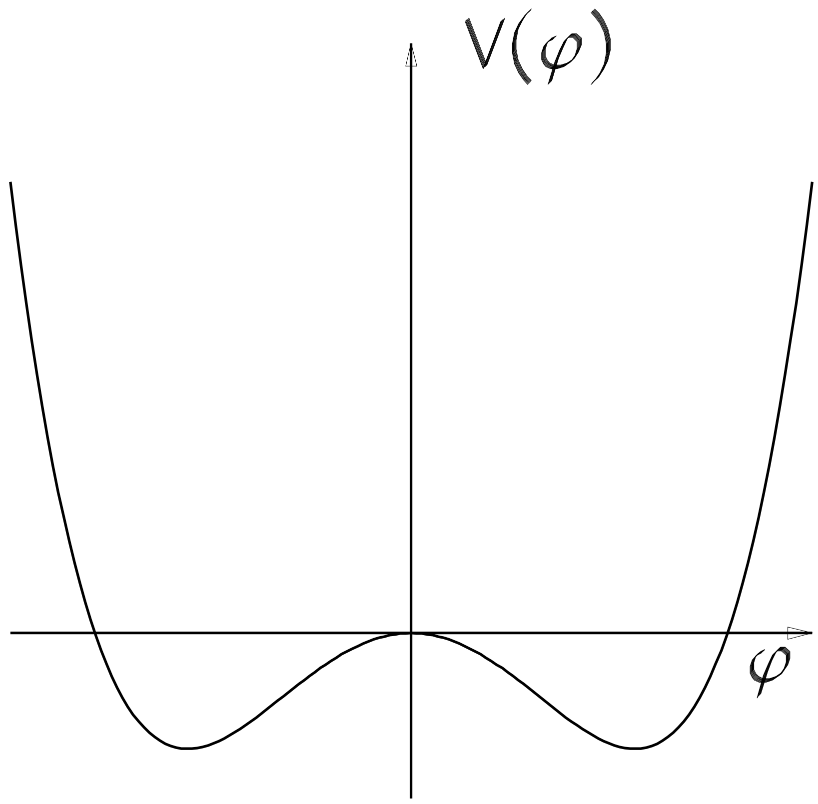
$\mu^2 > 0$



Minimum: $\phi = 0$

$\mu^2 < 0$



Minimum: $\phi^* \phi = |\phi|^2 = -\frac{\mu^2}{2\lambda} \equiv v^2$

Spontaneous Symmetry Breaking (SSB): Spectra

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- We want to find the spectra of the two degrees of freedom for the case of SSB
- The easiest way is to parameterize the complex field as

$$\phi(x) = e^{\frac{i}{\sqrt{2}v}\xi(x)} \left(v + \frac{\sigma(x)}{\sqrt{2}} \right)$$

where ξ and σ are real scalar fields.

- Then

$$\partial_\mu \phi = \frac{i}{\sqrt{2}v}\, \partial_\mu \xi\, \phi + e^{\frac{i}{\sqrt{2}v}\xi(x)}\, \partial_\mu \sigma, \quad \partial^\mu \phi^* = \frac{-i}{\sqrt{2}v}\, \partial^\mu \xi\, \phi^* + e^{-\frac{i}{\sqrt{2}v}\xi(x)}\, \partial^\mu \sigma$$

and we get

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\, \partial_\mu \sigma\, \partial^\mu \sigma + \frac{1}{2}\, \partial_\mu \xi\, \partial^\mu \xi + \frac{1}{2}\, \partial_\mu \xi\, \partial^\mu \xi \left(\sqrt{2} v \sigma + \frac{\sigma^2}{2} \right) \\ & - \mu^2 \left(v^2 + \sqrt{2} v \sigma + \frac{\sigma^2}{2} \right) - \lambda \left(v^4 + 2\sqrt{2} v^3 \sigma + 3\sqrt{2} v^2 \sigma^2 + \sqrt{2} v \sigma^3 + \frac{\sigma^4}{4} \right)\end{aligned}$$

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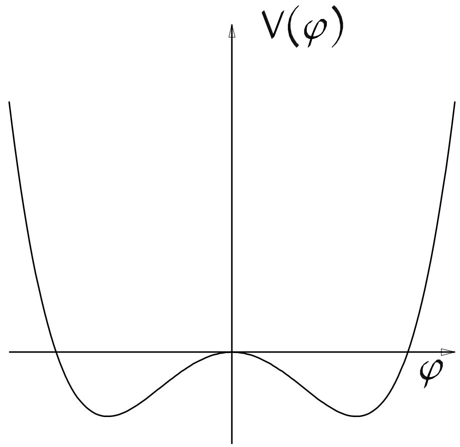
- Using the minimum condition we get

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\sigma\,\partial^\mu\sigma + \frac{1}{2}\,\partial_\mu\xi\,\partial^\mu\xi - \frac{1}{2}\,(-2\mu^2)\,\sigma^2 + \text{constant \& higher order terms}$$

- This Lagrangian describes two real scalar fields ξ and σ, one with mass $m_\sigma = \sqrt{-2\,\mu^2}$ and another massless $m_\xi = 0$.
- This is interpreted as follows: V is a potential with the form of the *Mexican hat* or *bottom of a champagne bottle*
  - The field σ refers to radial excitations and the field ξ to excitations around the bottom.
  - These do not cost energy and therefore ξ is massless.
  - This is a particular case of a general theorem known as Goldstone theorem.



SSB: An $SU(2) \times U(1)$ Example

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- As another example let us consider a Lagrangian invariant under the symmetry group $SU(2) \times U(1)$

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 \equiv \partial_\mu \Phi^\dagger \partial^\mu \Phi - V(\Phi^\dagger \Phi)$$

where

$$\Phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} \equiv \begin{pmatrix} \dfrac{\phi_1 + i\phi_2}{\sqrt{2}} \\ \dfrac{\phi_3 + i\phi_4}{\sqrt{2}} \end{pmatrix}$$

- This is invariant under the transformations

$$\Phi'(x) = e^{i \frac{\tau^a}{2} \alpha^a} \Phi(x), \quad \Phi' = e^{i\epsilon} \Phi$$

- We are interested in the case of a non trivial minimum

$$\Phi^\dagger \Phi = -\frac{\mu^2}{2\ \lambda} \equiv v^2$$

SSB: An $SU(2) \times U(1)$ Example ...

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- As before we can parameterize the field as

$$\phi(x) = e^{i\frac{\tau^a}{2}\theta^a(x)} \begin{pmatrix} 0 \\ v + \dfrac{\sigma(x)}{\sqrt{2}} \end{pmatrix}$$

- The 4 real scalar fields are now $\theta^a(x)$ and $\sigma(x)$.
- To get the mass spectrum we substitute and expand to get

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\,\partial_\mu\sigma\,\partial^\mu\sigma + v^2\partial_\mu\theta^a\,\partial^\mu\theta^a + \mu^2\sigma^2 + \text{constant \& higher order}\\ =& \frac{1}{2}\,\partial_\mu\sigma\,\partial^\mu\sigma + \frac{1}{2}\,\partial_\mu\hat{\theta}^a\,\partial^\mu\hat{\theta}^a + \mu^2\sigma^2 + \text{constant \& higher order}\end{aligned}$$

where $\hat{\theta}^a \equiv \sqrt{2}v\theta^a$

- We have therefore 3 massless fields $\theta^a(x)$ and one massive field with mass

$$m_\sigma = \sqrt{-2\,\mu^2}$$

SSB: Golsdtone's Theorem

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- Again this a general case of the Theorem of Goldstone

> *Consider a theory invariant under group G with n generators. If there is a spontaneous breaking of the symmetry such that the vacuum is invariant under $G' \subset G$ with m generators the number of massless particles is $n-m$, that is, equal to the number of broken generators*

- In the first example we had $U(1)$ with one generator and the vacuum had no symmetry, and therefore we got just one massless particle

- In the $SU(2) \times U(1)$ example we started with $3+1=4$ generators and got 3 massless particles. So there must exist a generator that leaves the vacuum invariant. This the combination $Q = \frac{1+\tau_3}{2}$ that we will identify with the *electric charge*. We have then

$$Q\Phi_{min} = \frac{1+\tau_3}{2} \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = 0$$

SSB and Gauge Theories: The Higgs Mechanism

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- At this point we can ask the following question. We started the discussion of SSB because we wanted to give masses to gauge theories where they were forbidden by symmetry. But all that we have achieved was to get more massless particles!

- Here comes the miracle, the **Higgs Mechanism**: If we have SSB in gauge theories, then the gauge fields can absorb the massless modes and become massive!

- To show how this is done let us go to our first example, the charged scalar field. Promoting the symmetry to a local one we have the Lagrangian

$$\mathcal{L} = (D_\mu \phi)^* (D^\mu \phi) - \mu^2 \phi^* \phi - \lambda(\phi^* \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

  where the covariant derivative is

$$D_\mu = \partial_\mu + ieA_\mu$$

- The Lagrangian is invariant under local gauge transformations

$$\phi(x) \to \phi'(x) = e^{i\epsilon(x)}\, \phi(x), \quad A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{e}\, \partial_\mu \epsilon(x)$$

Higgs Mechanism: $U(1)$ Example

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- If $\mu^2 > 0$ we just have scalar QED. However if $\mu^2 < 0$ we have SSB and we should analize the spectra more carefully
- The vacuum should then be (respecting Lorentz invariance)

$$\langle A_\mu \rangle = 0, \quad \langle \phi \rangle = v = \sqrt{-\frac{\mu^2}{2\lambda}} > 0$$

- To get the spectra we parametrize

$$\phi(x) = e^{i\frac{\xi(x)}{\sqrt{2}v}} \left(v + \frac{\sigma(x)}{\sqrt{2}} \right)$$

to get

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}\, \partial_\mu \sigma\, \partial^\mu \sigma + \frac{1}{2}\, \partial_\mu \xi\, \partial^\mu \xi \; + e^2 v^2 A_\mu A^\mu$$
$$+ \sqrt{2} v e A_\mu \partial^\mu \xi + \mu^2 \sigma^2 + \text{higher order terms}$$

- We have A_μ and $\partial_\mu \xi$ mixed and we cannot read the spectra

Higgs Mechanism: $U(1)$ Example ...

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- Solution: we use the gauge freedom to make a gauge transformation with parameter $\epsilon(x) = -\frac{\xi(x)}{\sqrt{2}v}$

- In this gauge, called *Unitary gauge*, we have

$$\phi(x) \to \phi'(x) = e^{-i\frac{\xi(x)}{\sqrt{2}v}}\phi(x) = v + \frac{\sigma(x)}{\sqrt{2}}$$

$$A_\mu(x) \to A'_\mu(x) + \frac{1}{e\sqrt{2}v}\partial_\mu \xi$$

- Substituting in the Lagrangian, $\mathcal{L}(\phi, A_\mu) = \mathcal{L}(\phi', A'_\mu)$

$$\mathcal{L}(\phi', A'_\mu) = -\frac{1}{4}\, F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}\,\partial_\mu\sigma\;\partial^\mu\sigma + e^2v^2A'_\mu A'^\mu - \frac{1}{2}\,\sigma^2(6\lambda v^2+\mu^2)$$
$$+\frac{1}{2}\,e^2A'_\mu A'^\mu\sigma\,(2\sqrt{2}v+\sigma) - \sqrt{2}\lambda v\sigma^3 - \frac{1}{4}\,\lambda\sigma^4$$

- The ξ field disappeared and we get the masses for σ and A_μ

$$m_\sigma = \sqrt{6\lambda v^2+\mu^2} = \sqrt{-2\mu^2}, \quad m_A = \sqrt{2}ev$$

The Higgs Mechanism in the Standard Model

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Massless SM
Higgs Mechanism
Massive SM
- Higgs Lagrangian
- W and Z Mass
- Electric Charge
- Fermion Masses
- V^{CKM}
- Full SM Lagrangian
- CC and NC
$Z^0 \to f\overline{f}$
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Muon Decay
Higgs & Unitarity

- We are now ready to apply the Higgs mechanism to the Standard Model. As the $SU_c(3)$ group of the strong interactions will remain unbroken, we will just consider the $SU(2)_L \times U(1)_Y$ of the electroweak interactions
- As we have seen, in order to preserve the $SU(2)_L \times U(1)_Y$ invariance all fields are massless. Their mass will be generated by the Higgs mechanism
- In order to implement this idea in the SM a $SU(2)_L$ scalar doublet Φ is introduced in the theory

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

with the quantum numbers

Particle	ϕ^+	ϕ^0
T_3	$\frac{1}{2}$	$-\frac{1}{2}$
Y	$\frac{1}{2}$	$\frac{1}{2}$
Q	1	0

$$Q = T_3 + Y$$

Higgs Boson Lagrangian

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• W and Z Mass
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- We start by writing the following $SU(2)_L \times U(1)_Y$ invariant Lagrangian

$$\mathcal{L}_H = (\mathcal{D}_\mu \Phi)^\dagger (\mathcal{D}^\mu \Phi) - \mu^2\, \Phi^\dagger \Phi - \lambda \left(\Phi^\dagger \Phi\right)^2$$

- As before, we choose the non-trivial vacuum for $\mu^2 < 0$, with the minimum of the potential at $|\Phi|^2 = -\frac{1}{2}\mu^2/\lambda$. For the quantized theory this is equivalent to

$$\langle |\Phi| \rangle = \frac{v}{\sqrt{2}} = \sqrt{-\frac{\mu^2}{2\lambda}}$$

- If we now perform perturbation theory around the true vacuum (θ^a and H are real and have zero vacuum expectation value)

$$\Phi = e^{i\frac{\theta^a(x)\sigma_a}{v}} \begin{pmatrix} 0 \\ \dfrac{v + H(x)}{\sqrt{2}} \end{pmatrix} \to \Phi'(x) = e^{-i\frac{2\theta^a(x)}{v}\frac{\sigma^a}{2}}\, \Phi = \begin{pmatrix} 0 \\ \dfrac{v + H(x)}{\sqrt{2}} \end{pmatrix}$$

where we have performed a gauge transformation into the unitary gauge

Higgs Boson Lagrangian ...

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• Fermion Masses
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$Z^0 \to f\overline{f}$
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Muon Decay
Higgs & Unitarity

- In the unitary gauge there is only one physical scalar field, the Higgs boson H, and the θ^a degrees of freedom become the longitudinal components of the 3 gauge bosons of $SU_L(2)$ which are now massive.

- Introducing into the Lagrangian and dropping the primes we get

$$\begin{aligned}\mathcal{L}_H =&\frac{1}{2}(\partial_\mu H)^2 + \mu^2 H^2 - \frac{1}{4}\lambda H^4 - \lambda\, v\, H^3 + \lambda\frac{v^4}{4}\\ &+ \frac{1}{8}(H+v)^2\left(g^2 W^3_\mu W^{3\mu} + g'^2 B_\mu B^\mu + 2gg' W^3_\mu B^\mu + 2g^2 W^+_\mu W^{-\mu}\right)\\ =&\frac{1}{2}(\partial_\mu H)^2 - \frac{1}{2}m_H^2 H^2 + m_W^2 W^+_\mu W^{-\mu}\\ &+ \frac{1}{8}v^2\left(g^2 W^3_\mu W^{3\mu} + g'^2 B_\mu B^\mu + 2gg' W^3_\mu B^\mu\right) + \cdots\end{aligned}$$

where

$$m_W = \frac{1}{2}gv, \quad m_H = \sqrt{-2\mu^2}, \quad W^\pm_\mu \equiv \frac{W^1_\mu \mp iW^2_\mu}{\sqrt{2}}$$

- We see that B_μ and W^3_μ mix

Higgs Boson Lagrangian: W and Z Mass

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- As we have cross terms between A^3_μ and B_μ to get the mass eigenstates we have to diagonalize the mass matrix

$$M^2 = \frac{1}{4}\, v^2 \begin{bmatrix} g^2 & gg' \\ gg' & g'^2 \end{bmatrix}$$

- The eigenvalues are 0 and $\frac{1}{4}\, v^2(g^2 + g'^2)$. If we call the massless eigenstate A_μ (photon) and the other Z_μ we can write

$$A_\mu = -\sin\theta_W A^3_\mu + \cos\theta_W B_\mu$$

$$Z_\mu = \cos\theta_W A^3_\mu + \sin\theta_W B_\mu$$

- The weak mixing angle θ_W is determined by requiring that A_μ is massless

$$\frac{1}{4} v^2 \begin{bmatrix} g^2 & gg' \\ gg' & g'^2 \end{bmatrix} \begin{bmatrix} -\sin\theta_W \\ \cos\theta_W \end{bmatrix} = 0 \quad \to \quad \tan\theta_W = \frac{g'}{g}$$

The Identification of the Electric Charge

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- The constants g, g' and θ_W are not independent. In fact

$$\tan\theta_W = \frac{g'}{g} \quad \to \quad g\sin\theta_W = g'\cos\theta_W$$

- But they are also related to the electric charge. To see that we write

$$\begin{aligned} D_\mu \Psi_L =& \left[\partial_\mu - i\,\frac{g}{\sqrt{2}}\begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix} - i\,g T_3 W_\mu^3 + i\,g' Y \mathbb{1} B_\mu\right]\Psi_L \\ =& \left[\partial_\mu - i\,\frac{g}{\sqrt{2}}\begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix} + i\,g\sin\theta_W Q A_\mu \right. \\ & \left. -i\,\frac{g}{\cos\theta_W}\left(T_3 - \sin^2\theta_W Q\right) Z_\mu\right]\Psi_L \\ D_\mu \Psi_R =& \partial_\mu + i\,g' Y \mathbb{1} B_\mu = \left[\partial_\mu + i\,g'\cos\theta_W Q A_\mu + i\,\frac{g}{\cos\theta_W}\sin^2\theta_W Q Z_\mu\right]\Psi_R \end{aligned}$$

- Therefore A_μ couples as the photon (minimal coupling) if

$$g\sin\theta_W = e \quad \to \quad e = g'\cos\theta_W$$

where $e > 0$ is the charge of the proton

Yukawa Lagrangian: Fermion Masses

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- We now turn to the Yukawa Lagrangian $\mathcal{L}_{\text{Yuk}}$. The most general $SU_(2) \times U_Y(1)$ invariant Lagrangian for Yukawa interactions is

$$\mathcal{L}_{\text{Yuk}} = -\sum_{ij} \left[Y^l_{ij}\, \overline{l'}_{iL}\, \Phi\, l'_{jR} + Y^u_{ij}\, \overline{u'}_{iL}\, \tilde{\Phi}\, u'_{jR} + Y^d_{ij}\, \overline{d'}_{iL}\, \Phi\, d'_{jR} + \text{h.c.} \right]$$

where $\tilde{\Phi} = i\sigma_2\, \Phi^*$ is a doublet with $Y = -1/2$ as needed by gauge invariance

- In the unitary gauge we have

$$\Phi = \begin{pmatrix} 0 \\ \dfrac{v + H(x)}{\sqrt{2}} \end{pmatrix}, \quad \tilde{\Phi} = \begin{pmatrix} \dfrac{v + H(x)}{\sqrt{2}} \\ 0 \end{pmatrix}$$

- We get

$$\mathcal{L}_{\text{Yuk}} = -\sum_{ij} \Bigg[\overline{l'}_{iL}\, M^l_{ij} l'_{jR} + \overline{u'}_{iL}\, M^u_{ij} u'_{jR} + \overline{d'}_{iL}\, M^d_{ij} d'_{jR}$$

$$+ \frac{H}{\sqrt{2}}\, \overline{l'}_{iL}\, Y^l_{ij} l'_{jR} + \frac{H}{\sqrt{2}}\, \overline{u'}_{iL}\, Y^u_{ij} u'_{jR} + \frac{H}{\sqrt{2}}\, \overline{d'}_{iL}\, Y^d_{ij} d'_{jR} + \text{h.c.} \Bigg]$$

Yukawa Lagrangian: Fermion Masses . . .

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- The mass matrices are

$$M^l_{ij} = Y^l_{ij} \frac{v}{\sqrt{2}} \quad , \quad M^u_{ij} = Y^u_{ij} \frac{v}{\sqrt{2}} \quad , \quad M^d_{ij} = Y^d_{ij} \frac{v}{\sqrt{2}}$$

- The mass eigenstates are then obtained via the rotations

$$\begin{aligned} l_{iL} &= \boldsymbol{R^l_L}_{ij} l'_{jL} & u_{iL} &= \boldsymbol{R^u_L}_{ij} u'_{jL} & d_{iL} &= \boldsymbol{R^d_L}_{ij} d'_{jL} \\ l_{iR} &= \boldsymbol{R^l_R}_{ij} l'_{jR} & u_{iR} &= \boldsymbol{R^u_R}_{ij} u'_{jR} & d_{iR} &= \boldsymbol{R^d_R}_{ij} d'_{jR} \end{aligned}$$

where the $\boldsymbol{R^f}$'s are unitary matrices.

- In the SM neutrinos are massless because the model does not contain right-handed neutrinos. For convenience we choose $\nu_{iL} = \boldsymbol{R^l_L}_{ij} \nu'_{jL}$. With this rotation we get

$$\mathcal{L}_{\text{Yuk}} = -\sum_i \left[m^l_i \, \overline{l_i} \, l_i + m^u_i \, \overline{u_i} \, u_i + m^d_i \, \overline{d_i} \, d_i \right] + \cdots$$

where m^f_i are the physical fermion masses

The Cabbibo-Kobayashi-Maskawa matrix

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- We write now $\mathcal{L}_F^{\text{kinetic}}$ in term of the physical mass eigenstates
- The derivative and the neutral current coupling remain diagonal since it always involves $\boldsymbol{R^f}^\dagger \boldsymbol{R^f} = \mathbb{1}$
- The diagonal form of the neutral current coupling implies that there are not flavour changing neutral currents. This is the Glashow, Iliopoulos and Miani (GIM) mechanism.
- The charged current Lagrangian for quarks becomes

$$\mathcal{L} = \frac{g}{2\sqrt{2}} \overline{u_i} \gamma^\mu (1 - \gamma^5) \boldsymbol{V}_{ij}^{\text{CKM}} d_j W_\mu^+ + \text{h.c.}$$

See F. Botella's Lectures

where $\boldsymbol{V}^{\text{CKM}} = \boldsymbol{R_L^u} \boldsymbol{R_L^d}^\dagger$ is the Cabbibo-Kobayashi-Maskawa. It contains 4 free parameters, 3 angles and a phase which leads to CP violating terms

- The charged currents for leptons remain diagonal because we have chosen the neutrino states to be rotated by the same matrix as the charge leptons. This is only possible because in the SM the neutrinos are massless

Full Lagrangian for the Standard Model after SSB

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Higgs & Unitarity

- We are now in position to write the full Lagrangian for the the SM

$$
\begin{aligned}
\mathcal{L}_{\text{SM}} =& -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \frac{1}{4} W^b_{\mu\nu} W^{b\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\
& + (\mathcal{D}_\mu \Phi)^\dagger (\mathcal{D}^\mu \Phi) - \mu^2\, \Phi^\dagger \Phi - \lambda \left(\Phi^\dagger \Phi\right)^2 \\
& + \sum_{\text{doublets}} i\, \overline{\Psi}_L \gamma^\mu\, D_\mu \Psi_L + \sum_{\text{singlets}} i\, \overline{\psi}_R \gamma^\mu\, D_\mu \psi_R \\
& - \sum_{ij} \left[Y^l_{ij}\, \overline{l'}_{iL}\, \Phi\, l'_{jR} + Y^u_{ij}\, \overline{u'}_{iL}\, \tilde{\Phi}\, u'_{jR} + Y^d_{ij}\, \overline{d'_{iL}}\, \Phi\, d'_{jR} + \text{h.c.} \right] \\
=& -e \sum_f Q^f \overline{\psi}_f \gamma^\mu \psi_f A_\mu + \frac{g}{\cos\theta_W} \sum_f \overline{\psi}_f \gamma^\mu \left(g^f_V - g^f_A \gamma_5 \right) \psi_f Z_\mu \\
& + \frac{g}{\sqrt{2}} \sum_{\text{doublets}} \overline{\psi_u} \gamma^\mu \frac{1-\gamma_5}{2} \psi_d W^+_\mu + \frac{g}{\sqrt{2}} \sum_{\text{doublets}} \overline{\psi_d} \gamma^\mu \frac{1-\gamma_5}{2} \psi_u W^-_\mu + \cdots
\end{aligned}
$$

where we have kept only the charged and neutral currents and

$$
\Psi_L = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} u \\ d \end{pmatrix}_L, \cdots, \quad g^f_V = \frac{1}{2} T^f_{3L} - \sin^2\theta_W Q^f, \quad g^f_A = \frac{1}{2} T^f_{3L}
$$

The Charge and Neutral Current Interactions

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$Z^0 \to f\bar{f}$
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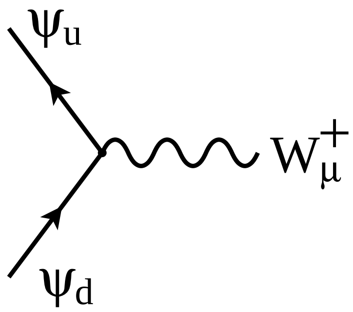
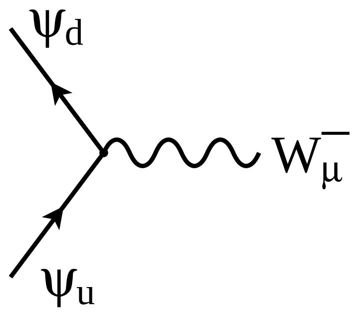
- The charged current Feynaman rules are



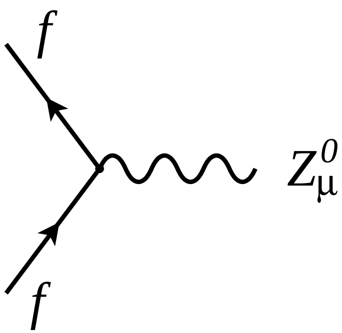
$$i\frac{g}{\sqrt{2}}\gamma^\mu \frac{1-\gamma_5}{2}$$



$$i\frac{g}{\sqrt{2}}\gamma^\mu \frac{1-\gamma_5}{2}$$

- For the neutral current we have

$$g_V^f = \frac{1}{2}T_{3L}^f - \sin^2\theta_W Q^f\,, \qquad g_A^f = \frac{1}{2}T_{3L}^f$$



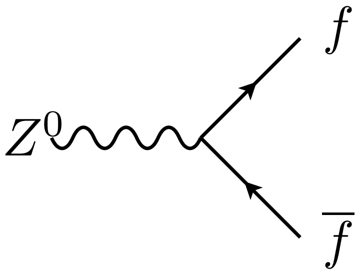
$$i\frac{g}{\cos\theta_W}\,\gamma^\mu(g_V^f - g_A^f\gamma_5)$$

Fermion	e^-, μ^-, τ^-	ν_e, ν_μ,ν_τ	u	d	c	s	t	b
T_{3L}^f	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
Q_f	-1	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$

Decay $Z^0 \to f\overline{f}$

- As a first example in the SM let us consider the process

$$Z^0 \to f\,\overline{f}$$



$$i\frac{g}{\cos\theta_W}\,\gamma^\mu(g_V^f - g_A^f\gamma_5)$$

where f is any fermion of the SM with the exception of the top.

- The amplitude is then

$$M = \frac{ig}{\cos\theta_W}\,\epsilon^\mu(k,\lambda)\,\overline{u}(q_1)\gamma^\mu\left(g_V^f - g_A^f\gamma_5\right)v(q_2)$$

- The decay width is obtained from the definition

$$\Gamma = \int \frac{1}{2M_Z}\,\overline{|M|^2}(2\pi)^4\delta^4(k-q_1-q_2)\prod_{i=1}^{2}\frac{d^3q_i}{(2\pi)^3 2E_i}$$

- In the Z^0 rest frame

$$\frac{d\Gamma}{d\Omega} = \frac{1}{64\pi^2}\,\frac{1}{M_Z}\,\overline{|M|^2}\sqrt{1-\frac{4m_f^2}{M_Z^2}}$$

Decay $Z^0 \to f\overline{f}$: The average amplitude squared $\overline{|M|^2}$

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- The next step is to evaluate $\overline{|M|^2}$

$$\overline{|M|^2} = \frac{1}{3} \sum_{\text{spins}} |M|^2 = \frac{1}{3} \left(\frac{g}{\cos\theta_W} \right)^2 \sum_\lambda \epsilon^\mu(k,\lambda) \epsilon^{*\nu}(k,\lambda)$$
$$\text{Tr}\left[(\not{q}_1 + m_f)\gamma_\mu \left(g_V^f - g_A^f \gamma_5 \right) (\not{q}_2 - m_f)\gamma_\nu \left(g_V^f - g_A^f \gamma_5 \right) \right]$$

- Using now

$$\sum_\lambda \epsilon^\mu(k,\lambda) \epsilon^{*\nu}(k,\lambda) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{M_Z^2}$$

$$\text{Tr}\,[\ldots] = 4 \left[\left({g_V^f}^2 + {g_A^f}^2 \right) \left(q_{1\mu} q_{2\nu} + q_{1\nu} q_{2\mu} - g_{\mu\nu}\ q_1 \cdot q_2 \right) \right.$$
$$\left. - g_{\mu\nu}\ m_f^2 \left({g_V^f}^2 - {g_A^f}^2 \right) - 2i \epsilon^{\alpha\beta}{}_{\mu\nu} q_{1\alpha} q_{2\beta}\ g_V^f g_A^f \right]$$

- We get

$$\overline{|M|^2} = \frac{4}{3} \left(\frac{g}{\cos\theta_W} \right)^2 M_Z^2 \left[{g_V^f}^2 + {g_A^f}^2 + 2 \left(\frac{m_f}{M_Z} \right)^2 \left({g_V^f}^2 - 2{g_A^f}^2 \right) \right]$$

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- Using the relation

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} = \left(\frac{g}{\cos\theta_W}\right)^2 \frac{1}{8M_Z^2}$$

where we have used the SM relation $M_W = M_Z \cos\theta_W$, we finally get

$$\Gamma = \frac{2G_F m_Z^3}{3\sqrt{2}\pi} \sqrt{1 - \frac{4m_f^2}{M_Z^2}} \left[g_V^{f\,2} + g_A^{f\,2} + 2\left(\frac{m_f}{M_Z}\right)^2 \left(g_V^{f\,2} - 2g_A^{f\,2}\right)\right]$$

- For the SM fermions is a good approximation to neglect $(m_f/M_Z)^2$ (even for the b quark is only 3×10^{-3}), so

$$\Gamma(Z \to f\overline{f}) = \frac{2G_F m_Z^3}{3\sqrt{2}\pi} \left(g_V^{f\,2} + g_A^{f\,2}\right)$$

and

$$\Gamma(Z \to e^+e^-) \simeq 83.4\ \text{MeV}$$

$e^-e^+ \to \mu^-\mu^+$ in the Standard Model

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$Z^0 \to f\overline{f}$
$e^-e^+ \to \mu^-\mu^+$
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- Cross Section
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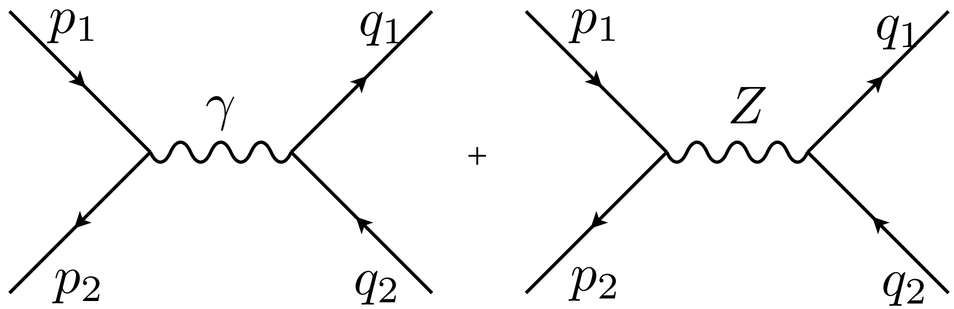
- We consider now the process $e^-e^+ \to \mu^-\mu^+$ in the SM. We have the diagrams



- We get for the amplitude (we include Γ_Z in the propagator)

$$M = (ie)^2 \frac{-ig^{\mu\nu}}{s} \; \overline{v}(p_2)\gamma_\mu u(p_1) \; \overline{u}(q_1)\gamma_\nu v(q_2)$$

$$+ \left(\frac{ig}{\cos\theta_W}\right)^2 \overline{v}(p_2)\gamma^\mu \left(g_V^e - g_A^e\gamma_5\right) u(p_1) \frac{-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_Z^2}\right)}{s - M_Z^2 + iM_Z\Gamma_Z} \overline{u}(q_1)\gamma^\nu \left(g_V^f - g_A^f\gamma_5\right)$$

- For the energies we are considering it is safe to take $m_e = 0$. Then we can neglect the terms in $k_\mu k_\nu / M_Z^2$ in the propagator after using the Dirac equation

$e^- e^+ \to \mu^- \mu^+$ in the Standard Model

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- We get for the amplitude

$$\begin{aligned} M =& i\frac{e^2}{s}\Big[\overline{v}(p_2)\gamma^\mu u(p_1)\ \overline{u}(q_1)\gamma_\mu v(q_2) \\ & + F(s)\overline{v}(p_2)\gamma^\alpha(g_V^e - g_A^e\gamma_5)u(p_1)\ \overline{u}(q_1)\gamma_\alpha(g_V^f - g_A^f\gamma_5)v(q_2)\Big] \end{aligned}$$

where

$$F(s) = \frac{1}{\sin^2\theta_W \cos^2\theta_W}\ \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z}$$

- After some trace calculations

$$\overline{|M|^2} = \frac{e^4}{4s^2}\left\{A + |F(s)|^2 B + 2Re\big[F(s)C\big]\right\}$$

- We get for A

$$\begin{aligned} A =& 32(p_1\cdot q_1\ p_2\cdot q_2 + p_1\cdot q_2\ p_2\cdot q_1 + m_f^2\ p_1\cdot p_2) \\ =& 4s^2\ \left[1 + \cos^2\theta + (1-\beta^2)\sin^2\theta\right] \end{aligned}$$

$e^- e^+ \to \mu^- \mu^+$ in the Standard Model

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- Cross Section
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- For B and C

$$\begin{aligned}B =&4s^2\Big\{g_V^{f\,2}(g_V^{e\,2}+g_A^{e\,2})\left[1+\cos^2\theta+(1-\beta^2)\sin^2\theta\right]\\&\qquad+g_A^{f\,2}(g_V^{e\,2}+g_A^{e\,2})\beta^2(1+\cos^2\theta)+8g_A^e g_V^e g_A^f g_V^f\beta\cos\theta\Big\}\\C =&4s^2\Big\{g_V^e g_V^f\left[1+\cos^2\theta+(1-\beta^2)\sin^2\theta\right]+2g_A^e g_A^f\beta\cos\theta\Big\}\end{aligned}$$

- For the differential cross section

$$\begin{aligned}\frac{d\sigma}{d\Omega} =&\frac{\alpha^2}{4s}\ \beta\Big\{Q_f^2\left[1+\cos^2\theta+(1-\beta^2)\sin^2\theta\right]\\&-2Q_f\chi_1(s)\Big[g_V^e g_V^f\left[1+\cos^2\theta+(1-\beta^2)\sin^2\theta\right]+2g_A^e g_A^f\beta\cos\theta\Big]\\&+\chi_2(s)\Big[g_V^{f\,2}(g_V^{e\,2}+g_A^{e\,2})\left[1+\cos^2\theta+(1-\beta^2)\sin^2\theta\right]\\&\qquad+g_A^{f\,2}(g_V^{e\,2}+g_A^{e\,2})\beta^2(1+\cos^2\theta)+8g_A^e g_V^e g_A^f g_V^f\beta\cos\theta\Big]\Big\}\end{aligned}$$

where $\chi_1(s) = Re(F(s)), \quad \chi_2(s) = |F(s)|^2$

$e^-e^+ \to \mu^-\mu^+$: Total Cross section

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$Z^0 \to f\overline{f}$
$e^-e^+ \to \mu^-\mu^+$
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• Cross Section
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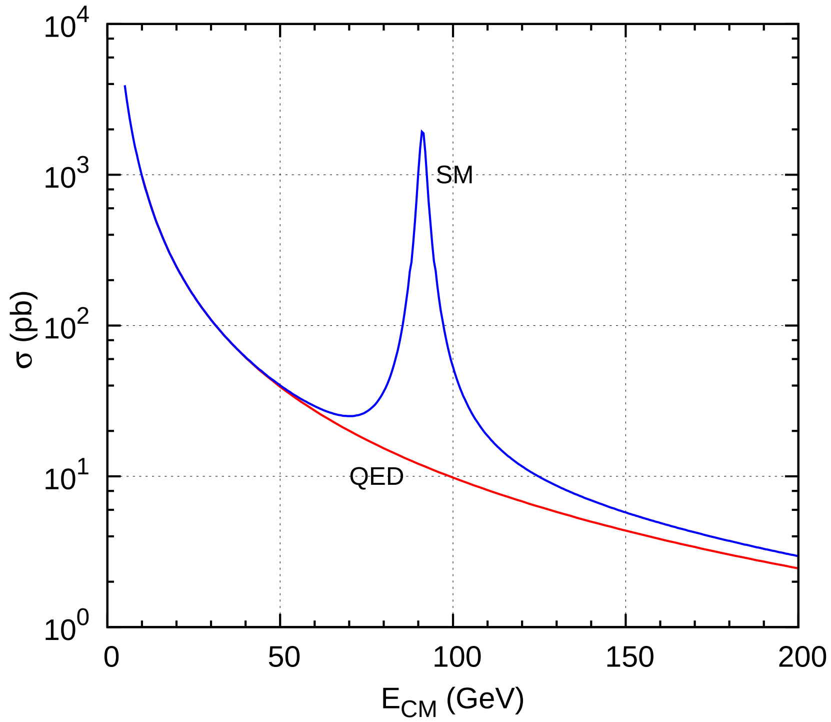
- Integrating in the solid angle Ω we get

$$\sigma = \frac{2\pi\alpha^2}{3s}\ \beta\Big\{3-\beta^2+2\chi_1(s)g_V^e g_V^f(3-\beta^2) + \chi_2(s)\big[{g_V^f}^2({g_V^e}^2+{g_A^e}^2)(3-\beta^2)+2{g_A^f}^2({g_V^e}^2+{g_A^e}^2)\beta^2\big]\Big\}$$

which should be compared with pure QED

$$\sigma_{\rm QED} = \frac{2\pi\alpha^2}{3s}\ \beta(3-\beta^2)$$

- In the figure we show the result comparing with pure QED. This was fully confirmed at LEP



Muon Decay

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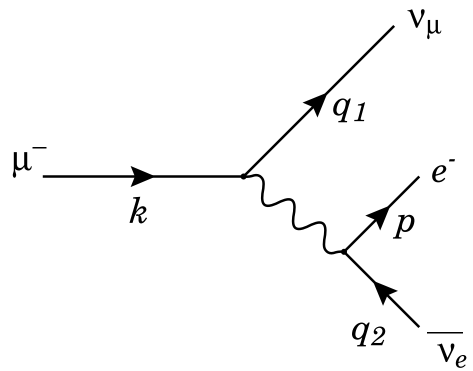
- The last example is the decay of the muon. Historically this process was very important for the acceptance of *Fermi theory* of four-fermion interaction
- Today this process is understood in the SM through the diagram



with an intermediate W boson

- The amplitude is

$$M = \left(\frac{ig}{\sqrt{2}}\right)^2 \overline{u}(q_1)\gamma^\mu \left(\frac{1-\gamma_5}{2}\right) u(k) \frac{-i\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M_W^2}\right)}{s - M_W^2 + iM_W\Gamma_W} \overline{u}(p)\gamma^\nu \left(\frac{1-\gamma_5}{2}\right) v(q_2)$$

$$= i\frac{g^2}{8}\overline{u}(q_1)\gamma^\mu (1-\gamma_5) u(k)\overline{u}(p)\gamma^\nu (1-\gamma_5) v(q_2) \frac{g_{\mu\nu} - \frac{q_\mu q_\nu}{M_W^2}}{s - M_W^2 + iM_W\Gamma_W}$$

Muon Decay

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- The width is given by the usual expression

$$\Gamma = \int \frac{1}{2m_\mu} \overline{|M|^2} (2\pi)^4 \delta^4(k - p - q_1 - q_2) \frac{d^3p}{(2\pi)^3 2p^0} \prod_{i=1}^{2} \frac{d^3 q_1}{(2\pi)^3 2q_i^0}$$

- The final state with three particles makes things complicated. However a simplification can be done due to the fact that $q^2 \ll M_W^2$

- This approximation corresponds to the effective Fermi theory and is equivalent to collapsing the propagator into a point

$$\frac{g_{\mu\nu} - \frac{q_\mu q_\nu}{M_W^2}}{q^2 - M_W^2} \to -\frac{g_{\mu\nu}}{M_W^2}$$

- In this approximation we get

$$M = -i \frac{g^2}{8M_W^2}\, \overline{u}(q_1)\gamma^\mu (1 - \gamma_5) u(k) \overline{u}(p) \gamma_\mu (1 - \gamma_5) v(q_2) \;\to\; \frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$$

Muon Decay: Total width ...

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❒ We get for averaged squared amplitude

$$\overline{|M|^2} = \frac{1}{2} \sum_{\text{spins}} |M|^2 = \frac{g^4}{128 m_W^4} \, \text{Tr}\left[\not{q}_1 \gamma^\mu (1-\gamma_5)(\not{k} + m_\mu)\gamma^\nu (1-\gamma_5)\right]$$

$$\text{Tr}\left[(\not{p} + m_e)\gamma_\mu (1-\gamma_5)\not{q}_2 \gamma_\nu (1-\gamma_5)\right]$$

$$= 2\left(\frac{g^4}{m_W^4}\right) \, k\cdot q_2 \; p\cdot q_1$$

❒ Then for the total width

$$\Gamma = \left(\frac{g}{M_W}\right)^4 \frac{1}{(2\pi)^5 m_\mu} \int \frac{d^3p}{2p^0} \int \frac{d^3q_1 \; d^3q_2}{2q_1^0 \; 2q_2^0} \delta^4(k-p-q_1-q_2) \; k\cdot q_2 \; p\cdot q_1$$

❒ We can use Lorentz invariance to show that ($\Delta = k - p$)

$$I_{\alpha\beta} = \int \frac{d^3q_1}{2q_1^0} \frac{d^3q_2}{2q_2^0} \, \delta^4(\Delta - q_1 - q_2) q_{1\alpha} q_{2\beta} = \frac{\pi}{24}\left(g_{\alpha\beta}\Delta^2 + 2\Delta_\alpha \Delta_\beta\right)$$

Muon Decay: Total width . . .

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- Putting everything together

$$\Gamma = \left(\frac{g}{M_W}\right)^4 \frac{1}{(2\pi)^5 m_\mu} \int \frac{d^3p}{2p^0} \frac{\pi}{24} \left[3k\cdot p(m_\mu^2 + m_e^2) - 4(k\cdot p)^2 - 2m_\mu^2 m_e^2\right]$$

$$= \frac{G_F^2}{32\pi^3} \int_{m_e}^{\frac{m_\mu^2+m_e^2}{2m_\mu}} dE_e \sqrt{E_e^2 - m_e^2} \left[3E_e(m_\mu^2 + m_e^2) - 4E_e^2 m_\mu - 2m_\mu m_e^2\right]$$

$$= \frac{G_F^2 m_\mu^5}{192\pi^3} \left[(1-x^2)(1-7x^2-7x^4+x^6) - 24x^4 \ln x\right]$$

where $x = \frac{m_e}{m_\mu}$

- Introducing the values from PDG we get

$$\Gamma = 2.95 \times 10^{-16}\ \text{MeV} = 4.48 \times 10^5\ \text{s}^{-1}, \quad \tau = \frac{1}{\Gamma} = 2.2 \times 10^{-6}\ \text{s}$$

IST

Higgs and Unitarity

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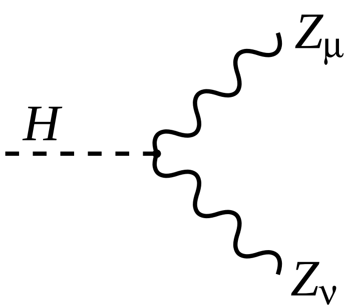
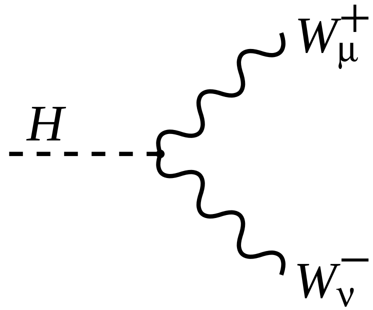
- In the last part of this lecture I will review the role of the Higgs boson in preserving unitarity of the scattering amplitudes in the Standard Model.
- We will look at the process
  - $W_L^- + W_L^+ \to W_L^- + W_L^+$

  for longitudinally polarized gauge bosons.
- We will need the couplings of the Higgs with the gauge bosons



$$i \frac{g}{\cos\theta_W} m_Z\, g_{\mu\nu}$$



$$i g\, m_W\, g_{\mu\nu}$$

- And the gauge boson self-couplings

Gauge Boson Self-Couplings

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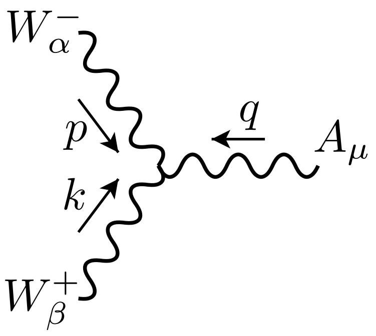
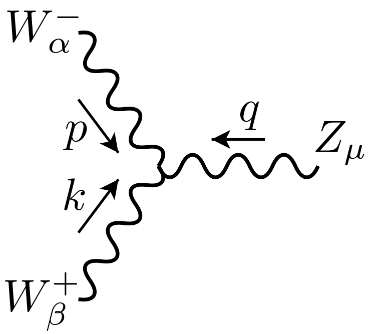
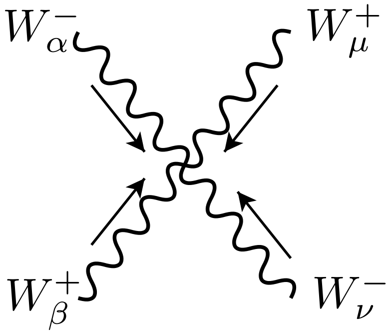


$$-ie\left[g_{\alpha\beta}(p-k)_\mu + g_{\beta\mu}(k-q)_\alpha + g_{\mu\alpha}(q-p)_\beta\right] \equiv -ie\Gamma_{\alpha\beta\mu}(p,k,q)$$



$$ig\cos\theta_W\left[g_{\alpha\beta}(p-k)_\mu + g_{\beta\mu}(k-q)_\alpha + g_{\mu\alpha}(q-p)_\beta\right] \equiv ig\Gamma_{\alpha\beta\mu}(p,k,q)$$



$$ig^2\left[2g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}\right]$$

$$\Gamma_{\alpha\beta\mu}(p,k,q) \equiv \left[g_{\alpha\beta}(p-k)_\mu + g_{\beta\mu}(k-q)_\alpha + g_{\mu\alpha}(q-p)_\beta\right]$$

Polarization Vectors

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- In many problems with massive gauge bosons we do not measure their polarization, and therefore we sum over all polarizations using the well known result,

$$\sum_\lambda \varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{M_W^2}$$

where we used the W boson as an example.

- As we will be considering the case of longitudinal polarized gauge bosons, we have to review the expressions for the polarization vectors.

- Let us start with the case of longitudinal polarization. In the gauge boson rest frame where $p^\mu = (M_W, 0, 0, 0)$, the longitudinal polarization vector is

$$\varepsilon_L^\mu(p) = (0,0,0,1), \quad \text{satisfying} \quad \varepsilon_L(p)\cdot\varepsilon_L(p) = -1, \varepsilon_L(p)\cdot p = 0\ .$$

- In the frame where the gauge boson is moving with velocity $\vec{\beta}$ we have

$$\varepsilon_L^\mu(p) = (\gamma\beta, \gamma\hat{\beta}), \quad \vec{\beta} = \vec{p}/E, \quad \gamma^{-1} = \sqrt{1-\beta^2}, \hat{\beta} = \vec{\beta}/\beta$$

Unitarity and Growth of the Amplitudes with $\sqrt{s}$

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- The last topic we want to discuss is the behaviour of the amplitudes with the growth of the center of mass energy.

- For these $1 + 2 \to 3 + 4$ processes, it can be shown that the amplitudes for large values of $\sqrt{s}$ should, at most, be constant with the energy,

$$\lim_{\sqrt{s}\to\infty} \mathcal{M} = \text{constant.}$$

- This in turn will imply that the cross sections given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 \, s} \frac{|\vec{p}_{3\mathrm{CM}}|}{|\vec{p}_{1\mathrm{CM}}|} \overline{|\mathcal{M}|}^2$$

will decrease for values of $\sqrt{s} \gg M$, where M is any mass in the problem.

The scattering $W_L^- + W_L^+ \to W_L^- + W_L^+$

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- Now we will consider the scattering of longitudinal $W_L^\pm$.

$$W_L^-(p_1) + W_L^+(p_2) \to W_L^-(q_1) + W_L^+(q_2)$$

where the momenta are as indicated and the subscript L means that the gauge bosons $W^\pm$ are longitudinally polarized.

- In the SM this process has seven tree-level diagrams

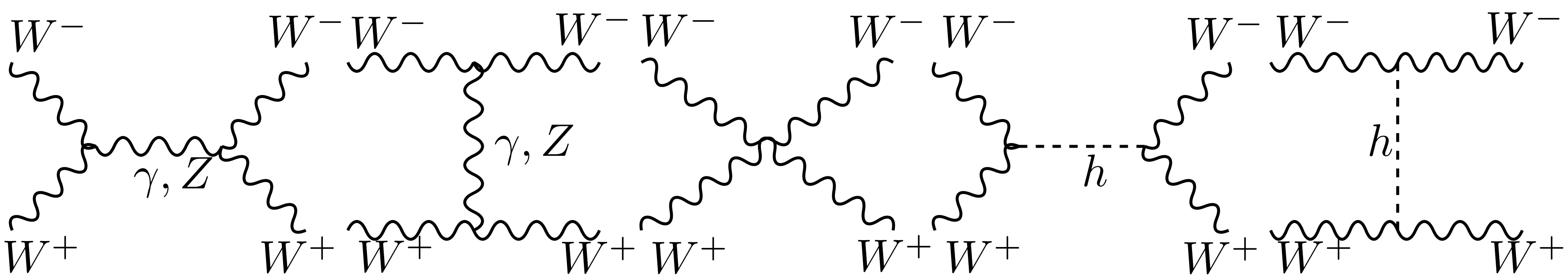

Figure 1: Diagrams contributing to $W_L^- + W_L^+ \to W_L^- + W_L^+$.

Kinematics for $W_L^- + W_L^+ \to W_L^- + W_L^+$

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$$\begin{cases} p_1 = \dfrac{\sqrt{s}}{2}(1,0,0,\beta) \\ p_2 = \dfrac{\sqrt{s}}{2}(1,0,0,-\beta) \\ q_1 = \dfrac{\sqrt{s}}{2}(1,\beta\sin\theta_{\text{CM}},0,\beta\cos\theta_{CM}) \\ q_2 = \dfrac{\sqrt{s}}{2}(1,-\beta\sin\theta_{\text{CM}},0,-\beta\cos\theta_{CM}) \end{cases} \begin{cases} \varepsilon_L(p_1) = \dfrac{\sqrt{s}}{2M_W}(\beta,0,0,1) \\ \varepsilon_L(p_2) = \dfrac{\sqrt{s}}{2M_W}(\beta,0,0,-1) \\ \varepsilon_L(q_1) = \dfrac{\sqrt{s}}{2M_W}(\beta,\sin\theta_{\text{CM}},0,\cos\theta_{CM}) \\ \varepsilon_L(q_2) = \dfrac{\sqrt{s}}{2M_W}(\beta,-\sin\theta_{\text{CM}},0,-\cos\theta_{CM}) \end{cases}$$

- $\beta = \sqrt{1 - 4M_W^2/s}$
- Notice that the invariant relations of the type $\varepsilon_L(p_1) \cdot \varepsilon_L(p_1) = -1$ and $\varepsilon_L(p_1) \cdot p_1 = 0$ are verified for all cases.
- This case is very interesting, because not only the gauge structure is necessary for the amplitudes to have the correct behaviour, but also the Higgs boson is fundamental.

The Amplitudes

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Let us denote, in an obvious notation, the amplitudes as

$$\mathcal{M} = \mathcal{M}^s_{\gamma+Z} + \mathcal{M}^t_{\gamma+Z} + \mathcal{M}_{4W} + \mathcal{M}^{s+t}_H$$

We have then,

$$\mathcal{M}^s_\gamma = \frac{g^2 s^2_W}{s}\ \epsilon^\alpha_L(p_1)\epsilon^\beta_L(p_2)\epsilon^\gamma_L(q_1)\epsilon^\delta_L(q_2)\ \Gamma_{\alpha,\beta,\mu}(p_1,p_2,-p_1-p_2) \\ \times \Gamma_{\delta,\gamma,\nu}(-q_2,-q_1,p_1+p_2)\ g^{\mu\nu}$$

$$\mathcal{M}^s_Z = \frac{g^2 c^2_W}{s - M^2_W/c^2_W}\ \epsilon^\alpha_L(p_1)\epsilon^\beta_L(p_2)\epsilon^\gamma_L(q_1)\epsilon^\delta_L(q_2)\ \Gamma_{\alpha,\beta,\mu}(p_1,p_2,-p_1-p_2) \\ \times \Gamma_{\delta,\gamma,\nu}(-q_2,-q_1,p_1+p_2)\left[g^{\mu\nu} - \frac{(p_1+p_2)^\mu (p_1+p_2)^\nu}{M^2_W/c^2_W}\right]$$

$$\mathcal{M}^t_\gamma = \frac{g^2 s^2_W}{t}\ \epsilon^\alpha_L(p_1)\epsilon^\beta_L(p_2)\epsilon^\gamma_L(q_1)\epsilon^\delta_L(q_2)\ \Gamma_{\alpha,\gamma,\mu}(p_1,-q_1,q_1-p_1) \\ \times \Gamma_{\delta,\beta,\nu}(-q_2,p_2,q_2-p_2)\ g^{\mu\nu}$$

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$$\mathcal{M}_Z^t = \frac{g^2 c_W^2}{t - M_W^2/c_W^2} \ \epsilon_L^\alpha(p_1)\epsilon_L^\beta(p_2)\epsilon_L^\gamma(q_1)\epsilon_L^\delta(q_2)\ \Gamma_{\alpha,\gamma,\mu}(p_1,-q_1,q_1-p_1)$$

$$\times\, \Gamma_{\delta,\beta,\nu}(-q_2,p_2,q_2-p_2)\left[g^{\mu\nu} - \frac{(p_1-q_1)^\mu (p_1-q_1)^\nu}{M_W^2/c_W^2}\right]$$

$$\mathcal{M}_{4W} = g^2\ \epsilon_L^\alpha(p_1)\epsilon_L^\beta(p_2)\epsilon_L^\gamma(q_1)\epsilon_L^\delta(q_2)\ \left[2 g_{\alpha\delta}\, g_{\beta\gamma} - g_{\alpha\beta}\, g_{\delta\gamma} - g_{\alpha\gamma}\, g_{\delta\beta}\right]$$

$$\mathcal{M}_H^s = -\frac{g^2 M_W^2}{s - M_H^2}\ \epsilon_L^\alpha(p_1)\epsilon_L^\beta(p_2)\epsilon_L^\gamma(q_1)\epsilon_L^\delta(q_2)\ g_{\alpha\beta}\, g_{\gamma\delta}$$

$$\mathcal{M}_H^t = -\frac{g^2 M_W^2}{t - M_H^2}\ \epsilon_L^\alpha(p_1)\epsilon_L^\beta(p_2)\epsilon_L^\gamma(q_1)\epsilon_L^\delta(q_2)\ g_{\alpha\gamma}\, g_{\beta\delta}$$

- Where $s_W^2 = \sin^2\theta_W$, $c_W^2 = \cos^2\theta_W$, and use the SM relations $M_W = c_W M_Z$ and $e = g s_W$.
- If we insert the $\epsilon_L^\mu \simeq p^\mu/M_W$ we see that the amplitudes can grow potentially as s^2 or even s^3.
- We calculate the amplitudes using the exact expressions for ϵ_L^μ

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$$\mathcal{M}_\gamma^s = \frac{g^2 s_W^2}{4M_W^4 s} \left(2M_W^2 + s\right)^2 \left(4M_W^2 - s - 2t\right)$$

$$\mathcal{M}_Z^s = \frac{g^2\, c_W^2}{4M_W^4 \left(s - M_W^2/c_W^2\right)} \left(2M_W^2 + s\right)^2 \left(4M_W^2 - s - 2t\right)$$

$$\mathcal{M}_\gamma^t = \frac{g^2 s_W^2}{4M_W^4 t \left(s - 4M_W^2\right)^2} \Big[256M_W^{10} - 64M_W^8(4s+t) + 16M_W^6 s(5s+14t)$$
$$- 4M_W^4 s\left(2s^2 + 21st + 20t^2\right) + 8M_W^2 s^2 t(s+3t) - s^2t^2(2s+t)\Big]$$

$$\mathcal{M}_Z^t = \frac{g^2 c_W^2}{4M_W^4 \left(s - 4M_W^2\right)^2 \left(t - M_W^2/c_W^2\right)} \Big[256M_W^{10} - 64M_W^8(4s+t) + 16M_W^6 s(5s+14t)$$
$$- 4M_W^4 s\left(2s^2 + 21st + 20t^2\right) + 8M_W^2 s^2 t(s+3t) - s^2t^2(2s+t)\Big]$$

$$\mathcal{M}_{4W} = \frac{g^2 s}{4M_W^4 \left(s - 4M_W^2\right)^2} \Big[-64M_W^6 + 48M_W^4(s+t) - 4M_W^2 s(3s+7t)$$
$$+ s\left(s^2 + 4st + t^2\right)\Big]$$

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And finally for the Higgs mediated diagrams,

$$\mathcal{M}_H^s = -\, g^2 \frac{\left(s - 2M_W^2\right)^2}{4M_W^2\,(s - M_H^2)}$$

$$\mathcal{M}_H^t = -\, g^2 \frac{\left(-8M_W^4 + 2M_W^2 s + st\right)^2}{4\,(t - M_H^2)\,(M_W s - 4M_W^3)^2}$$

- For $\sqrt{s} \gg M_W$ the first five amplitudes grow as s^2 and the last two (from the Higgs exchange) as s.
- We define the dimensionless variable $x = s/(4M_W^2)$, for $x \gg 1$ we should be able to write all amplitudes in the form

$$\mathcal{M}_i = A_i x^2 + B_i x + C_i + \mathcal{O}(1/x)$$

Unitarity and the cancellation of the bad behavior

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$$\mathcal{M}_i = A_i x^2 + B_i x + C_i + \mathcal{O}(1/x)$$

	A_i	B_i	C_i
$\mathcal{M}_\gamma^s$	$-g^2 4 s_W^2 \cos\theta$	0	$g^2 3 s_W^2 \cos\theta$
$\mathcal{M}_Z^s$	$-g^2 4 c_W^2 \cos\theta$	$-g^2 \cos\theta$	$g^2 \left[3\cos\theta\; c_W^2 - \dfrac{\cos\theta}{4c_W^2}\right]$
$\mathcal{M}_\gamma^t$	$g^2\, s_W^2 \left(-\cos^2\theta - 2\cos\theta + 3\right)$	$g^2\, 8 s_W^2 \cos\theta$	$\dfrac{g^2 s_W^2}{\cos\theta - 1}\left(-2\cos^2\theta - \cos\theta - 1\right)$
$\mathcal{M}_Z^t$	$g^2\, c_W^2 \left(-\cos^2\theta - 2\cos\theta + 3\right)$	$g^2 \left(8\cos\theta c_W^2 - \dfrac{\cos\theta}{2} - \dfrac{3}{2}\right)$	$\dfrac{g^2}{\cos\theta - 1}\left[-2\cos^2\theta c_W^2 - \dfrac{\cos^2\theta}{2} - \cos\theta c_W^2 - \dfrac{\cos\theta}{4c_W^2} + 3\cos\theta - c_W^2 - \frac{3}{4c_W^2} + \dfrac{3}{2}\right]$
$\mathcal{M}_{4W}$	$g^2 \left(\cos^2\theta + 6\cos\theta - 3\right)$	$g^2 \left(2 - 6\cos\theta\right)$	0
$\sum_{\gamma Z}$	0	$g^2 \dfrac{1+\cos\theta}{2}$	$g^2 3\cos\theta + \cdots$
$\mathcal{M}_H^s$	0	$-g^2$	$g^2 \left(1 - \dfrac{M_H^2}{4M_W^2}\right)$
$\mathcal{M}_H^t$	0	$g^2 \left(\dfrac{1}{2} - \dfrac{\cos\theta}{2}\right)$	$-g^2 \left[\dfrac{1+\cos\theta}{2} + \dfrac{M_H^2}{4M_W^2}\right]$
$\sum_{\gamma Z H}$	0	0	$\neq 0$

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- We see that the terms proportional to x^2 cancel among the first five diagrams involving only the gauge bosons.
- The term proportional do x remains after we sum over the gauge part. So, if we consider only a gauge theory of intermediate gauge bosons, we are in trouble.
- This trouble can be traced back to the fact that with mass the gauge invariance is lost, and the theory is inconsistent if the diagrams involving the Higgs boson field are not taken in account.
- In conclusion, the Higgs boson is crucial to make the SM consistent.

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$e^- e^+ \to \mu^- \mu^+$
Muon Decay
Higgs & Unitarity
• Couplings
• Pol. Vectors
• Unitarity
• $w_L^- + w_L^+ \to w_L^- + w_L^+$
• KInematics
• x
• Unitarity
• x

- As the amplitudes are pure c-numbers with no spinor part, the cross section is simply obtained by summing all the amplitudes and taking the absolute value of the result to obtain $\mathcal{M}|^2$.
- We get